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ABSTRACT. We begin with the classical study of the Riemann zeta function and Dirichlet L-functions. This includes a full exposition on one of the most useful ways of exploiting their connection with primes, namely, explicit formulae. We then proceed to introduce statistics of low-lying zeros of Dirichlet L-functions, discussing prior results of Fiorilli and Miller (2015) on the 1-level density of Dirichlet L-functions and their achievement in surpassing the prediction of the powerful Ratios Conjecture. Finally, we present our original work partially generalizing these results to the case of Hecke L-functions over imaginary quadratic fields.

CONTENTS

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Introduction

One of the most famous, and as yet unsolved, problems in mathematics is the Riemann hypothesis. Almost two hundred years old, and with hundreds more failed attempts at a proof (see [Wat] for a partial list), the problem continues to occupy a central role in number theory. The hypothesis posits the location of the zeros of the Riemann zeta function. While we will certainly consider this function, this thesis concerns itself more broadly with the classical study of some L-functions. By an L-function we mean a function arising from an L-series, which in turn is a complex function with an infinite series representation, given by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

converging in a half-plane $\Re(s) > \sigma$ for some $\sigma > 0$. Here, f is some object which determines the complex coefficients a_n. The Riemann zeta function, then, is precisely the L-function one obtains from setting every $a_n = 1$, that is,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1.$$

Indeed, the true objects of study are the continuations of these functions to the entire complex plane, but more on that in §1.

The theory of L-functions is a rich and complex area of mathematics that intersects number theory, complex analysis, and mathematical physics. These functions, and in particular the study of their zeros, play a crucial role in understanding the distribution of prime numbers and further have real-world implications for cryptography and chaos theory. More generally, their theory is one of the central animating forces in the Langlands program, a web of far-reaching and consequential conjectures linking seemingly disconnected subfields of mathematics, often dubbed "a Grand Unified Theory of Mathematics."

In this thesis, we begin by delving into the foundational aspects of L-functions, starting with the Riemann zeta function, and then Dirichlet L-functions. These latter objects are significant because they generalize the Riemann zeta function, allowing us to probe deeper into the mysteries of prime numbers and their patterns, and are especially important to the study of primes in arithmetic progressions.

The L-functions which are the subject of our original work in §4 are termed "Hecke L-functions from $\mathfrak{f}$-ray class characters," and are objects located at the interface of analytic and algebraic number theory. These L-functions are situated over the more general (and complicated) setting of number fields, and are essential to understanding the arithmetic of number fields. The goal is to explore the "1-level density," which is essentially an averaging statistic for the zeros of a family of L-functions. Surprisingly, this "spectral interpretation" of zeros finds its roots in random matrix theory. We discuss the deep and powerful connection between number theory and random matrix theory in detail in §3, and how it provides strong evidence for the Hilbert-Pólya conjecture[1]. This in turn may provide a possible approach to the Riemann hypothesis by means of spectral theory. Furthermore, a careful study of the 1-level density also serves to reveal the underlying symmetry of the family of L-functions in question.

[1]which Hilbert had nothing to do with, see [Odl].

Overview

In §1, we aim to provide a rigorous foundation in the most classical of L-functions, the Riemann zeta function and Dirichlet L-functions. Both are powerful mathematical entities, steeped in historical significance and complexity, and they offer profound insight into the mysterious nature of prime numbers and their distribution. Historically, the cornerstone of analytic number theory has been the Riemann zeta function, a function of complex variable that encodes deep properties of prime numbers within its zeros. The Riemann Hypothesis, a conjecture positing that all non-trivial zeros of the zeta function have a real part of $1/2$, has stood as a sentinel in the field of mathematics, guiding and challenging mathematical inquiry for over a century. This hypothesis, yet unproven, is not merely a mathematical curiosity, but a key that unlocks further understanding of the prime number theorem and the distribution of prime numbers.

We provide a complete and detailed exposition on the analytic continuation of Dirichlet L-functions in §1. Given that the Riemann zeta function is a special case of a Dirichlet L-function, we avoid proving the analytic continuation of the former, and have only opted for a more general approach. With the standard theory of Dirichlet L-functions in hand – namely, a solid understanding of their analytic continuation and functional equation – we can consider one of the most powerful tools for studying zeros of L-functions in §2: explicit formulae. These are key tools in understanding the relation between prime numbers (or more generally, primes in number fields) and zeros of L-functions. They explicitly relate a sum over zeros of an L-function to a sum over prime numbers. Such explicit formulae have also been applied to questions on bounding the discriminant of an algebraic number field and estimating the conductor of a number field. We provide complete proofs of the explicit formulae for the Riemann zeta function and Dirichlet L-functions.

With a firm grasp of explicit formulae, we are ready to move on to a discussion of our main object of study in §3, namely, the 1-level density, following the work of [FM15]. Indeed, when we wish to derive statistics of zeros such as the 1-level density, the explicit formula is key to further investigations. It allows us to pass from sums over coefficients of L-functions (which we can compute), to statistics of zeros (which we want to know). We supply an extensive motivation in §3.1, whereby we hope to convey not just the power of this statistic, but also provide its history.

For the sections described till this point, standard undergraduate coursework more generally, and in particular a strong background in complex and Fourier analysis, are sufficient to follow the key ideas in proofs. In §4, however, we make a significant jump to an area of active research. General familiarity with the techniques of analytic number theory will be beneficial for following the arguments, and to understand the setting of our study, so would familiarity with algebraic number theory.

1. Dirichlet L-Functions

The starting point for considering an L-function is its corresponding L-series, by which we mean an infinite series representation which converges on some half-plane. A *Dirichlet series* is given as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.0-1}$$

for $s \in \mathbb{C}$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. To fix notation, we henceforth write $s = \sigma + it$ with $\Re s = \sigma$ and $\Im s = t$.

The material for this section primarily references [Dav80], [MTB06], and [Tit86].

1.1. The Riemann zeta function.

The most classical example of a Dirichlet series is the Riemann-zeta function, where we set all coefficients $a_n = 1$.

Definition 1.1.1. The *Riemann zeta function* for $\Re s > 1$ is given by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The definition assumes convergence for $\Re s > 1$, and while this is not difficult to see, let us prove it formally.

Proposition 1.1.2. *For $\Re s > 1$, $\zeta(s)$ converges absolutely.*

Proof. Note

$$\left|\frac{1}{n^s}\right| = \frac{1}{|n^{\sigma+it}|} = \frac{1}{n^\sigma},$$

so that

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma},$$

which converges for $\sigma > 1$ by the p-series test (a corollary of the integral test). □

First things first, when dealing with complex functions, the foremost thing we would like to establish is their holomorphy (i.e., they are complex differentiable, or equivalently, analytic).

Lemma 1.1.3. *The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly to $\zeta(s)$ on $\Re s > 1 + \delta$ for any $\delta \in \mathbb{R}^+$.*

Proof. Set $f_n(x) = \frac{1}{n^s}$ and $M_n = \frac{1}{n^{1+\delta}}$. Then,

$$|f_n(x)| = \left|\frac{1}{n^s}\right| = \frac{1}{n^{\Re s}} \leq \frac{1}{n^{1+\delta}}$$

for $\Re s > 1 + \delta$. Further,

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$$

is a series of non-negative numbers that clearly converges. By the Weierstrass M-Test, our series converges uniformly. □

In particular, this means that the series converges locally uniformly to $\zeta(s)$ on $\Re s > 1$. Observe that $n^{-s} = e^{-s\log n}$ is entire. Recalling that the uniform limit of holomorphic functions which converge locally uniformly is holomorphic, we have proven the following.

Proposition 1.1.4. *On the half-plane $\Re s > 1$, $\zeta(s)$ is holomorphic.*

Well, holomorphic functions are lovely to work with, and now we have all the tools of complex analysis at our disposal to apply to $\zeta(s)$ if we so desire. But $\zeta(s)$ is not just any analytic function. Theorem 1.1.5 below gives us our first hint at the deep and fundamental connection between $\zeta(s)$ and the prime numbers, but we will have to wait till Theorem 2.1.4 (the explicit formula) to see how to precisely exploit this.

A short note on notation: some authors use P, $\mathbb{Z}_p$, or $\mathbb{P}$ to denote the set of primes, but the downfalls of such notation (especially the second and third being $\mathbb{Z}/p\mathbb{Z}$ and projective space, respectively) are clear. Rather, we eschew any such notation, and rely on convention that p *always* denotes a prime number. So, $\displaystyle\sum_p$ or $\displaystyle\prod_p$ represent the sum and product over all primes, respectively.

Theorem 1.1.5 (Euler Product for $\zeta(s)$)**.** *For $\Re s > 1$, we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Proof. Recall the general fact that for a sequence $\{1 + a_n\}$ of non-zero complex numbers, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if $\prod_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent (see [Ahl79, Chapter 5, Theorem 6]).

Observe

$$\sum_p \left|-p^{-s}\right| = \sum_p p^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma},$$

where the final series certainly converges for $\Re s = \sigma > 1$ as discussed earlier.

It follows that $\prod_p(1 - p^{-s})$ is convergent to a non-zero value. It then follows that the reciprocal, $\prod_p(1 - p^{-s})^{-1}$ is also convergent.

Now we prove the series agree, that is, $\sum_{n=1}^{\infty} n^{-s}$ and $\prod_p(1-p^{-s})^{-1}$ converge to the same number. We begin with expressing each term of the infinite product as a geometric series, as follows:

$$\begin{aligned}\prod_{p\leq M} \left(1 - p^{-s}\right)^{-1} &= \prod_{p\leq M} \frac{1}{1 - \frac{1}{p^s}} \\ &= \prod_{p\leq M} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right).\end{aligned}$$

Using absolute convergence of the geometric series to freely rearrange terms, we observe that multiplying out the final product above will allow us to group the result into two sums. Since we have multiplied all primes $\leq M$, we can certainly obtain all integers $n \leq M$, but also have another remaining sum consisting of terms $n > M$ such that the prime factors of n are $\leq M$. Hence,

$$\prod_{p\leq M} \left(1 - p^{-s}\right)^{-1} = \sum_{n\leq M} n^{-s} + \sum_{n\in X} n^{-s}, \quad \text{where } X = \{n > M \,|\, p \text{ divides } n \text{ implies } p \leq M\}.$$

Taking the limit as $M \to \infty$ gives the desired result. $\square$

To unlock the full power of the Riemann zeta function, we must overcome the limitation of $\zeta(s)$ being convergent only on a half-plane. Indeed, we can find a meromorphic continuation of $\zeta(s)$ with a single singularity, namely, a simple pole at $s = 1$. Before we present the theorem, we must embark on a brief interlude to understand the Gamma function.

Definition 1.1.6. *The Gamma function* is defined as the Mellin transform of the negative exponential function, and given by

$$\Gamma(s) \coloneqq \mathcal{M}(e^{-t})(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \qquad \Re s > 0.$$

Remark 1.1.7. When presented in the context of a real analysis or statistics course, the function is defined only with s as a real variable. There, some of its relevant properties include $\Gamma(1) = 1$, $\Gamma(s+1) = s\Gamma(s)$, and the fact that $\log \Gamma(s)$ is a convex function. Here, however, we consider the function more generally in the complex analytic context.

First, we verify that the Gamma function is safe to work with by checking its convergence and holomorphy.

Proposition 1.1.8. *The Gamma function is well-defined as a holomorphic function in the half-plane $\Re s > 0$.*

Proof. We consider

$$I_n(s) \coloneqq \int_{1/n}^{n} t^{s-1} e^{-t}\, dt.$$

We will show that the functions $I_n(s)$ converge uniformly to $\Gamma(s)$ on each strip of the form

$$S = \{s \in \mathbb{C} \mid a \le \Re s \le b\}, \quad 0 < a < b.$$

Observe that for $t \le 1$,

$$\left|e^{-t} t^{s-1}\right| \le t^{a-1}, \quad s \in S,$$

and for $t \ge 1$,

$$\left|e^{-t} t^{s-1}\right| \le e^{-t} t^{b-1}, \quad s \in S. \tag{1.1-1}$$

Note that $e^{-t} t^{b+1}$ is continuous and goes to 0 at infinity, and we can hence write for some positive K that

$$\left|e^{-t} t^{s-1}\right| \le e^{-t} t^{b-1} \le K t^{-2}, \quad t \ge 1, s \in S. \tag{1.1-2}$$

Now, putting (1.1-1) and (1.1-2) together we have

$$\int_0^\infty \left|t^{s-1} e^{-t}\right| dt = \int_0^1 \left|t^{s-1} e^{-t}\right| dt + \int_1^\infty \left|t^{s-1} e^{-t}\right| dt \le \int_0^1 t^{a-1}\, dt + \int_1^\infty K t^{-2}\, dt < \infty,$$

where in the final step we note that t^{a-1} is integrable on $(0, 1]$ exactly when $a > 0$, and that Kt^{-2} is integrable on $[1, \infty]$. So, the improper defining $\Gamma(s)$ converges absolutely for any $s \in S$.

Finally, we show that $\Gamma(s)$ is holomorphic. Certainly $I_n(s)$ is holomorphic (by Morera's theorem, for example). Since the uniform limit of holomorphic functions is holomorphic, it will suffice to show that the functions $I_n(s)$ converge uniformly on each strip of the form S as $n \to \infty$. Indeed, using (1.1-1) and (1.1-2) we have

$$|\Gamma(s) - \Gamma_n(s)| \leq \int_0^{1/n} t^{a-1}\,dt + \int_n^\infty K t^{-2}\,dt \leq \frac{1}{n^a\,a} + \frac{K}{n}.$$

Given ϵ, we can clearly choose n (independent of s) large enough so that the right-hand side of the above inequality is less than ϵ. This completes the proof. □

Next, we would like to obtain a meromorphic continuation of $\Gamma(s)$ to the complex plane. There are a multitude of ways we might approach this problem. Indeed, there are similarly numerous ways we may define the Gamma function. The integral form given in Definition 1.1.6 is just one possible approach. For other methods, see [PM22]. We opt for the functional equation method presented in Theorem 1.1.11. This method has the added benefit of emphasizing the Gamma function as a natural generalization of the factorial, as we shall now see in the following proposition and its corollary.

Proposition 1.1.9. *For $\Re s > 0$, the Gamma function satisfies the functional equation*

$$\Gamma(s+1) = s\,\Gamma(s).$$

Proof. By definition, we can write

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^s\,dt.$$

Integrating by parts yields

$$\begin{aligned}\Gamma(s+1) &= \left[-t^s e^{-t}\right]_0^\infty + \int_0^\infty s\,t^{s-1} e^{-t}\,dt \\ &= s\int_0^\infty t^{s-1} e^{-t}\,dt \\ &= s\,\Gamma(s).\end{aligned}$$

□

Corollary 1.1.10. *If $n \in \mathbb{Z}^+$, then*

$$\Gamma(n) = (n-1)!.$$

Proof. Observe that

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} = 1\,dt.$$

Using Proposition 1.1.9, we conclude inductively that $\Gamma(n) = (n-1)!$. □

We are now in a position to demonstrate the meromorphic continuation of the Gamma function by means of the functional equation.

Theorem 1.1.11. *The Gamma function as given in Definition 1.1.6 admits an analytic continuation to a meromorphic function on the complex plane whose singularities are simple poles at $s = 0, -1, -2\ldots,$ with corresponding residues $\frac{(-1)^n}{n!}$.*

Proof. It suffices to demonstrate continuation to $\Re s > -m$, for all $m > 0$. So, we begin by setting $\Gamma_0(s) = \Gamma(s)$ as defined in Definition 1.1.6. We then further set

$$\Gamma_1(s) = \frac{\Gamma_0(s+1)}{s}, \quad \Re s > -1.$$

This is exactly the unique continuation of $\Gamma_0(s)$ to $\Re s > -1$, since by the functional equation of Proposition 1.1.9 we have $\Gamma_1(s) = \Gamma_0(s)$ for $\Re s > 0$. Since $\Gamma_0(s+1)$ is analytic with no singularities, $\Gamma_1(s)$ has a simple pole at $s = 0$. The residue is given by

$$\operatorname*{Res}_{s=0} \Gamma_1(s) = \lim_{s\to 0} s\,\Gamma_1(s) = \Gamma_0(1) = 1.$$

Similarly, we can set

$$\Gamma_2(s) = \frac{\Gamma_1(s+1)}{s} = \frac{\Gamma_0(s+2)}{s(s+1)}, \quad \Re s > -2.$$

Indeed, inductively, we obtain

$$\Gamma_m(s) = \frac{\Gamma_0(s+m)}{\prod_{j=0}^{j=m-1}(s+j)}, \quad \Re s > -m,$$

so that $\Gamma_m(s)$ is meromorphic with simple poles at $s = 0, -1, \ldots, -(m-1)$.
Finally, the residue at $s = -(m-1)$ is given by

$$\operatorname*{Res}_{s=-(m-1)} \Gamma_m(s) = \lim_{s\to-(m-1)} (s+m-1)\,\Gamma_m(s) = \frac{\Gamma_0(1)}{\prod_{j=0}^{j=m-2}(j-(m-1))} = \frac{(-1)^{m-1}}{(m-1)!}.$$

Successive applications of the functional equation as in Proposition 1.1.9 show that $\Gamma_m(s) = \Gamma(s)$ for $\Re s > 0$, and this shows $\Gamma(s)$ admits a meromorphic continuation to $\Re s > -m$ for any $m \in \mathbb{Z}^+$. □

For the rest of this thesis, unless specified, we call the function we obtain in the above proof as *the* Gamma function, defined for all $s \in \mathbb{C} \setminus \{0, -1, -2 \ldots\}$. Note that by our construction, we also have $\Gamma(s+1) = s\,\Gamma(s)$ for all $s \in \mathbb{C}$, as expected.

Finally, we state one more property of the Gamma function.

Proposition 1.1.12 (Euler's reflection formula)**.** *For all* $s \in \mathbb{C}$*,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{1.1-3}$$

We omit a proof, since the techniques are similar to those we have presented so far. Essentially, we must derive another functional equation by means of Euler's beta function. A complete proof may be found in [Tay11, Theorem 9.1.7]. Note that Euler's reflection formula provides an alternate means of locating the poles of $\Gamma(s)$ and computing the corresponding residues, by studying the denominator $\sin(\pi s)$. The particular consequence of Euler's reflection formula we are interested in is the following.

Corollary 1.1.13. *The Gamma function vanishes nowhere. That is, it has no zeros.*

Proof. Suppose $\Gamma(s) = 0$. Since the right-hand side of (1.1-3) is always non-zero, we conclude that $1 - s$ must be a pole of the Gamma function. But as we showed in 1.1.11, this means $1 - s = -n$, or equivalently $s = n + 1 \in \mathbb{Z}^+$. But we know from Corollary 1.1.10 that for $s = n+1$ we have $\Gamma(s) = n!$, contradicting $\Gamma(s) = 0$. □

With a solid understanding of the Gamma function, we can now state one of the most important theorems in number theory.

Theorem 1.1.14. *Define the completed zeta function by*

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s).$$

Originally defined only for $\Re s > 1$, $\xi(s)$ has an analytic continuation to an entire function. Furthermore, it satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Since the Riemann zeta function is but a special case of a Dirichlet L-function, in particular that associated with a Dirichlet character modulo 1, we defer the proof of this theorem to §1.4.

The form of the completed zeta function reveals that the only possible zeros of $\zeta(s)$ in $\Re s > 1$ and $\Re s < 0$ could arise from the poles of $\Gamma\big(\frac{s}{2}\big)$. As noted in Theorem 1.1.11, these will be precisely at $s = 0, -2, -4, \ldots$. Since the pole at $s = 0$ is cancelled by the s factor in the definition of $\xi(s)$, we are left with zeros of $\zeta(s)$ at $s = -2, -4, -6, \ldots$ These are called the *trivial zeros* of $\zeta(s)$ because they are easy to determine. The *non-trivial zeros* are those which lie in the *critical strip*, defined by the region $0 < \Re s < 1$. Finally, note that the factor of $(s-1)$ in the definition of $\xi(s)$ provides $\zeta(s)$ with exactly one simple pole at $s = 1$.

We would be remiss to not mention the seminal Riemann hypothesis, which posits more strictly the location of the non-trivial zeros.

Conjecture 1.1.15 (Riemann hypothesis)**.** *All non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re s = \dfrac{1}{2}$.*

We now consider a more general Dirichlet series in the form of Dirichlet L-functions, whereby the coefficients a_n in (1.0-1) come from Dirichlet characters. Before considering the L-functions themselves, we must understand Dirichlet characters and their properties.

1.2. **Dirichlet Characters.**

See Appendix B for the general theory of characters; in particular, recall the notion of a group character from Definition B.1. We now describe the class of characters which will occupy our attention for the remainder of this section.

Definition 1.2.1. Let $q \in \mathbb{N}$. A Dirichlet character modulo q is a character of the multiplicative group $\mathbb{Z}_q^\times$. We extend to an arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ setting

$$\chi(n) = \begin{cases} \chi(n \bmod q) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem B.6, we can see that the number of Dirichlet characters modulo q is exactly $\varphi(q)$, with φ the Euler totient function. From Proposition B.5, we note that $|\chi(n)| = 1$ for $(n,q) = 1$.

The *trivial character* or *principal character* modulo q is

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Further recall from Theorem B.6 that the set of characters modulo q form a multiplicative group (indeed, isomorphic to $\mathbb{Z}_q^\times$) with $\varphi(q)$ elements. Here, χ_0 is the identity element, and the inverse of χ is given by the character $\overline{\chi}(n) := \overline{\chi(n)}$.

Remark 1.2.2. An alternative (but equivalent) definition would be to state a Dirichlet character modulo q is a completely multiplicative arithmetic function (see Appendix A) $\chi : \mathbb{Z} \to \mathbb{C}$ of period q such that $\chi(n) = 0$ if and only if $(n, q) > 1$. The added benefit of this definition is that it is more clear what we *might* mean by a Dirichlet character modulo 1. It is only in the case of modulo 1 characters that $\chi(0) \neq 0$, whereby periodicity forces $\chi(n) = 1$ for all n. We will see later that, in a sense, it is a consequence of this fact that $L(s, \chi_0)$ has a simple pole at 1, since for any q, χ_0 is induced by the character ψ modulo 1, in which case $L(s, \psi) = \zeta(s)$. That being said, we must be careful when Dirichlet characters modulo 1 come up, particularly in questions of their primitivity (see Remark 1.2.11).

Now for one of the most important properties of Dirichlet characters, which is particularly useful when we would like to take averages over families of L-functions (see §3.2). Indeed, as can be seen from Theorem B.7, Dirichlet characters exhibit *orthogonality*.

Proposition 1.2.3 (Orthogonality of Dirichlet Characters)**.** *Let $q \in \mathbb{N}$. Then the Dirichlet characters modulo q satisfy*

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise}, \end{cases}$$

and further

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \bmod q, \\ 0 & \text{otherwise}. \end{cases}$$

An equivalent form that emphasizes the orthogonality is given below in Corollary 1.2.4. The first relation is central to the theory of the discrete Fourier transform. The second relation is useful for detecting congruence conditions in counting problems. In particular, for fixed $a \in \mathbb{Z}$, it provides a way to filter for $n \equiv a \bmod m$, and can hence be used to find primes congruent to $a \bmod m$, as shown in the following corollary.

Corollary 1.2.4. *Let $q \in \mathbb{N}$. Let χ, χ' denote Dirichlet characters modulo q. They satisfy the relations*

$$\frac{1}{\varphi(q)} \sum_{n \bmod q} \chi(n)\overline{\chi'}(n) = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise}, \end{cases}$$

and further, for any $a, n \in \mathbb{Z}$, we have

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \bmod q, \\ 0 & \text{otherwise}. \end{cases}$$

We now proceed to consider how one character can be "built" from another.

Definition 1.2.5. Suppose $d \mid q$ and χ^* is a character modulo d. Consider

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } (n, q) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Then, it can be seen that χ is a character modulo q. If we can write χ using χ^* as above, then we say that χ^* *induces* χ.

Remark 1.2.6. If d and q share all prime factors, then we can see that $\chi(n) = \chi^*(n)\ \forall n$, (so χ and χ^* are actually the same) and χ has period d. In fact, the converse is also true, as alluded to earlier. Indeed, suppose d, q did not share all prime factors. Then, we can find n such that $(n, d) = 1$ but $(n, q) > 1$. We can then choose k such that $(n + kd, q) = 1$ (see proof of Theorem 1.2.12 for how). Clearly then $\chi(n) = 0 \neq \chi(n + kd)$ and so χ cannot have period d.

As one may have gleaned from the above discussion, Dirichlet characters modulo q are of course periodic with period q, but their periodicity is in fact more subtle than one may assume on first glance. Hence the following definitions.

Definition 1.2.7 (Quasiperiod)**.** Let χ be a Dirichlet character modulo q. We call d a *quasiperiod* of χ if

$$\chi(m) = \chi(n), \text{whenever } m \equiv n \bmod d \text{ and } (mn, q) = 1.$$

The quasiperiod only takes into account the periodicity in the nonzero values of χ, which are of course the values which matter the most.

Definition 1.2.8 (Conductor)**.** The smallest quasiperiod of χ is called its *conductor*.

A key property of the conductor is captured by the following proposition.

Proposition 1.2.9. *Let χ be a Dirichlet character modulo q. The conductor of χ must divide q.*

Proof. Suppose d is the conductor of χ. Consider $g = (d, q)$. Then,

$$m \equiv n \bmod g \iff m - n = kg \iff m - n = dx + qy,$$

where in the final step we used Bezout's identity. Then, for $(mn, q) = 1$, we can write

$$\chi(m) = \chi(n + dx + qy) = \chi(n + dx) = \chi(n).$$

The second equality follows because q is a period. The third equality is true because d is a quasiperiod, and $(n(n + dx), q) = (n(m - qy), q) = (n, m) = 1$.

What we have established is that g is also a quasiperiod of χ, but since d was the least such quasiperiod, we must have $d = g = (d, q)$, and so $d \mid q$. □

Not all Dirichlet characters are built equal. Indeed, some are in a sense more fundamental than others, and can be used to construct all other characters. We distinguish these types of characters in the following definition.

Definition 1.2.10 (Primitivity)**.** Let χ be a Dirichlet character modulo q. Then, χ is said to be *primitive* if it has conductor q. If not, then χ is *imprimitive*.

Remark 1.2.11. It can now be seen that *any* principal character χ_0 modulo q with $q \neq 1$ is imprimitive (it has quasiperiod 1). That being said, *we will also consider χ modulo 1 to be an imprimitive character by convention,* so as to not have this special case break down our later results (see Remark 1.2.2).

The final result of this subsection shows that all characters are built out of primitive ones, and so it suffices to work with primitive characters (which is often easier) and generalize later to imprimitive characters.

Theorem 1.2.12. *Let χ be a Dirichlet character modulo q with conductor d. Then there exists a unique primitive character χ^* modulo d that induces χ.*

Proof. It has been shown in Proposition 1.2.9 that $d \mid q$. Next, put

$$r = \prod_{\substack{p^a \| q \\ p \nmid d}} p^a.$$

Consider any $n \in \mathbb{Z}$. If $(n, q) = 1$ then define $\chi^*(n) = \chi(n)$.

If $(n, q) > 1$ but $(n, d) = 1$, we choose $k \in \mathbb{Z}$ such that $(n+kd, q) = 1$ and define $\chi^*(n) = \chi(n+kd)$. First, note that such k exists. It suffices to have $(n + kd, r) = 1$, because by choice we also have $(n, d) = 1$ which implies $(n + kd, d) = 1$, and combining $(n + kd, d) = 1$ with $(n + kd, r) = 1$ gives us $(n+kd, q) = 1$. Next, since there are many choices of k, we must ensure that $\chi(n+kd)$ is well-defined. Indeed, for $(n+kd, q) = 1$ and $(n+k'd, q) = 1$, since d is a quasiperiod of χ and $n+kd \equiv n+k'd \bmod d$ with $((n + k'd)(n + kd), q) = 1$, we conclude $\chi(n + kd) = \chi(n + k'd)$.

Finally, for $(n, d) > 1$ we of course set $\chi^*(n) = 0$. To recap, we have defined

$$\chi^*(n) = \begin{cases} \chi(n) & \text{if } (n, q) = 1, \\ \chi(n + kd) & \text{if } (n, q) > 1 \text{ but } (n, d) = 1 \text{ (with } k \text{ as earlier)}, \\ 0 & (n, d) > 1. \end{cases}$$

We can now easily see that χ^* is a Dirichlet character modulo d that χ^* induces χ. The primitivity of χ^* follows from the fact that d was the conductor, i.e., the smallest quasiperiod. A smaller quasiperiod for χ^* would imply a smaller one for χ.

It remains to establish uniqueness. Suppose χ' is another primitive Dirichlet character modulo d that induces χ. Consider $(n, d) = 1$, with k as before such that $(n + kd, q) = 1$ so that

$$\chi^*(n) = \chi^*(n + kd) = \chi(n + kd) = \chi'(n + kd) = \chi'(n).$$

□

1.3. **Dirichlet L-functions.** We are now ready to conduct a detailed study of Dirichlet L-functions.

Definition 1.3.1. The Dirichlet L-function of a Dirichlet character modulo q is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re s > 1.$$

As in the case of the Riemann zeta function, the series converges absolutely, and furthermore, $L(s, \chi)$ is holomorphic on the half-plane $\Re s > 1$. The proofs are identical to that of the corresponding Propositions 1.1.2 and 1.1.4. Indeed, all we must note is that $|\chi(n)| = 1$ when $(n, q) = 1$ (see Proposition B.5) and of course $|\chi(n)| = 0$ otherwise, by definition.

These functions (and virtually all L-functions that we work with) also have an Euler product. By virtue of the total multiplicativity of Dirichlet characters, the proof is very similar to that of Theorem 1.1.5.

Theorem 1.3.2. *Let χ be a Dirichlet character modulo q. For $\Re s > 1$,*

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1}.$$

Proof. Recall that for a sequence $\{1+a_n\}$ of non-zero complex numbers, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent (see proof of Theorem 1.1.5).

Observe
$$\sum_p \left|-\chi(p)p^{-s}\right| \leq \sum_p p^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma},$$
where the final series certainly converges for $\Re s = \sigma > 1$. The second equality used the fact that $|\chi(n)| \leq 1$ (see Proposition B.5).

It follows that $\prod_p(1-\chi(p)p^{-s})$ is convergent. It then follows that the reciprocal, $\prod_p(1-p^{-s})^{-1}$ is also convergent.

Now we prove the series agree, that is $\sum_{n=1}^{\infty}\chi(n)n^{-s}$ and $\prod_p(1-\chi(p)p^{-s})^{-1}$ converge to the same number. We begin with expressing each term of the infinite product as a geometric series, that is,
$$\begin{aligned}\prod_{p\leq M}\left(1-\chi(p)p^{-s}\right)^{-1} &= \prod_{p\leq M}\frac{1}{1-\frac{\chi(p)}{p^s}}\\ &= \prod_{p\leq M}\left(1+\frac{\chi(p)}{p^s}+\frac{\chi(p)^2}{p^{2s}}+\dots\right).\end{aligned}$$

Using absolute convergence of the geometric series to freely rearrange terms, we observe that multiplying out the final product above will allow us to group the result into two sums. Since we have multiplied all primes $\leq M$, we can certainly obtain all integers $n \leq M$, but also have another remaining sum consisting of terms $n > M$ such that the prime factors of n are $\leq M$. It is precisely the total multiplicativity of χ which ensures that the numerator "matches" with the denominator, allowing us to conclude that the form of the resulting terms is $\chi(n)n^{-s}$.

Hence,
$$\prod_{p\leq M}\left(1-\chi(p)p^{-s}\right)^{-1} = \sum_{n\leq M}\chi(n)n^{-s} + \sum_{n\in X}\chi(n)n^{-s},$$
where $X = \{n > M \,|\, p \text{ divides } n \text{ implies } p \leq M\}$. Taking the limit as $M \to \infty$ of our final equality gives the desired result. $\square$

It is usually easier to prove statements about $L(s,\chi)$ when χ is primitive. These can, however, be generalized to the case of imprimitive characters, and the key bridge is the following Proposition.

Proposition 1.3.3. *Let χ be a character modulo m, induced by a character ψ. Then, for $\Re s > 1$,*
$$L(s,\chi) = L(s,\psi)\prod_{p|m}\left(1-\frac{\psi(p)}{p^s}\right).$$

(We have used m instead of q so as to not confuse q being a prime.)

Proof. We have
$$L(s,\chi) = \prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1},$$

but since $\chi(p) = 0$ for $p \mid m$ and $\chi(p) = \psi(p)$ for $p \nmid m$,

$$L(s,\chi) = \prod_{p \nmid m} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1}.$$

Then,

$$\begin{aligned} L(s,\chi) \prod_{p \mid m} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} &= \prod_{p \nmid m} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} \prod_{p \mid m} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} \\ &= \prod_{p} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} \\ &= L(s,\psi). \end{aligned}$$

Note we can freely rearrange terms in the products since they converge absolutely. □

1.4. **Analytic Continuation and Functional Equation.** We now come to a key property of Dirichlet L-functions, namely, their analytic continuation to the entire complex plane.

Proving the following theorem will take the entirety of this subsection. It is analogous to that stated in Theorem 1.1.14, i.e., the analytic continuation for the Riemann zeta function. The key difference to note now is that the analytic continuation of $L(s,\chi)$ with χ primitive has no poles.

Theorem 1.4.1. *Let χ be a primitive Dirichlet character modulo q. The function $L(s,\chi)$ originally defined only for $\Re s > 1$ has an analytic continuation to an entire function. Further, if we set*

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) L(s,\chi),$$

with

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

then

$$\Lambda(s,\chi) = (-i)^\epsilon \frac{\tau(\chi)}{q^{1/2}} \Lambda(1-s,\overline{\chi}),$$

where $\tau(\chi)$ is the Gauss sum defined as

$$\tau(\chi) \coloneqq \sum_{a=0}^{q-1} \chi(a) e^{2\pi i a/q}.$$

Note that for the coefficient in the functional equation we have

$$\left| (-i)^\epsilon \frac{\tau(\chi)}{q^{1/2}} \right| = 1,$$

which we will see follows from Proposition 1.4.6.

In order to prove the above theorem, we must recall some Fourier analysis (see Appendix C). Denote by $\hat{f}$ the Fourier transform of $f \in L^1(\mathbb{R})$. Now, one of our key tools in proving functional equations will be the Poisson summation formula (see Theorem C.7). However, since $L(s,\chi)$ is $\zeta(s)$ "twisted" by χ, we would like to twist Poisson summation by χ. We achieve this in Theorem 1.4.8, but it will take some work to get there.

Recall (see Theorem C.7) that Poisson summation was obtained from

$$F(x) = \sum_{m=-\infty}^{\infty} f(x+m), \tag{1.4-1}$$

which had Fourier series expansion given by

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}. \tag{1.4-2}$$

Setting $x = 0$ gave us the basic Poisson summation formula. We will now use the Fourier series for fractional x.

Proposition 1.4.2. *Consider $c : \mathbb{Z}_q^\times \to \mathbb{C}$ and $f \in L^1(\mathbb{R})$. Then,*

$$\sum_{m=-\infty}^{\infty} c(m) f(m/q) = \sum_{n=-\infty}^{\infty} \hat{c}(-n) \hat{f}(n),$$

where $\hat{c}$ is the discrete Fourier transform of c, defined by

$$\hat{c}(n) := \sum_{a \bmod q} c(a) e^{-2\pi i n a/q}. \tag{1.4-3}$$

Proof. Note that absolute convergence of the sums (because $f \in L^1$) allows us to freely rearrange terms. We have

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} c(m) f(m/q) &= \sum_{a \bmod q} \sum_{m=-\infty}^{\infty} c(a+mq) f((a+mq)/q) \\
&= \sum_{a \bmod q} c(a) \sum_{m=-\infty}^{\infty} f(a/q+m) \\
&= \sum_{a \bmod q} c(a) F(a/q) \qquad \text{(using (1.4-1))} \\
&= \sum_{a \bmod q} c(a) \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n a/q} \qquad \text{(using (1.4-2))} \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n) \sum_{a \bmod q} c(a) e^{2\pi i n a/q}
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \hat{c}(-n)\hat{f}(n).$$

□

We have just determined what Poisson summation is twisted by an arbitrary $c : \mathbb{Z}_q^\times \to \mathbb{C}$. Of course, for us, c is not just any function. It will be a primitive character χ. The additional specializations of homomorphy of χ and primitivity allow us to simplify the discrete Fourier transform. Hence, we now focus on understanding the discrete Fourier transform of χ.

Definition 1.4.3. Generalized Gauss sum Define

$$\tau_n(\chi) = \sum_{a=0}^{q-1} \chi(a)e^{2\pi ina/q}.$$

Then, the *Gauss sum*, as defined in Theorem 1.4.1, is a special case of τ_n with $n = 1$, given by

$$\tau(\chi) := \tau_1(\chi) = \sum_{a=0}^{q-1} \chi(a)e^{2\pi ia/q}.$$

In the following proposition, we show that for primitive χ, the discrete Fourier transform decomposes nicely.

Proposition 1.4.4. *For χ primitive mod q, we have*

$$\tau_n(\chi) = \overline{\chi}(n)\tau(\chi). \tag{1.4-4}$$

Proof. Set $\omega = e^{2\pi i/q}$.

There are two cases. First, for $\gcd(n, q) = 1$, we can set $a = n^{-1}b$, so that

$$\tau_n(\chi) = \sum_{a=0}^{q-1} \chi(a)\omega^{na} = \sum_{b=0}^{q-1} \chi(n^{-1}b)\omega^{b} = \chi(n^{-1}) \sum_{b=0}^{q-1} \chi(b)\omega^{b} = \overline{\chi}(n)\tau(\chi).$$

Next, for $\gcd(n, q) \neq 1$, $\chi(n) = 0$, so that the right-hand side of (1.4-4) vanishes. So, what we need to show is

$$\tau_n(\chi) = \sum_{a=0}^{q-1} \chi(a)\omega^{na} = 0, \text{ for } \gcd(n, q) \neq 1.$$

Consider

$$\overline{\chi}(b)\tau_n(\chi) = \overline{\chi}(b) \sum_{a=0}^{q-1} \chi(a)\omega^{na}$$

$$= \sum_{a=0}^{q-1} \chi(ab^{-1})e^{2\pi ina/m}$$

$$= \sum_{c=0}^{q-1} \chi(c) e^{2\pi i c b \frac{n}{q}} \qquad \text{set } c = ab^{-1}$$

$$= \sum_{c=0}^{q-1} \chi(c) e^{2\pi i c \frac{bn'}{q'}}, \tag{1.4-5}$$

where $n' = \dfrac{n}{\gcd(n,q)}, m' = \dfrac{q}{\gcd(n,q)}$ and $\gcd(n', q') = 1$.

Now, χ is primitive, so for $q' \mid q$, there exists α, β with $\alpha \equiv \beta \bmod q'$, and $\gcd(\alpha\beta, q) = 1$, such that

$$\chi(\alpha) \neq \chi(\beta).$$

Setting $b = \alpha\beta^{-1}$, we have

$$b \equiv \alpha\beta^{-1} \equiv 1 \bmod q' \text{ and } \chi(b) \neq 1 \text{ because } \chi(\alpha) \neq \chi(\beta).$$

In conclusion, we choose b such that

$$bn' \equiv n' \bmod q', \text{ and } \chi(b) \neq 1.$$

So, continuing from (1.4-5), we observe that

$$\overline{\chi}(b)\tau_n(\chi) = \sum_{c=0}^{q-1} \chi(c) e^{2\pi i c \frac{bn'}{q'}}$$

$$= \sum_{c=0}^{q-1} \chi(c) e^{2\pi i c \frac{n'}{q'}} \quad \text{because } bn' \equiv n' \bmod q'$$

$$= \sum_{c=0}^{q-1} \chi(c) e^{2\pi i c \frac{n}{q}}$$

$$= \tau_n(\chi)$$

Hence,

$$\overline{\chi}(b)\tau_n(\chi) = \tau_n(\chi) \text{ and } \chi(b) \neq 1,$$

which implies

$$\tau_n(\chi) = 0 \text{ for } \gcd(n,q) \neq 1,$$

finishing the proof. □

Remark 1.4.5. Incidentally, what we have just shown also allows us to interpolate χ from a function on $\mathbb{Z}$ to one on $\mathbb{R}$. Rearranging, we obtain

$$\overline{\chi}(n) = \frac{1}{\tau(\chi)} \tau_n(\chi).$$

We can replace χ with $\overline{\chi}$ since χ is arbitrary to see this more clearly. So,

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\tau_n(\overline{\chi}),$$

where the right-hand side is defined for any $n \in \mathbb{R}$.

Following is another useful property of Gauss sums. In particular, we can be reassured that $\tau(\chi)$ is non-zero when χ is primitive.

Proposition 1.4.6. *Let χ be a primitive character modulo q. Then,*

$$|\tau(\chi)| = q^{1/2}.$$

Proof. We begin with

$$\begin{aligned}
\tau(\chi)\overline{\tau(\chi)} &= \tau(\chi)\sum_{a=0}^{q-1}\overline{\chi}(a)e^{-2\pi ia/q} \\
&= \sum_{a=0}^{q-1} e^{-2\pi ia/q}\overline{\chi}(a)\tau(\chi) \\
&= \sum_{a=0}^{q-1} e^{-2\pi ia/q}\tau_a(\chi) \qquad \text{(using Proposition 1.4.4)} \\
&= \sum_{a=0}^{q-1} e^{-2\pi ia/q}\sum_{b=0}^{q-1}\chi(b)e^{2\pi iab/q} \qquad \text{(by definition of } \tau_a\text{)} \\
&= \sum_{b=0}^{q-1}\chi(b)\sum_{a=0}^{q-1}e^{-2\pi ia(b-1)/q}.
\end{aligned}$$

We now consider the inner sum

$$\sum_{a=0}^{q-1}e^{-2\pi ia(b-1)/q}.$$

If $b = 1$, then clearly

$$\sum_{a=0}^{q-1}e^{-2\pi ia(b-1)/q} = q.$$

If $b \neq 1$, then

$$\sum_{a=0}^{q-1}e^{-2\pi ia(b-1)/q} = \frac{e^{-2\pi ia(b-1)}-1}{e^{-2\pi ia(b-1)/q}-1} = 0.$$

Continuing from our earlier expression for $\tau(\chi)\overline{\tau(\chi)}$, we have

$$\tau(\chi)\overline{\tau(\chi)} = \sum_{b=0}^{q-1}\chi(b)\sum_{a=0}^{q-1}e^{-2\pi ia(b-1)/q} = \chi(1)q = q.$$

So,

$$\tau(\chi)\overline{\tau(\chi)} = q, \text{ which means } |\tau(\chi)| = q^{\frac{1}{2}}.$$

□

Remark 1.4.7. Sums like Gauss sums are ubiquitous in number theory. Beyond appearing in functional equations for some L-functions, they can also be useful tools in proving quadratic, cubic, and quartic reciprocity. Further, Gauss sums can be used to calculate the number of solutions to polynomial equations over finite fields, and hence also find utility in the computation of certain zeta functions. See, for example, [Ste].

Of course, we studied τ_n precisely because it is *almost* the discrete Fourier transform of χ. To be exact, using (1.4-3), we see that

$$\hat{\chi}(-n) = \sum_{a \bmod q} \chi(a)e^{2\pi i n a/q} = \tau_n(\chi). \tag{1.4-6}$$

Now that we have understood the discrete Fourier transform of χ, our Poisson summation "twisted by χ" follows as a quick corollary of our earlier work.

Theorem 1.4.8 (Poisson summation twisted by χ)**.** *For any $f : \mathbb{R} \to \mathbb{C}$ which is in $L^1(\mathbb{R})$ and primitive character χ modulo q, we have*

$$\sum_{m=-\infty}^{\infty} \chi(m)f(m/q) = \tau(\chi) \sum_{n=-\infty}^{\infty} \overline{\chi}(n)\hat{f}(n).$$

Proof. We use Proposition 1.4.2 with $c = \chi$ to write

$$\sum_{m=-\infty}^{\infty} \chi(m)f(m/q) = \sum_{n=-\infty}^{\infty} \hat{\chi}(-n)\hat{f}(n).$$

Using (1.4-6), this becomes

$$\sum_{m=-\infty}^{\infty} \chi(m)f(m/q) = \sum_{n=-\infty}^{\infty} \tau_n(\chi)\hat{f}(n).$$

Finally, applying Proposition 1.4.4, we obtain

$$\sum_{m=-\infty}^{\infty} \chi(m)f(m/q) = \tau(\chi) \sum_{n=-\infty}^{\infty} \overline{\chi}(n)\hat{f}(n),$$

as desired. □

We can now move onto the next step of determining the correct theta function and understanding its properties. The upcoming proof of Theorem 1.4.1 will make it clear that the natural choice is as follows.

Definition 1.4.9. For χ an even primitive character modulo q and $s \in \mathbb{C}$ with $\Re s > 0$, define the *theta function* as

$$\theta(s,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 \frac{s}{q}}.$$

Indeed, as the statement of Theorem 1.4.1 indicated, our proofs will be different for the case of even and odd characters, and the theta function is precisely the point where the proofs bifurcate. We will return to the case of odd characters later.

Proposition 1.4.10. *The function $\theta(s,\chi)$ as defined above is holomorphic on $\Re s > 0$.*

Proof. Let K be a compact subset of the right-half plane $\Re s > 0$. We can consider $\sigma_0 = \inf_{s\in K} \Re s$, and using compactness of K it is seen that $\sigma_0 > 0$.

Now, for any $s = \sigma + it \in K$ we have $\sigma \geq \sigma_0$. We can choose n_0 large enough so that $\sigma_0 > \frac{1}{n_0}$. Then,

$$\begin{aligned}
\sum_{n\geq n_0}^{\infty} \left|\chi(n)e^{-\pi n^2 \frac{s}{q}}\right| &\leq \sum_{n\geq n_0}^{\infty} e^{-\pi n^2 \frac{\sigma}{q}} \\
&\leq \sum_{n\geq n_0}^{\infty} e^{-\pi n^2 \frac{\sigma_0}{q}} \\
&\leq \sum_{n\geq n_0}^{\infty} e^{-\pi \frac{n^2}{n_0} \frac{1}{q}} \\
&\leq \sum_{n\geq n_0}^{\infty} e^{-\pi n \frac{1}{q}},
\end{aligned}$$

where we concluded with a geometric series that converges. It follows from the Weierstrass M-test that $\theta(s,\chi)$ converges uniformly on any compact subset of the right half-plane, which means it is holomorphic there. □

Now that we have verified the safety of working with our theta function $\theta(s,\chi)$, let us study some of its properties. Perhaps the most useful property is its functional equation. Indeed, the functional equations of L-functions are precisely the logical consequences of the functional equations of the corresponding theta functions.

Before we can prove the functional equation, we need a quick technical lemma.

Lemma 1.4.11. *The Fourier transform of*

$$f(x) = e^{-\pi x^2 \lambda}$$

with $\lambda \in \mathbb{R}\setminus\{0\}$ is given by

$$\hat{f}(y) = \frac{1}{\sqrt{\lambda}} e^{-\pi \frac{y^2}{\lambda}}.$$

Proof. Let $g(x) = e^{-\pi x^2}$. Observe that

$$f(x) = g(x\sqrt{\lambda}).$$

Using the scaled Fourier transform from Lemma C.4, we can write

$$\hat{f}(y) = \frac{1}{\sqrt{\lambda}} \hat{g}\left(\frac{y}{\sqrt{\lambda}}\right).$$

Finally, recalling that g is its own Fourier transform (see Proposition C.2), we obtain

$$\hat{f}(y) = \frac{1}{\sqrt{\lambda}} e^{-\pi \frac{y^2}{\lambda}}.$$

$\square$

With this lemma and our work with Poisson summation earlier, the functional equation of $\theta(s, \chi)$ follows quickly.

Proposition 1.4.12. *The theta function as in Definition 1.4.9 satisfies the functional equation*

$$\theta(s, \chi) = \frac{\tau(\chi)}{\sqrt{sq}}\, \theta\left(\frac{1}{s}, \overline{\chi}\right). \tag{1.4-7}$$

Proof. This is a direct application of our Poisson summation twisted by χ in Theorem 1.4.8. Set $f(x) = e^{-\pi x^2 sq}$. We can then write

$$\begin{aligned}
\theta(s, \chi) = \sum_{m=-\infty}^{\infty} \chi(m) e^{-\pi m^2 \frac{s}{q}} &= \sum_{m=-\infty}^{\infty} \chi(m) f(m/q) \\
&= \tau(\chi) \sum_{n=-\infty}^{\infty} \overline{\chi}(n) \hat{f}(n) \qquad \text{(by Theorem 1.4.8)} \\
&= \frac{\tau(\chi)}{\sqrt{sq}} \sum_{n=-\infty}^{\infty} \overline{\chi}(n) e^{-\pi \frac{n^2}{sq}} \qquad \text{(by Lemma 1.4.11 with } \lambda = sq) \\
&= \frac{\tau(\chi)}{\sqrt{sq}} \theta\left(\frac{1}{s}, \overline{\chi}\right).
\end{aligned}$$

$\square$

Finally, we need to study the limiting behavior of $\theta(s, \chi)$ as s approaches infinity.

Proposition 1.4.13. *Let χ be an even primitive Dirichlet character modulo q. As the real number $s \to \infty$, the theta function $\theta(s, \chi)$ decays rapidly to zero. In particular, for s large enough*

$$|\theta(s, \chi)| < e^{-\frac{s}{q}}.$$

Proof. First, rewrite $\theta(s, \chi)$ as

$$\theta(s, \chi) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 \frac{s}{q}} = 2 \sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 \frac{s}{q}},$$

where we used the evenness of χ and the fact that $\chi(0) = 0$.

Choose s large enough so that

$$e^{-\pi \frac{s}{q}} < \frac{1}{2} \quad \text{and} \quad 4 e^{-\pi \frac{s}{q}} < e^{-\frac{s}{q}}.$$

Then,

$$\begin{aligned}|\theta(s,\chi)| &= \left|2\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2\frac{s}{q}}\right| \\ &\le 2\sum_{n=1}^{\infty}e^{-\pi n^2\frac{s}{q}} \\ &\le 2\sum_{n=1}^{\infty}e^{-\pi n\frac{s}{q}} \\ &\le 2e^{-\pi\frac{s}{q}}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \\ &= 4e^{-\pi\frac{s}{q}} \\ &< e^{-\frac{s}{q}}.\end{aligned}$$

□

Remark 1.4.14. Note that because of the functional equation (1.4-7), the proposition above simultaneously encodes the behavior of θ as $s \to 0$, which could hence be derived as a corollary. Instead of doing this, we will satisfy ourselves with only the behavior at infinity, and in the proof of Theorem 1.4.1 conduct a transformation $s \to \dfrac{1}{s}$ to bypass directly studying the behavior at 0. Note that both approaches are of course equivalent and give exactly the same result.

We are now, as promised, in a position to provide proof of the fact that $L(s,\chi)$ can be analytically continued to the entire complex plane.

Proof of Theorem 1.4.1 for primitive even characters. We consider the case of even characters, so that $\chi(-n) = \chi(n)$.

Consider $\Re s > 1$. Recall the Gamma function is given by

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}\,dt.$$

Conduct a change of variables as

$$s \to \frac{s}{2} \quad \text{and} \quad t \to \frac{n^2}{q}\pi x,$$

so that

$$\Gamma\Big(\frac{s}{2}\Big) = \int_0^{\infty} e^{-\frac{n^2}{q}\pi x}\left(\frac{n^2}{q}\pi x\right)^{\frac{s}{2}-1}\frac{n^2}{q}\pi\,dx.$$

Rearranging, we have

$$\Big(\frac{q}{\pi}\Big)^{s/2}\Gamma\Big(\frac{s}{2}\Big)\frac{1}{n^s} = \int_0^{\infty} e^{-\pi n^2\frac{x}{q}}x^{\frac{s}{2}-1}\,dx.$$

Multiply this with $\chi(n)$ and sum over $n \in \mathbb{Z}^+$,

$$\sum_{n=1}^{\infty}\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}\chi(n)e^{-\pi n^2 \frac{x}{q}}\,dx,$$

so that for $\Re s > 1$, we have

$$\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}\chi(n)e^{-\pi n^2 \frac{x}{q}}\,dx.$$

Absolute convergence allows us to interchange the order of integration and summation (apply the Fubini-Tonelli Theorem with counting measure on $\mathbb{N}$ and Lebesgue measure on $\mathbb{R}$). So,

$$\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \int_0^{\infty} x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2 \frac{x}{q}}\,dx. \tag{1.4-8}$$

Recalling our theta function from Definition 1.4.9, we note that

$$\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2 \frac{x}{q}} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\chi(n)e^{-\pi n^2 \frac{x}{q}} = \frac{1}{2}\theta(x,\chi), \tag{1.4-9}$$

where in the first equality we used the fact that χ is even and $\chi(0) = 0$.

Substituting (1.4-9) in (1.4-8), and then splitting the integral, we find

$$\begin{aligned} 2\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi) &= \int_0^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx \\ &= \int_0^{1} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx + \int_1^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx \\ &= \int_1^{\infty} x^{-\frac{s}{2}-1}\theta\left(\frac{1}{x},\chi\right)dx + \int_1^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx, \end{aligned}$$

where in the final step we have changed $x \to \frac{1}{x}$ in the first integral. Applying the functional equation for θ as in Proposition 1.4.12, we obtain

$$\begin{aligned} \left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi) &= \frac{1}{2}\int_1^{\infty} x^{-\frac{s}{2}-1}\sqrt{\frac{x}{q}}\,\tau(\chi)\theta(x,\overline{\chi})\,dx + \frac{1}{2}\int_1^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx \\ &= \frac{1}{2}\frac{\tau(\chi)}{\sqrt{q}}\int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}}\theta(x,\overline{\chi})\,dx + \frac{1}{2}\int_1^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)\,dx. \end{aligned} \tag{1.4-10}$$

The absolute convergence of the right-hand side in (1.4-10) for any s follows from the rapid decay of θ. Since all integrands are analytic and convergence is absolute, the integrals represent entire functions of s. If we set

$$\Lambda(s,\chi) \coloneqq \left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi),$$

then we have analytically continued Λ to an entire function of s. Rearranging the expression above, we can write

$$L(s,\chi) = \frac{\Lambda(s,\chi)}{\left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)} \tag{1.4-11}$$

Recall from Theorem 1.1.11 that the Gamma function is meromorphic on the complex plane, with the only singularities being simple poles; further, Corollary 1.1.13 tells us the Gamma function vanishes nowhere. Hence, (1.4-11) shows that we have successfully found the analytic continuation for $L(s,\chi)$. Note the continuation is in fact entire and not merely meromorphic, since both terms in the denominator above are non-vanishing.

Finally, we obtain the functional equation by changing $s \to 1-s$ in our integral expression for $\Lambda(s,\chi)$. Consider

$$\begin{aligned}
\frac{\tau(\chi)}{\sqrt{q}}\,\Lambda(1-s,\overline{\chi}) &= \frac{\tau(\chi)}{\sqrt{q}}\left(\frac{1}{2}\frac{\tau(\overline{\chi})}{\sqrt{q}}\int_1^\infty x^{-\frac{1-s}{2}-\frac{1}{2}}\,\theta(x,\chi)\,dx + \frac{1}{2}\int_1^\infty x^{\frac{1-s}{2}-1}\theta(x,\overline{\chi})\,dx\right)\\
&= \frac{1}{2}\frac{\tau(\chi)\overline{\tau(\chi)}}{q}\int_1^\infty x^{\frac{s}{2}-1}\,\theta(x,\chi)\,dx + \frac{1}{2}\frac{\tau(\chi)}{\sqrt{q}}\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\theta(x,\overline{\chi})\,dx\\
&= \frac{1}{2}\int_1^\infty x^{\frac{s}{2}-1}\,\theta(x,\chi)\,dx + \frac{1}{2}\frac{\tau(\chi)}{\sqrt{q}}\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\theta(x,\overline{\chi})\,dx\\
&= \Lambda(1-s,\chi).
\end{aligned}$$

The second equality follows from observing that for even χ, $\overline{\tau(\chi)} = \tau(\overline{\chi})$. The third equality used $|\tau(\chi)| = q^{1/2}$ as in Proposition 1.4.6. □

We now proceed to demonstrate analytic continuation for odd primitive characters. Immediately, we notice that we cannot define θ as before. Indeed, the terms being summed would simply cancel out because of the oddness of χ. We must start from scratch and develop a new theta function. We will call this one *the phi function* henceforth to avoid confusion. Note this is not conventional, these objects are in fact particular examples of the general family of functions known as Jacobi theta functions (see [Wei]).

The simplest way to turn an odd function into an even function is of course to multiply it by the simplest odd function, namely, the identity function $x \mapsto x$. This motivates the following definition.

Definition 1.4.15. For χ an odd primitive character modulo q, and $s \in \mathbb{C}, \Re s > 0$, define

$$\phi(s,\chi) := \sum_{n=-\infty}^{\infty} n\,\chi(n)\,e^{-\pi n^2 \frac{s}{q}}.$$

At this point, we would like to study the holomorphy and limiting behavior of our phi function, just as we did with our earlier theta function. However, the work is similar enough that we do not present the proofs and simply state the results. Indeed, the only difference appears in the fact that we must use a hypergeometric series $\sum_n nr^n$ in the final steps as opposed to a regular geometric series, but since in either case we have absolutely convergent series, this is no problem.

Proposition 1.4.16. *The function $\phi(s,\chi)$ as defined above is holomorphic on $\Re s > 0$.*

Proposition 1.4.17. *The function $\phi(s,\chi)$ as defined above decays rapidly as the real number $s \to \infty$.*

Finally, we derive the functional equation for ϕ. To apply the twisted Poisson summation to $\phi(s,\chi)$, we first need to compute the appropriate Fourier transform.

Lemma 1.4.18. *Let $s, q \neq 0$. The Fourier transform of*

$$f(x) = xq\, e^{-\pi x^2 sq}$$

is given by

$$\hat{f}(y) = -\frac{iy}{q^{1/2}s^{3/2}} e^{-\pi \frac{y^2}{sq}}.$$

Proof. Notice that $f(x)$ is in fact proportional to the derivative of $h(x) = e^{-\pi x^2 sq}$, whereby we have already computed the Fourier transform of the latter in Lemma 1.4.11. More precisely, we have

$$f(x) = -\frac{1}{2\pi s} h'(x).$$

So, we can compute the Fourier transform quickly using Lemma C.6. Taking the Fourier transform of both sides, we find

$$\begin{aligned}
\hat{f}(y) &= -\frac{1}{2\pi s}\hat{h'}(x) \\
&= -\frac{iy}{s}\hat{h}(y) \\
&= -\frac{iy}{s}\frac{1}{\sqrt{sq}} e^{-\pi \frac{y^2}{sq}} \\
&= -\frac{iy}{q^{1/2}s^{3/2}} e^{-\pi \frac{y^2}{sq}}.
\end{aligned}$$

□

Proposition 1.4.19. *The phi function $\phi(s,\chi)$ satisfies the functional equation*

$$\phi(s,\chi) = -\frac{i}{q^{1/2}s^{3/2}}\tau(\chi)\phi\left(\frac{1}{s}, \overline{\chi}\right).$$

Proof. Set $f(x) = xq\, e^{-\pi x^2 sq}$ so that $f(m/q) = m\, e^{-\pi m^2 \frac{s}{q}}$.

We can then apply Theorem 1.4.8, and observe

$$\begin{aligned}
\phi(s,\chi) &= \sum_{m=-\infty}^{\infty} m\,\chi(m) e^{-\pi m^2 \frac{s}{q}} \\
&= \sum_{m=-\infty}^{\infty} \chi(m) f(m/q)
\end{aligned}$$

$$
\begin{aligned}
&= \tau(\chi) \sum_{n=-\infty}^{\infty} \overline{\chi}(n) \hat{f}(n) \\
&= -\frac{i}{q^{1/2} s^{3/2}} \tau(\chi) \sum_{n=-\infty}^{\infty} n\, \overline{\chi}(n) e^{-\pi \frac{n^2}{sq}} \\
&= -\frac{i}{q^{1/2} s^{3/2}} \tau(\chi) \phi\left(\frac{1}{s}, \overline{\chi}\right).
\end{aligned}
$$

□

We are now ready to analytically continue $L(s,\chi)$ for odd χ. The proof is similar with appropriate modifications to that for even χ, so we will be more brief in our exposition.

Proof of Theorem 1.4.1 for primitive odd characters. We consider the case of odd characters, so that χ is a Dirichlet character modulo q with $\chi(-n) = -\chi(n)$.

Again, for $\Re s > 1$, we begin with the Gamma function,

$$
\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt,
$$

But this time we change variables by

$$
s \to \frac{s+1}{2} \quad \text{and} \quad t \to \frac{n^2}{q}\pi x,
$$

to obtain

$$
\Gamma\left(\frac{s+1}{2}\right) = \int_0^\infty e^{-\frac{n^2}{q}\pi x} \left(\frac{n^2}{q}\pi x\right)^{\frac{s-1}{2}} \frac{n^2}{q} \pi\, dx
$$

so that

$$
\left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \frac{1}{n^s} = \int_0^\infty n\, e^{-\pi n^2 \frac{x}{q}} x^{\frac{s-1}{2}}\, dx.
$$

Multiplying with $\chi(n)$, summing over $n \in \mathbb{N}$ with $\Re s > 1$, and interchanging the order of summation and integration, gives us

$$
\begin{aligned}
\left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) &= \int_0^\infty x^{\frac{s-1}{2}} \sum_{n=1}^{\infty} n\, \chi(n) e^{-\pi n^2 \frac{x}{q}}\, dx \\
&= \frac{1}{2} \int_0^\infty x^{\frac{s-1}{2}} \phi(x,\chi)\, dx. \qquad (1.4\text{-}12)
\end{aligned}
$$

Splitting the integral in (1.4-12), we find

$$
\begin{aligned}
\left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) &= \frac{1}{2} \int_0^1 x^{\frac{s-1}{2}} \phi(x,\chi)\, dx + \frac{1}{2} \int_1^\infty x^{\frac{s-1}{2}} \phi(x,\chi)\, dx \\
&= \frac{1}{2} \int_1^\infty x^{-\frac{s}{2}-\frac{3}{2}} \phi(x,\chi)\, dx + \frac{1}{2} \int_1^\infty x^{\frac{s-1}{2}} \phi(x,\chi)\, dx,
\end{aligned}
$$

where we have changed $x \to \frac{1}{x}$ in the first integral. Applying the functional equation for ϕ as in Proposition 1.4.19 yields

$$\left(\frac{q}{\pi}\right)^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = -\frac{1}{2}\,i\,\frac{\tau(\chi)}{q^{1/2}}\int_1^\infty x^{-\frac{s}{2}}\phi(s,\overline{\chi})\,dx + \frac{1}{2}\int_1^\infty x^{\frac{s-1}{2}}\phi(x,\chi)\,dx.$$

The absolute convergence of the right-hand side above for any s follows from the rapid decay of ϕ. Since all integrands are analytic, and convergence is absolute, the integrals represent entire functions of s. By setting

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi),$$

we have successfully analytically continued Λ to an entire function of s.

Finally, we obtain the functional equation by changing $s \to 1-s$ and $\chi \to \overline{\chi}$, which gives us

$$\begin{aligned}
-i\frac{\tau(\chi)}{q^{1/2}}\,\Lambda(1-s,\overline{\chi}) &= -i\frac{\tau(\chi)}{q^{1/2}}\left(-\frac{1}{2}\,i\,\frac{\tau(\overline{\chi})}{q^{1/2}}\int_1^\infty x^{-\frac{1-s}{2}}\phi(s,\chi)\,dx + \frac{1}{2}\int_1^\infty x^{\frac{(1-s)-1}{2}}\phi(x,\overline{\chi})\,dx\right)\\
&= \frac{1}{2}\,\frac{\tau(\chi)\overline{\tau(\chi)}}{q}\int_1^\infty x^{\frac{s-1}{2}}\phi(s,\chi)\,dx - \frac{1}{2}\,i\,\frac{\tau(\chi)}{q^{1/2}}\int_1^\infty x^{-\frac{s}{2}}\phi(x,\overline{\chi})\,dx\\
&= \frac{1}{2}\int_1^\infty x^{\frac{s-1}{2}}\phi(s,\chi)\,dx - \frac{1}{2}\,i\,\frac{\tau(\chi)}{q^{1/2}}\int_1^\infty x^{-\frac{s}{2}}\phi(x,\overline{\chi})\,dx\\
&= \Lambda(s,\chi).
\end{aligned}$$

The second equality follows from observing that for odd χ, $\overline{\tau(\chi)} = -\tau(\overline{\chi})$. The third equality used $|\tau(\chi)| = q^{1/2}$ as in Proposition 1.4.6. □

2. Explicit formulae

One of the most powerful ways to understand the zeros of an L-function is to use an *explicit formula*. Indeed, it allows us to relate sums of a test function ϕ over the zeros, to sums of the product of the L-function's coefficients with the Fourier transform $\hat{\phi}$ evaluated at said coefficients. In terms of the random matrix analogy, this is the number theory analogue for the Eigenvalue Trace formula (see [MTB06, Part 5]). In that case, we can pass from knowledge of the randomly chosen matrix coefficients, to the knowledge about what we wish to study, the eigenvalues. Here, we will pass from sums over coefficients of L-functions (which we can compute), to statistics of zeros (which we want to know). The explicit formula for L-functions also tells us the correct scale to study the zeros, i.e., what the normalizing factor should be when we wish to rescale our zeros to have mean spacing of 1.

In the case of number theory, however, it is comparatively more difficult to evaluate the sums over coefficients (or sums over primes) of an L-function. For random matrices, the corresponding sums over entries can be evaluated (in the averaging limit) because of the process we used to generate them. We hence require powerful averaging formulas to execute the prime sums. In the case of the L-functions we study in §3 and §4, this is not too difficult (partly thanks to the orthogonality relations of characters).

The primary reference for the work of this section is [Dav80], and we also draw from [MTB06].

2.1. **Explicit formula for $\zeta(s)$.** We begin with the completed zeta function from Theorem 1.1.14.

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = \xi(1-s). \tag{2.1-1}$$

We can then write the logarithmic derivative as

$$\frac{\xi'(s)}{\xi(s)} = -\frac{1}{2}\log\pi + \frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'\big(\frac{s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)} + \frac{\zeta'(s)}{\zeta(s)} = -\frac{\xi'(1-s)}{\xi(1-s)}. \tag{2.1-2}$$

Lemma 2.1.1. *Contour integration of* (2.1-2) *results in residue contributions only by the zeros of $\zeta(s)$ in the critical strip.*

Proof. Recall $\xi(s)$ is entire, and so $\frac{\xi'(s)}{\xi(s)}$ is meromorphic. Contour integration and application of the residue theorem to (2.1-2) will then give us information about the poles of $\frac{\xi'(s)}{\xi(s)}$, or equivalently the zeros of $\xi(s)$. So, let us determine the zeros of $\xi(s)$ by studying (2.1-1) term by term. First, note that the zero contributed by s is cancelled by the simple pole of $\Gamma\big(\frac{s}{2}\big)$ at $s = 0$. Second, the zero of $s-1$ is cancelled by the simple pole of $\zeta(s)$ at $s = 1$. The Gamma function and $\pi^{-s/2}$ are zero nowhere. Finally, the trivial zeros of $\zeta(s)$ at the negative even integers are cancelled by the simple poles of $\Gamma\big(\frac{s}{2}\big)$. All that remains are the non-trivial zeros of $\zeta(s)$. Since we have exhausted the poles from the remaining terms, these will not be cancelled, and will contribute residues. □

We will now witness the power of the Euler product, and how it serves to connect a sum over zeros to a sum over coefficients of ζ. First, we perform a standard operation and take the logarithm, since it is easier to deal with sums. For $\Re s > 1$,

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Taking logarithms, we can convert the product to a sum.

$$\log\zeta(s) = -\sum_p \log(1-p^{-s}).$$

Differentiating both sides yields

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \log p \frac{p^{-s}}{1-p^{-s}}.$$

Finally, we can rewrite as a geometric series,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \log p \sum_{n=1}^{\infty} \frac{1}{(p^s)^n}. \tag{2.1-3}$$

We will soon observe how contour integration of the left side in (2.1-3) gives us a sum over zeros of ζ, while the right-hand side will become a sum over the primes. Indeed, we can see for the first time an explicit hint (hence the build-up to explicit formula) at the connection between the zeros of $\zeta(s)$ and the distribution of prime numbers. The Euler product was the key bridge; this is why we like to study L-functions that have Euler products.

Before we construct our test functions, a review of Schwartz functions may be helpful, and can be found in Appendix C.

Let g be an even Schwartz function (see Definition C.5) of compact support. Set

$$\phi(r) := \int_{-\infty}^{\infty} g(u)e^{iru}\, du. \tag{2.1-4}$$

We emphasize that this ϕ is completely different from that in Definition 1.4.15.

Next, put

$$H(s) := \phi\left(\frac{s-\frac{1}{2}}{i}\right). \tag{2.1-5}$$

Here, we let r, s be complex variables.
For $y \in \mathbb{R}$, we note that $\phi(y)$ can be written as a Fourier transform, namely,

$$\phi(y) = \hat{g}\Big(-\frac{y}{2\pi}\Big). \tag{2.1-6}$$

For $s = x + iy$, we can write

$$H(x+iy) = \int_{-\infty}^{\infty} \Big[g(u)e^{\left(x-\frac{1}{2}\right)u}\Big] e^{iuy}\, du, \tag{2.1-7}$$

and since g is compactly supported, $H(s)$ is well-defined for all s. The Cauchy-Riemann equations verify that ϕ and H are holomorphic.

Lemma 2.1.2. *The test function $H(s)$ from* (2.1-5) *is rapidly decreasing in the imaginary direction (y-direction).*

Proof. Define for a given $x \in \mathbb{R}$ the function

$$g_x(u) = g(u)e^{\left(x-\frac{1}{2}\right)u}.$$

Since g is a compactly supported Schwartz function, g_x is also the same. By (2.1-7), we can write H as a Fourier transform of g_x, namely,

$$H(x+iy) = \hat{g}_x\left(\frac{-y}{2\pi}\right).$$

The Fourier transform of a Schwartz function is also Schwartz (see Appendix C), i.e., it is rapidly decreasing, which proves the lemma. □

Lemma 2.1.3. *The test function ϕ from* (2.1-4) *is even and rapidly decreasing in the real direction.*

Proof. From (2.1-4) we have

$$\phi(-r) = \int_{-\infty}^{\infty} g(u)e^{-iru}\, du = -\int_{\infty}^{-\infty} g(-v)e^{irv}\, dv = \int_{-\infty}^{\infty} g(v)e^{irv}\, dv = \phi(r).$$

So, ϕ is even.
Next, if $r = x + iy$,

$$\begin{aligned}\phi(r) = \phi(x+iy) &= \int_{-\infty}^{\infty} g(u)e^{i(x+iy)u}\,du \\ &= \int_{-\infty}^{\infty} g(u)e^{i(x+iy)u}\,du \\ &= \int_{-\infty}^{\infty} g(u)e^{-yu}e^{ixu}\,du \\ &= \hat{g}_y\Big(-\frac{x}{2\pi}\Big)\end{aligned}$$

where

$$g_y(u) = g(u)e^{-yu}.$$

So, ϕ is the Fourier transform of a Schwartz function, which establishes ϕ as rapidly decreasing in the x-direction. $\square$

Let us now consider

$$I = \frac{1}{2\pi i}\int_{\Re s = \frac{3}{2}} \frac{\xi'(s)}{\xi(s)} H(s)\,ds.$$

Since $\dfrac{\xi'(s)}{\xi(s)}$ only grows as logarithmically, and $H(s)$ is rapidly decreasing along the imaginary axis by Lemma 2.1.2, I is well-defined.

We now shift contours from $\Re s = \frac{3}{2}$ to $\Re s = -\frac{1}{2}$ as follows.

$$\begin{aligned}&\int_{\frac{3}{2}-it}^{\frac{3}{2}+it} \frac{\xi'(s)}{\xi(s)} H(s)\,ds + \int_{\frac{3}{2}+it}^{-\frac{1}{2}+it} \frac{\xi'(s)}{\xi(s)} H(s)\,ds \\ &\qquad + \int_{-\frac{1}{2}+it}^{-\frac{1}{2}-it} \frac{\xi'(s)}{\xi(s)} H(s)\,ds + \int_{-\frac{1}{2}-it}^{\frac{3}{2}-it} \frac{\xi'(s)}{\xi(s)} H(s)\,ds = 2\pi i \sum_{|\rho|<t} \operatorname*{Res}_{s=\rho}\Big(\frac{\xi'(s)}{\xi(s)} H(s)\Big). \qquad (2.1\text{-}8)\end{aligned}$$

By a sum over ρ we will always mean the sum over non-trivial zeros in the critical strip. Note that this is precisely the sum of residues we need as shown in Lemma 2.1.1. Now, in a sufficiently small neighborhood of ρ, with f analytic in the neighborhood and $f(\rho) \neq 0$, we can write

$$\xi(s) = (s-\rho)^{\mathrm{ord}_\xi(\rho)} f(s),$$

where $\mathrm{ord}_\xi(\rho)$ is the order of the zero ρ. Taking the logarithmic derivative gives us

$$\frac{\xi'(s)}{\xi(s)} = \frac{\mathrm{ord}_\xi(\rho)}{(s-\rho)} + \frac{f'(s)}{f(s)}.$$

Therefore, we can take residues to obtain

$$\operatorname*{Res}_{s=\rho}\left(\frac{\xi'(s)}{\xi(s)}H(s)\right) = \operatorname{ord}_{\xi}(\rho)H(\rho), \tag{2.1-9}$$

by applying Cauchy's integral formula and Cauchy's Theorem, and noting that f is analytic in the neighborhood in question with $f(\rho) \neq 0$.

The right-hand side of (2.1-8) then becomes

$$\int_{\frac{3}{2}-it}^{\frac{3}{2}+it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds + \int_{\frac{3}{2}+it}^{-\frac{1}{2}+it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds$$

$$+ \int_{-\frac{1}{2}+it}^{-\frac{1}{2}-it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds + \int_{-\frac{1}{2}-it}^{\frac{3}{2}-it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds = 2\pi i \sum_{|\rho|<t} H(\rho), \tag{2.1-10}$$

where multiplicities of zeros are taken into account by there being multiple zeros that are equal, i.e., there will be n equal terms in the sum for a zero of order n.

Next, we consider the limit of (2.1-10) as $t \to \infty$. Note that the integrals

$$\int_{\frac{3}{2}+it}^{-\frac{1}{2}+it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds \qquad \text{and} \qquad \int_{-\frac{1}{2}-it}^{\frac{3}{2}-it} \frac{\xi'(s)}{\xi(s)}H(s)\,ds$$

will vanish in the limit because $\dfrac{\xi'(s)}{\xi(s)}$ only grows logarithmically, and $H(s)$ is rapidly decreasing in the imaginary direction by Lemma 2.1.2. The equation (2.1-10) then becomes

$$\begin{aligned}
\sum_{\rho} H(\rho) &= I - \frac{1}{2\pi i}\int_{\Re s=-\frac{1}{2}} \frac{\xi'(s)}{\xi(s)}H(s)\,ds \\
&= I + \frac{1}{2\pi i}\int_{\Re s=-\frac{1}{2}} \frac{\xi'(1-s)}{\xi(1-s)}H(s)\,ds && \text{(by (2.1-2))} \\
&= I + \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\xi'(s)}{\xi(s)}H(1-s)\,ds && \text{(changing s to $1-s$)} \\
&= \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\xi'(s)}{\xi(s)}[H(s)+H(1-s)]\,ds && \text{(definition of I)}
\end{aligned}$$

and now recalling our expression for the logarithmic derivative of $\xi(s)$ (2.1-2) we can write

$$\sum_{\rho} H(\rho) = \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)}[H(s)+H(1-s)]\,ds$$

$$+ \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\log\pi\right)[H(s)+H(1-s)]\,ds. \tag{2.1-11}$$

We shift contours in the second integral from $\Re s = \frac{3}{2}$ to $\Re s = \frac{1}{2}$. Since $H(s)$ has no poles (see (2.1-7)), the only residue is

$$\operatorname*{Res}_{s=1}\left(\frac{1}{s-1}\,[H(s)+H(1-s)]\right) = H(1)+H(0).$$

We note that $H(0) = \phi\left(\frac{i}{2}\right) = \phi\left(-\frac{i}{2}\right) = H(1)$. So, (2.1-11) becomes

$$\begin{aligned}\sum_{\rho} H(\rho) &= \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)}[H(s)+H(1-s)]\,ds + 2\phi\left(\frac{i}{2}\right) \\ &\quad + \frac{1}{2\pi i}\int_{\Re s=\frac{1}{2}}\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\log\pi\right)[H(s)+H(1-s)]\,ds.\end{aligned} \tag{2.1-12}$$

Further, for $\Re s = \frac{1}{2}$, i.e., $s = \frac{1}{2}+iy$, we can use the evenness of ϕ (Lemma 2.1.3) to note that $H(s) = \phi(y) = \phi(-y) = H(1-s)$. Continuing from (2.1-12), we can now write

$$\begin{aligned}\sum_{\rho} H(\rho) &= \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)}[H(s)+H(1-s)]\,ds + 2\phi\left(\frac{i}{2}\right) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\frac{1}{2}+iy}+\frac{1}{-\frac{1}{2}+iy}+\frac{1}{2}\frac{\Gamma'\left(\frac{1}{4}+\frac{iy}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)} - \frac{1}{2}\log\pi\right)\phi(y)\,dy.\end{aligned} \tag{2.1-13}$$

Note that 2 in the denominator in front of the second integral of (2.1-12) cancels because $H(s) + H(1-s) = 2\phi(y)$ for $s = \frac{1}{2}+iy$, and the i cancels because $ds = i\,dy$. Additionally,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\frac{1}{2}+iy}+\frac{1}{-\frac{1}{2}+iy}\right)\phi(y)\,dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{2iy}{-y^2-\frac{1}{4}}\right)\phi(y)\,dy = 0$$

because the integrand is an odd function (again, ϕ is even). We also note that

$$-\frac{\log\pi}{2\pi}\int_{-\infty}^{\infty}\phi(y)\,dy = -\frac{\log\pi}{2\pi}\int_{-\infty}^{\infty}\hat{g}\left(-\frac{y}{2\pi}\right)dy = -g(0)\log\pi,$$

where the second equality followed from (2.1-6), and the third from Fourier inversion.

Substituting the above observations in (2.1-13), we obtain

$$\begin{aligned}\sum_{\rho} H(\rho) &= \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)}[H(s)+H(1-s)]\,ds + 2\phi\left(\frac{i}{2}\right) - g(0)\log\pi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iy}{2}\right)\phi(y)\,dy,\end{aligned} \tag{2.1-14}$$

where $\dfrac{\Gamma'}{\Gamma}$ is the digamma function.

Let us now focus on part of the first integral in (2.1-14), and apply the Euler product for the logarithmic derivative of ζ as in (2.1-3). We find

$$\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\frac{\zeta'(s)}{\zeta(s)}H(s)\,ds = -\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\sum_p\sum_{n=1}^{\infty}\frac{\log p}{(p^s)^n}H(s)\,ds$$

$$= -\frac{1}{2\pi}\sum_p\sum_{k\geq 1}\frac{\log p}{p^{\frac{3}{2}k}}\int_{-\infty}^{\infty}\frac{H\left(\frac{3}{2}+iy\right)}{p^{iyk}}\,dy$$

$$= -\frac{1}{2\pi}\sum_p\sum_{k\geq 1}\frac{\log p}{p^{\frac{3}{2}k}}\int_{-\infty}^{\infty}\frac{\phi(y-i)}{p^{iyk}}\,dy, \tag{2.1-15}$$

where the last equality follows from $H\left(\frac{3}{2}+iy\right)=\phi\left(\frac{\frac{3}{2}+iy-\frac{1}{2}}{i}\right)=\phi(y-i)$. Since the sums in (2.1-15) converge absolutely on the integral path, as can be seen in (2.1-3), we were justified in interchanging the order of integration and summation.

Next, consider the integral inside the double sum from (2.1-15). We would like to shift this integral upwards in the complex plane. Take the contour determined by the four points $(\pm t,0),(\pm t,i)$, then by Cauchy's theorem we have

$$\int_{(-t,0)}^{(t,0)}\frac{\phi(y-i)}{p^{iyk}}\,dy+\int_{(t,0)}^{(t,i)}\frac{\phi(y-i)}{p^{iyk}}\,dy$$

$$+\int_{(t,i)}^{(-t,i)}\frac{\phi(y-i)}{p^{iyk}}\,dy+\int_{(-t,i)}^{(-t,0)}\frac{\phi(y-i)}{p^{iyk}}\,dy=0.$$

The second and fourth integrals are seen to vanish in the limit as $t\to\infty$ by using the fact that ϕ is rapidly decaying in the real direction (Lemma 2.1.3). (The explicit computation to demonstrate this vanishing is similar to the work for Proposition C.2.) Hence,

$$\lim_{t\to\infty}\int_{(-t,0)}^{(t,0)}\frac{\phi(y-i)}{p^{iyk}}\,dy=\lim_{t\to\infty}\int_{(-t,i)}^{(t,i)}\frac{\phi(y-i)}{p^{iyk}}\,dy, \tag{2.1-16}$$

which allows us to write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi(y-i)}{p^{iyk}}\,dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi(y)}{p^{i(y+i)k}}\,dy$$

$$=\frac{p^k}{2\pi}\int_{-\infty}^{\infty}\phi(y)p^{-iyk}\,dy$$

$$=\frac{p^k}{2\pi}\int_{-\infty}^{\infty}\phi(y)e^{-iyk\log p}\,dy$$

$$=\frac{p^k}{2\pi}\int_{-\infty}^{\infty}\hat{g}\Big(-\frac{y}{2\pi}\Big)e^{-iyk\log p}\,dy \qquad \text{(using (2.1-6))}$$

$$= p^k \int_{-\infty}^{\infty} \hat{g}(u) e^{2\pi i u k \log p}\, du \qquad \left(\text{set } u = -\frac{y}{2\pi}\right)$$

$$= p^k g(k \log p). \tag{2.1-17}$$

Returning to (2.1-15), we obtain

$$\frac{1}{2\pi i} \int_{\Re s = \frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)} H(s)\, ds = -\sum_p \sum_{k \geq 1} \frac{\log p}{p^{\frac{1}{2}k}} g(k \log p). \tag{2.1-18}$$

In similar fashion, we have

$$\frac{1}{2\pi i} \int_{\Re s = \frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)} H(1-s)\, ds = -\frac{1}{2\pi} \sum_p \sum_{k \geq 1} \frac{\log p}{p^{\frac{3}{2}k}} \int_{-\infty}^{\infty} \frac{H\left(1 - \frac{3}{2} - iy\right)}{p^{iyk}}\, dy$$

$$= -\frac{1}{2\pi} \sum_p \sum_{k \geq 1} \frac{\log p}{p^{\frac{3}{2}k}} \int_{-\infty}^{\infty} \frac{\phi(y - i)}{p^{iyk}}\, dy,$$

where we used $H\left(-\frac{1}{2} + iy\right) = \phi\left(\frac{-\frac{1}{2} - iy - \frac{1}{2}}{i}\right) = \phi(i - y) = \phi(y - i)$. The rest of the process is identical, and we find

$$\frac{1}{2\pi i} \int_{\Re s = \frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)} H(1-s)\, ds = -\sum_p \sum_{k \geq 1} \frac{\log p}{p^{\frac{1}{2}k}} g(k \log p). \tag{2.1-19}$$

Using (2.1-18) and (2.1-19), we have shown that

$$\frac{1}{2\pi i} \int_{\Re s = \frac{3}{2}} \frac{\zeta'(s)}{\zeta(s)} [H(s) + H(1-s)]\, ds = -2 \sum_p \sum_{k \geq 1} \frac{\log p}{p^{\frac{1}{2}k}} g(\log p^k) = -2 \sum_n \frac{\Lambda(n)}{\sqrt{n}} g(\log n), \tag{2.1-20}$$

where Λ is the von Mangoldt function, defined as in Example A.6. Finally, substituting (2.1-20) in (2.1-14), we have proven our first explicit formula.

Theorem 2.1.4 (Riemann-Weil explicit formula)**.** *For each non-trivial zero ρ of ζ let $\rho = \frac{1}{2} + i\gamma_\rho$, with $\gamma_\rho \in \mathbb{C}$. We have*

$$\sum_\rho H(\rho) = \sum_\rho \phi(\gamma_\rho) = 2\phi\left(\frac{i}{2}\right) - g(0) \log \pi + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iy}{2}\right) \phi(y)\, dy - 2 \sum_n \frac{\Lambda(n)}{\sqrt{n}} g(\log n). \tag{2.1-21}$$

Here, g and ϕ are even Schwartz complex-valued functions of a real variable, related by the following integral transforms.

$$\phi(r) = \int_{-\infty}^{\infty} g(u) e^{iru}\, du, \quad \textit{and} \quad g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(r) e^{-iru}\, dr.$$

The truth of the Riemann hypothesis would of course imply that γ_ρ is real, so that $\sum_\rho \phi(\gamma_\rho)$ becomes a real sum. Of course, this is precisely why the relation between H and ϕ was defined (as in (2.1-5)) in the manner that it was.

A version of the above formula first appeared in [Wei52], albeit in considerably more general form. It is sometimes more instructive to present the right-hand side of (2.1-21) as a sum over primes rather than as the equivalent sum over integers using the von Mangoldt function, as in the left-hand side of (2.1-20). This is because the former emphasizes the true power of the explicit formula: a clear relation between the zeros of the Riemann Zeta function, and the distribution of prime numbers.

2.2. **Explicit formula for $L(s,\chi)$.** The work here is very similar to the development of the explicit formula for $\zeta(s)$ as in §2.1. As such, we will be able to achieve Theorem 2.2.1 quickly, but be sure to highlight modifications along the way.

Let our test functions g, ϕ, and H be the same as in equations (2.1-4) through (2.1-7).

Using Theorem 1.4.1, we begin with the logarithmic derivative of the completed Dirichlet L-function, given as

$$\frac{\Lambda'(s,\chi)}{\Lambda(s,\chi)} = \frac{1}{2}\log\Big(\frac{q}{\pi}\Big) + \frac{1}{2}\frac{\Gamma'\big(\frac{s+\epsilon}{2}\big)}{\Gamma\big(\frac{s+\epsilon}{2}\big)} + \frac{L'(s,\chi)}{L(s,\chi)} = -\frac{\Lambda'(1-s,\overline{\chi})}{\Lambda(1-s,\overline{\chi})}. \tag{2.2-1}$$

Again, the Euler product is key. For $\Re s > 1$,

$$L(s,\chi) = \prod_p \big(1-\chi(p)p^{-s}\big)^{-1}.$$

So,

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_p \frac{d}{ds}\log\big(1-\chi(p)p^{-s}\big) = -\sum_p\sum_{k=1}^{\infty}\frac{\chi(p^k)\log p}{(p^k)^s}. \tag{2.2-2}$$

We again consider the integral over the line $\Re s = \frac{3}{2}$,

$$I = \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\Lambda'(s,\chi)}{\Lambda(s,\chi)} H(s)\,ds.$$

An argument similar to that of Lemma 2.1.1 reveals that the only zeros of $\Lambda(s,\chi)$ are those of $L(s,\chi)$ in the critical strip. We now shift contours from $\Re s = \frac{3}{2}$ to $\Re s = -\frac{1}{2}$. In the manner of (2.1-9) the residues we pick up will be $H(\rho)$, where ρ indexes the zeros of $L(s,\chi)$ in the critical strip (taking into account multiplicity). We can hence write

$$\begin{aligned}
\sum_\rho H(\rho) &= I - \frac{1}{2\pi i}\int_{\Re s=-\frac{1}{2}} \frac{\Lambda'(s,\chi)}{\Lambda(s,\chi)} H(s)\,ds \\
&= I + \frac{1}{2\pi i}\int_{\Re s=-\frac{1}{2}} \frac{\Lambda'(1-s,\overline{\chi})}{\Lambda(1-s,\overline{\chi})} H(s)\,ds \qquad \text{(by (2.2-1))} \\
&= I + \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \frac{\Lambda'(s,\overline{\chi})}{\Lambda(s,\overline{\chi})} H(1-s)\,ds \qquad \text{(changing } s \text{ to } 1-s\text{)} \\
&= \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}} \left[\frac{\Lambda'(s,\chi)}{\Lambda(s,\chi)} H(s) + \frac{\Lambda'(s,\overline{\chi})}{\Lambda(s,\overline{\chi})} H(1-s)\right] ds \qquad \text{(definition of } I\text{).}
\end{aligned}$$

We now expand out the logarithmic derivative of $\Lambda(s,\chi)$ and $\Lambda(s,\overline{\chi})$ using (2.2-1). We will retain the $\dfrac{L'}{L}$ terms under the $\Re s = \frac{3}{2}$ integral, and shift contours for the remaining terms to $\Re s = \frac{1}{2}$; observe that there are no residues to pick up. Finally, note that for $\Re s = \frac{1}{2}$, i.e., $s = \frac{1}{2} + iy$, we have $H(s) = H(1-s) = \phi(y)$. We obtain

$$\begin{aligned}\sum_\rho H(\rho) = \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\left[\frac{L'(s,\chi)}{L(s,\chi)}H(s) + \frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}H(1-s)\right]ds \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\log\left(\frac{q}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{iy}{2}+\frac{1}{4}+\frac{\epsilon}{2}\right)\right)\phi(y)\,dy.\end{aligned} \tag{2.2-3}$$

But using (2.1-6) and Fourier inversion, we have

$$\frac{1}{2\pi}\log\left(\frac{q}{\pi}\right)\int_{-\infty}^{\infty}\phi(y)\,dy = \frac{1}{2\pi}\log\left(\frac{q}{\pi}\right)\int_{-\infty}^{\infty}\hat{g}\left(-\frac{y}{2\pi}\right)dy = g(0)\log\left(\frac{q}{\pi}\right).$$

So, we can simplify (2.2-3) to

$$\begin{aligned}\sum_\rho H(\rho) = \frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\left[\frac{L'(s,\chi)}{L(s,\chi)}H(s) + \frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}H(1-s)\right]ds + g(0)\log\left(\frac{q}{\pi}\right) \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{iy}{2}+\frac{1}{4}+\frac{\epsilon}{2}\right)\phi(y)\,dy.\end{aligned} \tag{2.2-4}$$

Using the Euler product as in (2.2-2), we can write

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\frac{L'(s,\chi)}{L(s,\chi)}H(s)\,ds &= -\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\sum_p\sum_{k=1}^{\infty}\frac{\chi(p^k)\log p}{(p^k)^s}H(s)\,ds \\ &= -\frac{1}{2\pi}\sum_p\sum_{k=1}^{\infty}\frac{\chi(p^k)\log p}{p^{\frac{3}{2}k}}\int_{-\infty}^{\infty}\frac{H\left(\frac{3}{2}+iy\right)}{p^{iyk}}\,dy \\ &= -\frac{1}{2\pi}\sum_p\sum_{k=1}^{\infty}\frac{\chi(p^k)\log p}{p^{\frac{3}{2}k}}\int_{-\infty}^{\infty}\frac{\phi(y-i)}{p^{iyk}}\,dy.\end{aligned}$$

Shifting the inner integral upwards in the complex plane (as in (2.1-16) and (2.1-17)), we find that

$$\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\frac{L'(s,\chi)}{L(s,\chi)}H(s)\,ds = -\sum_p\sum_{k=1}^{\infty}\frac{\chi(p^k)\log p}{p^{\frac{1}{2}k}}g(k\log p).$$

Similarly,

$$\frac{1}{2\pi i}\int_{\Re s=\frac{3}{2}}\frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}H(s)\,ds = -\sum_p\sum_{k=1}^{\infty}\frac{\overline{\chi}(p^k)\log p}{p^{\frac{1}{2}k}}g(k\log p).$$

Substituting back into (2.2-4), we obtain the theorem below.

Theorem 2.2.1. *Let $L(s,\chi)$ be a Dirichlet L-function from a primitive character modulo q. For each non-trivial zero ρ of $L(s,\chi)$ let $\rho = \frac{1}{2} + i\gamma$. Let $\sum_\rho$ denote the over the zeros of $L(s,\chi)$ in the critical strip. We have*

$$\sum_\rho \phi(\gamma_\rho) = -\sum_p \frac{\log p}{p^{k/2}} g(\log p^k)\Big(\chi(p^k) + \overline{\chi}(p^k)\Big) + g(0)\log\Big(\frac{q}{\pi}\Big) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{iy}{2} + \frac{1}{4} + \frac{\epsilon}{2}\right)\phi(y)\,dy.$$

Here, g and ϕ are Schwartz complex-valued functions of a real variable, related by the following integral transforms.

$$\phi(r) = \int_{-\infty}^{\infty} g(u)e^{iru}\,du, \quad \textit{and} \quad g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(r)e^{-iru}\,dr,$$

and

$$\epsilon = \begin{cases} 0 & \textit{if } \chi(-1) = 1, \\ 1 & \textit{if } \chi(-1) = -1. \end{cases}$$

3. 1-level density

3.1. Motivation.

In this subsection, we motivate the study of the "low-lying zeros" of L-functions, i.e., the zeros near the critical point at $s = \frac{1}{2}$. The specific means by which we study these zeros, namely the 1-level density, is deferred to §3.2. We draw heavily from the excellent history provided in [AAI$^+$15] and [MTB06, Part 5: Random Matrix Theory and L-functions].

For the purposes of this exposition, it suffices to consider an L-function as a function on the complex plane arising from the meromorphic continuation of an L-series.

Remark 3.1.1. The nice properties we hope a general L-function to have (but are often only conjectural) include an Euler product, analytic continuation, a functional equation, a generalized Riemann hypothesis (GRH), and special values. Of course, the "most conjectural" property here is the GRH, since even in the simplest case of an L-function over a number field – namely, the Riemann hypothesis for the Riemann zeta function – the problem appears to remain out of reach of current methods. Note, however, that the analogue in the case of ζ-functions of algebraic varieties over finite fields was proven by Pierre Deligne in [Del74, Del80].

For now, consider well-behaved L-functions arising from L-series with understood meromorphic continuations, an Euler product and coefficients with arithmetic meaning. For example, the L-series may be a Dirichlet series, as in

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} = \prod_p L_p\big(p^{-s}, f\big)^{-1}, \quad \Re s > s_0.$$

For our purposes, we can think of f as some object which determines the coefficients $a_n(f)$. When $a_n(f) = 1$ for all n, we obtain $\zeta(s)$. Alternatively, our L-series may be a sum over the integral ideals of a number field so that analytic continuation gives us a Hecke L-function; indeed, this will be the type of L-function we study in §4.

Next, for us, a family of L-functions is a collection of "similar enough" L-functions (for a more rigorous construction of families, see [SST16], but this is still largely a loosely defined term). In particular, all Dirichlet L-functions with fixed conductor q, or Dirichlet L-functions with conductor $q \in [Q/2, Q]$, and similarly $\mathfrak{f}$-ray class L-functions with $\mathrm{N}(f) \in [F/2, F]$, will all qualify as a family of L-functions. Since elements of a particular family share certain key properties, we might expect them to share other more complicated properties [SST16].

Assuming GRH, non-trivial zeros of L-functions lie on the critical line ($\Re s = 1/2$). This allows us to compute statistics of its normalized (so that the mean spacing is 1) zeros; of course we can compute said statistics regardless, but GRH allows for spectral interpretation of the spacings between zeros. The study of these statistics oftentimes bifurcates into two families: zeros high above the critical point and zeros near the critical point ($s = 1/2$).

In the case of high-lying zeros, the first classical ensemble of random matrices to play a key role was the Gaussian Unitary Ensemble (GUE), which refers to the space of complex Hermitian matrices with entries chosen independently from Gaussians (different for diagonal and off-diagonal entries since diagonal entries must be real). The history begins with Montgomery's pair correlation conjecture [Mon73], whereby it was observed by Freeman Dyson (in a chance encounter, see [Hay03] for an entertaining account) that his pair correlation between zeros of the Riemann zeta function was the same as that of normalized eigenvalues of random Hermitian matrices. Given an increasing sequence of real numbers α_j (for us, this is the imaginary parts of the zeros $\rho_j = \frac{1}{2} + i\alpha_j$) and $B \subset \mathbb{R}^{n-1}$ a compact box, the n-level correlation is defined as

$$\lim_{N \to \infty} \frac{\#\{(\alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n}) \in B, j_i \leq N\}}{N}.$$

The pair correlation then refers to the 2-level correlation.

Montgomery's results further inspired the work [Odl87] of Odlyzko, who starting at the $10^{20\text{th}}$ zero, demonstrated agreement (to the point of indistinguishability) between the spacings of adjacent normalized zeros of $\zeta(s)$ and those of adjacent normalized eigenvalues of complex Hermitian matrices. More investigations followed, for example in [RS96, Hej94], whereby agreement was demonstrated for the n-level correlation in the case of automorphic cuspidal L-functions. Indeed, it can be shown that knowledge of *all* n-level correlations contains enough information to determine exactly the neighborhood spacings. All this lent great credence to the GUE conjecture. Namely, that in the limit the GUE is a good model for the zeros of L-functions, in the sense that the spacings between zeros and the spacings between eigenvalues share the same limiting distribution, as we go higher up the critical point and consider larger matrices, respectively.

It quickly became clear, however, that something was lacking. In particular, the n-level correlation is easily seen to be insensitive to finitely many zeros. That is, the deletion of any finite set of zeros leaves the statistic unchanged. But many problems care only about a few zeros – for example, the Birch and Swinnerton-Dyer conjecture, which was first hypothesized in [SDB63, SDB65].

Indeed, Katz and Sarnak showed that the GUE does not play the central role one would assume, since the n-level correlations of the GUE are identical to those of classical compact groups. So, we can just as well claim the high-lying zeros of $\zeta(s)$ behave like the eigenvalues of orthogonal matrices instead of the GUE.

This led Katz and Sarnak to introducing new statistics for characterizing the behavior of low-lying zeros near the critical point. First, we note that Katz and Sarnak have conjectured that for any (well-behaved and irreducible) family of L-functions, there is an associated symmetry group $G(\mathcal{F})$; this is partly dependent on the function field analysis where $G(\mathcal{F})$ is the associated monodromy group (see [KS98]). There are five options for $G(\mathcal{F})$: unitary U, symplectic USp, orthogonal O, SO(even), SO(odd). The "Katz-Sarnak philosophy," then, is the notion that many statistics for low-lying zeros of L-functions should (in the limit) agree with statistics for random matrices arising from the classical compact group $G(\mathcal{F})$. See [KS98] for the origins of this in the case of L-functions for varieties over finite fields and also their fairly accessible review in [KS99]. For the case of L-functions over number fields, a small sampling of resources are [RS96], [Mil04] and [Kow08]; a textbook introduction may be found in [MTB06, Part 5: Random Matrix Theory and L-Functions].

Amongst other predictions, one expects that the distribution of critical zeros of the family of L-functions near $\frac{1}{2}$ reflects the distribution of the eigenvalues of $G(\mathcal{F})$ near 1. So, understanding the statistics of low-lying zeros allows us to identify the symmetry of the family.

3.2. **Surpassing the Ratios Conjecture for Dirichlet L-functions.** We spend this subsection providing an overview of the work, results and proof strategies of Fiorilli and Miller in [FM15], using the same notation as in the paper. It is this work which we then partially generalize to $\mathfrak{f}$-ray class L-functions in §4.

A powerful and convenient way to study low-lying zeros is through the 1-level density.

Definition 3.2.1. Let $\eta \in L^1(\mathbb{R})$ be an even real function whose Fourier transform is C^2 and has compact support. Let $\mathcal{F}_N$ be a finite family of L-functions with a GRH. The *1-level density associated to $\mathcal{F}_N$* is given by

$$D_{1;\mathcal{F}_N}(\eta) := \frac{1}{|\mathcal{F}_N|} \sum_{g \in \mathcal{F}_N} \sum_j \eta\left(\frac{\log c_g}{2\pi} \gamma_g^{(j)}\right).$$

Here, $\rho = \frac{1}{2} + i\gamma_g^{(j)}$ indexes the non-trivial zeros of $L(s,g)$, where multiplicity is incorporated by considering several zeros that are equal. The factor of c_g is referred to as the analytic conductor of the family and serves to provide the right scale factor for low-lying zeros (i.e., normalize the spacing). Since η decays, only low-lying zeros will have significant contribution.

Using Definition 3.2.1, we have that the 1-level density for Dirichlet L-functions with modulus q is given by

$$D_{1;q}(\eta) := \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \sum_{\gamma_\chi} \eta\left(\frac{\log Q}{2\pi} \gamma_\chi\right), \tag{3.2-1}$$

where $\varphi(q)$ is the Euler totient function, which is precisely the size of the family as discussed in §1.2.

It is fruitful to study the *averaged 1-level density*, defined as

$$D_{1;Q/2,Q}(\eta) := \frac{1}{Q/2} \sum_{Q/2 < q \le Q} D_{1;q}(\eta), \tag{3.2-2}$$

where we averaged the 1-level density over the moduli $Q/2 < q \le Q$. Generally, additional averaging makes quantities easier to compute and understand.

Techniques from random matrix theory have had success in predicting not just the main terms, but also arithmetic lower-order terms. As hinted at in §3.1, the main terms capture the symmetry type of the family and hence it is in the lower-order terms that we observe the arithmetic of the family. It is precisely in the prediction of these lower-order terms that the L-functions Ratios Conjecture demonstrates its remarkable power and utility. With it, we can make precise predictions for the aforementioned main and lower-order terms (see [CS07] for how and other wonderful applications of the Ratios Conjecture). We then attempt to prove these predictions directly.

There are many papers, such as [CS07, CS08], to name a few, which provide evidence for the veracity of the L-functions Ratios Conjecture, by demonstrating explicitly agreement between various L-function statistics and the corresponding Ratios Conjecture predictions.

The achievement of [FM15] is to compute the averaged 1-level density and demonstrate the existence of a lower-order arithmetical (*not error*) term which is not predicted by the Ratios Conjecture, suggesting that the powerful conjecture may need some refinement.

Indeed, the Ratios Conjecture prediction [FM15, Conjecture 1.1] for $D_{1;Q/2,Q}(\eta)$ is given by

$$\hat{\eta}(0)\left(1-\frac{\log(4\pi e^{\gamma})+1}{\log Q}-\frac{1}{\log Q}\sum_{p}\frac{\log p}{p(p-1)}\right)+\int_0^\infty\frac{\hat{\eta}(0)-\hat{\eta}(t)}{Q^{t/2}-Q^{-t/2}}\,dt+O_\epsilon\Big(Q^{-1/2+\epsilon}\Big). \tag{3.2-3}$$

The main result of Fiorilli and Miller in [FM15] is the following.

Theorem 3.2.2. [FM15, Theorem 1.2] *Assume GRH. If the Fourier transform of the test function η is supported in $\left(-\frac{3}{2},\frac{3}{2}\right)$, then $D_{1;Q/2,Q}(\eta)$ equals*

$$\hat{\eta}(0)\left(1-\frac{\log(4\pi e^{\gamma})+1}{\log Q}-\frac{1}{\log Q}\sum_{p}\frac{\log p}{p(p-1)}\right)+\int_0^\infty\frac{\hat{\eta}(0)-\hat{\eta}(t)}{Q^{t/2}-Q^{-t/2}}\,dt+\frac{Q^{-1/2}}{\log Q}S_\eta(Q), \tag{3.2-4}$$

where

$$S_\eta(Q)=C_1\hat{\eta}(1)+C_2\frac{\eta'(1)}{\log Q}+O\left(\left(\frac{\log\log Q}{\log Q}\right)^2\right),$$

with

$$C_1=(2-\sqrt{2})\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right)$$

and

$$C_2=C_1\left(\frac{\sqrt{2}+4}{3}-\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right)-\sum_p\frac{\log p}{(p-1)p^{1/2}+1}\right)\right).$$

Compare (3.2-3) and (3.2-4). Note the agreement, so that the work of [FM15] successfully verifies the prediction of the Ratios Conjecture. *The difference* lies in the presence of the lower-order arithmetical term $\frac{Q^{-1/2}}{\log Q}S_\eta(Q)$ in (3.2-4). In addition to suggesting possible refinement of the Ratios Conjecture, the term also confirms that the error term $Q^{-1/2+o(1)}$ in the Ratios Conjecture is best possible.

4. The case of $\mathfrak{f}$-ray class L-functions

For this section, we will be working with Hecke L-functions associated with ray class characters over imaginary quadratic fields. Our goal is to obtain a result similar to that of [FM15, Theorem 1.2], but the work is made difficult by our departure from the setting of the rationals, towards the more general setting of number fields.

4.1. **Notations and Preliminaries.** Let K be an imaginary quadratic field and $\mathcal{O}_K$ its ring of integers. Let $I(K)$ and $P(K)$ be the groups of fractional and principal ideals, respectively.

Fix an integral ideal $\mathfrak{f}$. Define

$$I_{\mathfrak{f}}(K) = \{\mathfrak{a} \subset I(K) \,|\, (\mathfrak{a}, \mathfrak{f}) = 1\}$$

and the "ray"

$$P_{\mathfrak{f}}(K) = \{\langle a \rangle \in P(K) \,|\, a \equiv 1\,(\mathfrak{f}), a \gg 0\}.$$

The $\mathfrak{f}$-ray class group is given by the quotient

$$\mathrm{Cl}_{\mathfrak{f}}(K) = I_{\mathfrak{f}}(K) \Big/ P_{\mathfrak{f}}(K) \,.$$

Some authors call this the narrow ray class group. The $\mathfrak{f}$-ray class number $h_{\mathfrak{f},K}$ is the size of the group. That is,

$$h_{\mathfrak{f},K} = |\mathrm{Cl}_{\mathfrak{f}}(K)|\,.$$

We let χ denote a character of the group $\mathrm{Cl}_{\mathfrak{f}}(K)$ and extend the domain to $I(K)$ by setting $\chi(\mathfrak{a}) = 0$ whenever $(\mathfrak{a}, \mathfrak{f}) \neq 1$. This is the $\mathfrak{f}$-ray class character.

The Hecke L-function associated with the $\mathfrak{f}$-ray class character χ is then given by

$$L(s, \chi) \coloneqq \sum_{\mathfrak{a} \in \mathcal{O}_K} \chi(a) \mathrm{N}(\mathfrak{a})^{-s} = \sum_{(\mathfrak{a},\mathfrak{f})=1} \chi(a) \mathrm{N}(\mathfrak{a})^{-s}.$$

This is what we mean by an $\mathfrak{f}$*-ray class L-function.*

Next, define the number field analogue of the von Mangoldt function (see Appendix A) for an integral ideal $\mathfrak{a}$ by

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathrm{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ for some prime ideal } \mathfrak{p} \text{ and } m \in \mathbb{Z}^+ \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we briefly introduce the asymptotic notation we will be using in this section. Indeed, we often wish to compare the value of arithmetic functions to functions whose growth properties we understand better, and the following notation is helpful.

If F and G are two real functions with $G(x) > 0$ for x large, we write

$$F(x) = O(G(x))$$

if there exist constants $M, x_0 > 0$ such that

$$|F(x)| \leq MG(x), \quad \text{for all } x > x_0.$$

We follow the standard convention of setting $F(x) \ll G(x)$ to mean $F(x) = O(G(x))$.

When the growth rate of F is *strictly* less than that of G, then this is characterized by saying that

$$\text{if } \lim_{x\to\infty} \frac{F(x)}{G(x)} = 0 \text{ then } F(x) = o(G(x)).$$

Little-o notation makes a stronger statement than the corresponding big-O notation, namely, every function that is little-o of G is also big-O of G, but not every function that is big-O of G is also little-o of G.

4.2. **Explicit formula.** We begin by considering Weil's explicit formula for Hecke L-functions associated with ray class characters.

Let $\eta \in L^1(\mathbb{R})$ be an even real function satisfying the following conditions.

A. For some $\epsilon > 0$, $\eta(x)\exp\left(\left(\frac{1}{2}+\epsilon\right)x\right)$ is integrable and is of bounded variation.

B. The function $\dfrac{\eta(x)-\eta(0)}{x}$ is of bounded variation.

Here and throughout, we will set $g := \hat{\eta}$ and assume g is compactly supported and C^2. Moreover, note that $\hat{g} = \eta$.

We set

$$\Phi(s) = \int_{\mathbb{R}} g(x)e^{(s-1/2)x}\,dx, \tag{4.2-1}$$

so that $\Phi(s) = \hat{g}\left(\dfrac{s-\frac{1}{2}}{2\pi i}\right)$.

With appropriate substitutions in the work for [Lan94, Chapter XVII, Theorem 3.1], we obtain the following.

Theorem 4.2.1. *Let K be a number field of degree $n = r_1 + 2r_2$ and let $\mathfrak{f}$ be an integral ideal in K. For an $\mathfrak{f}$-ray class character χ, the following formula holds.*

$$\sum_{\rho_\chi}\Phi(\rho_\chi) - \delta_\chi(\Phi(0)+\Phi(1)) = g(0)\log\bigl((2\pi)^{-n}2^{r_1}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\bigr)$$

$$- \sum_{\substack{m\geq 1,\\ \mathfrak{p}\,:\,prime}} \left(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}\right)\frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{m/2}} g(\log \mathrm{N}(\mathfrak{p}^m)) - \sum_{\nu\in S_\infty} W_{\nu,\chi}(g),$$

where $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ runs over each of the zeros of $L(s,\chi)$ inside the strip $0 \leq \Re(s) \leq 1$. Here, S_∞ is the collection of Archimedean places of K. Further,

$$\delta_\chi = \begin{cases} 1 & \textit{if } \chi = \chi_0 \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$W_{\nu,\chi}(g) = \begin{cases} \displaystyle\int_0^\infty 2g(x)\frac{e^{-\left(\frac{1}{2}+m_\nu\right)x}}{1-e^{-2x}} - e^{-2x}\frac{g(0)}{x}\,dx & \textit{if } \nu \textit{ is real} \\ \displaystyle\int_0^\infty 2g(x)\frac{e^{-\frac{x}{2}}}{1-e^{-x}} - 2e^{-x}\frac{g(0)}{x}\,dx & \textit{if } \nu \textit{ is complex.} \end{cases}$$

With the above theorem as our starting point, we derive our explicit formula, as follows.

Theorem 4.2.2 (Explicit formula)**.** *Using the same notation as in Theorem 4.2.1 and assuming K has only complex places with $n = 2r$, we have*

$$\sum_{\rho_\chi}\hat{g}\left(\gamma_\chi\frac{\log F}{2\pi}\right) = \frac{g(0)}{\log F}\log\bigl((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\bigr) - \frac{2}{\log F}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re(\chi(\mathfrak{a}))\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)$$

$$-\frac{r}{\log F}\int_0^\infty \frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dx+\delta_\chi\bigg(\int_{-\infty}^\infty 2g(x)\cosh\Big(\frac{x}{2}\Big)\,dx\bigg).$$

Proof. Let us focus on the sum $\sum_{v\in S_\infty} W_{\nu,\chi}(g)$ in Theorem 4.2.1. First, note that by our assumption on K, the Archimedean place ν is always a complex place and there are r such places. Further, $W_{\nu,\chi}(g)$ will then be constant with respect to ν. So, we have

$$\begin{aligned}\sum_{v\in S_\infty} W_{\nu,\chi}(g) &= r\,W_{\nu,\chi}(g)\\ &= r\int_0^\infty 2g(x)\frac{e^{-\frac{x}{2}}}{1-e^{-x}}-2e^{-x}\frac{g(0)}{x}\,dx\\ &= r\int_0^\infty g(x)\frac{2}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}-2g(0)\frac{e^{-x}}{x}\,dx\\ &= r\int_0^\infty g(x)\frac{1}{\sinh(\frac{x}{2})}\,dx-2\,r\,g(0)\int_0^\infty\frac{e^{-x}}{x}\,dx.\end{aligned}$$

Now, let us recall the relation

$$\int_0^\infty \frac{1}{2\sinh(x/2)}\,dx=\gamma+2\log 2+\int_0^\infty\frac{e^{-x}}{x}\,dx.$$

We can use this to substitute for $\displaystyle\int_0^\infty\frac{e^{-x}}{x}\,dx$ in the above sequence of equations. So,

$$\begin{aligned}\sum_{v\in S_\infty} W_{\nu,\chi}(g) &= r\int_0^\infty g(x)\frac{1}{\sinh(\frac{x}{2})}\,dx-2\,r\,g(0)\int_0^\infty\frac{e^{-x}}{x}\,dx\\ &= r\int_0^\infty g(x)\frac{1}{\sinh(\frac{x}{2})}\,dx-2\,r\,g(0)\left(\int_0^\infty\frac{1}{2\sinh(x/2)}\,dx-\gamma-2\log 2\right)\\ &= r\int_0^\infty\frac{g(x)-g(0)}{\sinh(\frac{x}{2})}\,dx+2\,r\,g(0)\,(\gamma+2\log 2)\,.\end{aligned}$$

So, we have shown that

$$\sum_{v\in S_\infty} W_{\nu,\chi}(g)=r\int_0^\infty\frac{g(x)-g(0)}{\sinh(\frac{x}{2})}\,dx+2\,r\,g(0)\,(\gamma+2\log 2)\,.$$

Before we use this, let us also rewrite $\Phi(0)+\Phi(1)$. Using the definition of Φ from (4.2-1), we can write

$$\begin{aligned}\Phi(0)+\Phi(1) &= \int_{-\infty}^\infty g(x)e^{-\frac{x}{2}}\,dx+\int_{-\infty}^\infty g(x)e^{\frac{x}{2}}\,dx\\ &= \int_{-\infty}^\infty g(x)\Big(e^{-\frac{x}{2}}+e^{\frac{x}{2}}\Big)\,dx\end{aligned}$$

$$= \int_{-\infty}^{\infty} 2g(x)\cosh\Big(\frac{x}{2}\Big)\, dx.$$

Setting $A = (2\pi)^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})$, we substitute back into the formula in Theorem 4.2.1 and obtain

$$\begin{aligned}
\sum_{\rho_\chi}\Phi(\rho_\chi) &= g(0)\log A - \sum_{\substack{m\geq 1,\\ \mathfrak{p}\,:\,\mathrm{prime}}} \big(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}\big)\frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{m/2}} g(\log \mathrm{N}(\mathfrak{p}^m)) \\
&\quad - \sum_{\nu\in S_\infty} W_{\nu,\chi}(g) + \delta_\chi(\Phi(0)+\Phi(1)) \\
&= g(0)\log A - \sum_{\substack{m\geq 1,\\ \mathfrak{p}\,:\,\mathrm{prime}}} \big(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}\big)\frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{m/2}} g(\log \mathrm{N}(\mathfrak{p}^m)) \\
&\quad - \bigg(r\int_0^\infty \frac{g(x)-g(0)}{\sinh(\frac{x}{2})}\, dx + 2\,r\,g(0)\,(\gamma + 2\log 2)\bigg) + \delta_\chi\bigg(\int_{-\infty}^{\infty} 2g(x)\cosh\Big(\frac{x}{2}\Big)\, dx\bigg) \\
&= g(0)(\log A - 4r\log 2 - 2r\gamma) - \sum_{\substack{m\geq 1,\\ \mathfrak{p}\,:\,\mathrm{prime}}} \big(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}\big)\frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{m/2}} g(\log \mathrm{N}(\mathfrak{p}^m)) \\
&\quad - r\int_0^\infty \frac{g(x)-g(0)}{\sinh(\frac{x}{2})}\, dx + \delta_\chi\bigg(\int_{-\infty}^{\infty} 2g(x)\cosh\Big(\frac{x}{2}\Big)\, dx\bigg).
\end{aligned}$$

Next, we replace $g(x)$ with $\dfrac{1}{\log F} g\Big(\dfrac{x}{\log F}\Big)$. This implies that

$$\Phi(\rho_\chi) = \int_{\mathbb{R}} \frac{1}{\log F} g\Big(\frac{x}{\log F}\Big) e^{(\rho_\chi - 1/2)x}\, dx = \hat{g}\left(\frac{\rho_\chi - \frac{1}{2}}{2\pi i}\log F\right) = \hat{g}\Big(\gamma_\chi \frac{\log F}{2\pi}\Big).$$

So, we obtain

$$\begin{aligned}
\sum_{\rho_\chi}\hat{g}\Big(\gamma_\chi\frac{\log F}{2\pi}\Big) &= \frac{g(0)}{\log F}\log\big(2^{-4r}e^{-2r\gamma}A\big) - \frac{2}{\log F}\sum_{\substack{m\geq 1,\\ \mathfrak{p}\,:\,\mathrm{prime}}} \Re(\chi(\mathfrak{p}^m))\frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{m/2}} g\Big(\frac{\log \mathrm{N}(\mathfrak{p}^m)}{\log F}\Big) \\
&\quad - \frac{r}{\log F}\int_0^\infty \frac{1}{\sinh(\frac{x}{2})}\bigg(g\Big(\frac{x}{\log F}\Big) - g(0)\bigg)\, dx + \delta_\chi\bigg(\int_{-\infty}^{\infty} 2g(x)\cosh\Big(\frac{x}{2}\Big)\, dx\bigg).
\end{aligned}$$

Finally, we rewrite the sum over powers of prime ideals as a sum over all integral ideals by using the appropriate von Mangoldt function. Also, in the first integral we substitute $t = \frac{x}{\log F}$. We find

$$\sum_{\rho_\chi}\hat{g}\Big(\gamma_\chi\frac{\log F}{2\pi}\Big) = \frac{g(0)}{\log F}\log\big(2^{-4r}e^{-2r\gamma}A\big) - \frac{2}{\log F}\sum_{\mathfrak{a}\subset\mathcal{O}_K} \Re(\chi(\mathfrak{a}))\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}} g\Big(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\Big)$$

$$-\frac{r}{\log F}\int_0^\infty \frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt + \delta_\chi\left(\int_{-\infty}^\infty 2g(x)\cosh\left(\frac{x}{2}\right)dx\right).$$

Recalling $A=(2\pi)^{-2r}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})$, we have

$$2^{-4r}e^{-2r\gamma}A = (8\pi e^{\gamma})^{-2r}\mathrm{N}(\mathfrak{d}_K\mathfrak{f}) = (8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f}),$$

and substituting this we have the statement of the proposition. $\square$

With the explicit formula in hand, we are in position to evaluate the 1-level density, which is the subject of the next subsection.

4.3. **1-Level Density.** The goal of this subsection will be the culmination of our thesis, namely, to find an explicit formula for the 1-level density of $\mathfrak{f}$-ray class L-functions. We use the general form of the 1-level density defined in (3.2-1) to define the following.

Definition 4.3.1. The 1-Level Density associated to $\mathfrak{f}$-ray class functions is

$$D_{1;\mathfrak{f}}(\eta) = \frac{1}{h_{\mathfrak{f},K}}\sum_{\chi\in\widehat{\mathrm{Cl}_\mathfrak{f}(K)}}\sum_{\gamma_\chi}\eta\left(\gamma_\chi\frac{\log F}{2\pi}\right).$$

Here, $h_{\mathfrak{f},K}$ is the $\mathfrak{f}$-ray class number and $\widehat{\mathrm{Cl}_\mathfrak{f}(K)}$ is the character group of the $\mathfrak{f}$-ray class group $\mathrm{Cl}_\mathfrak{f}(K)$. We denote by $\rho_\chi=\frac{1}{2}+i\gamma_\chi$ the non-trivial zeros of $L(s,\chi)$. Finally, we choose F to be a scaling parameter close to $\mathrm{N}(\mathfrak{f})$ (for example, in §4.4 we will be considering $F/2\le \mathrm{N}(\mathfrak{f})\le F$).

We are now ready to present the main result of our thesis.

Theorem 4.3.2. *Let $\mathfrak{f}$ be a fixed prime ideal. The 1-level density $D_{1;\mathfrak{f}}(\hat{g})$ equals*

$$\frac{1}{h_{\mathfrak{f},K}}\sum_{\chi\in\widehat{\mathrm{Cl}_\mathfrak{f}(K)}}\sum_{\gamma_\chi}\hat{g}\left(\gamma_\chi\frac{\log F}{2\pi}\right) = \frac{g(0)}{\log F}\log\big((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\big) - \frac{r}{\log F}\int_0^\infty\frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt$$

$$-\frac{2}{\log F}\left(\sum_{\mathfrak{a}\in P_\mathfrak{f}(K)} - \frac{1}{h_{\mathfrak{f},K}}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log\mathrm{N}(\mathfrak{a})}{\log F}\right) + O\left(\frac{1}{h_{\mathfrak{f},K}}\right).$$

Proof. We begin with the explicit formula from Proposition 4.2.2 and sum over $\chi\ne\chi_0, \chi\in\widehat{\mathrm{Cl}_\mathfrak{f}(K)}$ to obtain

$$\sum_{\chi\ne\chi_0}\sum_{\gamma_\chi}\hat{g}\left(\gamma_\chi\frac{\log F}{2\pi}\right) = h_{\mathfrak{f},K}\frac{g(0)}{\log F}\log\big((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\big) - h_{\mathfrak{f},K}\frac{r}{\log F}\int_0^\infty\frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt$$

$$-\frac{2}{\log F}\sum_{\chi\ne\chi_0}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re(\chi(\mathfrak{a}))\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log\mathrm{N}(\mathfrak{a})}{\log F}\right) \tag{4.3-1}$$

$$-\frac{g(0)}{\log F}\log\big((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\big) + \frac{r}{\log F}\int_0^\infty\frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt.$$

We would like to exclude the principal character for now, i.e., replace $h_{\mathfrak{f},K}-1$ with $h_{\mathfrak{f},K}$. This is why we have written the final two terms separately. Note that because $\mathrm{N}(\mathfrak{f})$ is close to F, we will have

$$\left(\frac{\log \mathrm{N}(\mathfrak{f})}{\log F}\right) = O(1).$$

Also, because g is bounded,

$$\frac{r}{\log F}\int_0^\infty \frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt = O(1).$$

With this, both our final two terms are $O(1)$ and we have

$$\begin{aligned}\sum_{\chi\neq\chi_0}\sum_{\gamma_\chi}\hat{g}\left(\gamma_\chi\frac{\log F}{2\pi}\right) &= h_{\mathfrak{f},K}\frac{g(0)}{\log F}\log\big((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\big) - h_{\mathfrak{f},K}\frac{r}{\log F}\int_0^\infty \frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt \\ &\quad - \frac{2}{\log F}\sum_{\chi\neq\chi_0}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re(\chi(\mathfrak{a}))\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)+O(1).\end{aligned} \tag{4.3-2}$$

We now focus on rewriting the last term in the sum.

$$\begin{aligned}&\frac{2}{\log F}\sum_{\chi\neq\chi_0}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re(\chi(\mathfrak{a}))\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re\left(\sum_{\chi\neq\chi_0}\chi(\mathfrak{a})\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\Re\left(\sum_{\chi\in\widehat{\mathrm{Cl}_{\mathfrak{f}}(K)}}\chi(\mathfrak{a})-\chi_0(\mathfrak{a})\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\left(h_{\mathfrak{f},K}\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \sum_{\mathfrak{a}\subset\mathcal{O}_K}\chi_0(\mathfrak{a})\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\left(h_{\mathfrak{f},K}\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \sum_{(\mathfrak{a},\mathfrak{f})=1}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\left(h_{\mathfrak{f},K}\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \sum_{(\mathfrak{a},\mathfrak{f})=1} - \sum_{(\mathfrak{a},\mathfrak{f})\neq1} + \sum_{(\mathfrak{a},\mathfrak{f})\neq1}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\left(h_{\mathfrak{f},K}\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \sum_{\mathfrak{a}\subset\mathcal{O}_K}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right) + \frac{2}{\log F}\sum_{(\mathfrak{a},\mathfrak{f})\neq1}\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\end{aligned}$$

$$= \frac{2}{\log F}\left(h_{\mathfrak{f},K}\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \sum_{\mathfrak{a}\subset\mathcal{O}_K}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right) + O(1), \tag{4.3-3}$$

where in the final step we have claimed

$$\frac{2}{\log F}\sum_{(\mathfrak{a},\mathfrak{f})\neq 1}\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right) = O(1). \tag{4.3-4}$$

Let us justify (4.3-4). Recall that $\mathfrak{f}$ is prime and so because of the presence of $\Lambda(\mathfrak{a})$ only prime powers of $\mathfrak{f}$ will contribute. We have

$$\begin{aligned}
\frac{2}{\log F}\sum_{(\mathfrak{a},\mathfrak{f})\neq 1}\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right) &= \frac{2}{\log F}\sum_{\substack{\mathfrak{a}=\mathfrak{f}^m\\ m\in\mathbb{N}}}\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right)\\
&= \frac{2}{\log F}\sum_{m=1}^{\infty}\frac{\Lambda(\mathfrak{f}^m)}{\mathrm{N}(\mathfrak{f}^m)^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{f}^m)}{\log F}\right)\\
&= \frac{2\log \mathrm{N}(\mathfrak{f})}{\log F}\sum_{m=1}^{\infty}\frac{1}{(\sqrt{\mathrm{N}(f)})^m}g\left(\frac{\log \mathrm{N}(\mathfrak{f}^m)}{\log F}\right)\\
&\ll \frac{2\log \mathrm{N}(\mathfrak{f})}{\log F}\sum_{m=1}^{\infty}\frac{1}{(\sqrt{\mathrm{N}(f)})^m} \qquad \text{(because } g \text{ is bounded)}\\
&\ll \frac{2\log \mathrm{N}(\mathfrak{f})}{\log F}\,\frac{1}{1-\frac{1}{\sqrt{\mathrm{N}(f)}}}\\
&\ll \frac{2\log \mathrm{N}(\mathfrak{f})}{\log F}\\
&\ll 1,
\end{aligned}$$

where in the final step we recalled that F was close to $\mathrm{N}(\mathfrak{f})$.

Next, we substitute (4.3-3) back into our first equation (4.3-1). Then, divide by $h_{\mathfrak{f},K}$ to obtain

$$\begin{aligned}
\frac{1}{h_{\mathfrak{f},K}}\sum_{\substack{\widehat{\mathrm{Cl}_{\mathfrak{f}}(K)}\\ \chi\neq\chi_0}}\sum_{\gamma_\chi}\hat{g}\left(\gamma_\chi\frac{\log F}{2\pi}\right) &= \frac{g(0)}{\log F}\log\left((8\pi e^{\gamma})^{-n}\mathrm{N}(\mathfrak{d}_K\mathfrak{f})\right) - \frac{r}{\log F}\int_0^\infty \frac{g(t)-g(0)}{F^{t/2}-F^{-t/2}}\,dt\\
&\quad - \frac{2}{\log F}\left(\sum_{\mathfrak{a}\in P_{\mathfrak{f}}(K)} - \frac{1}{h_{\mathfrak{f},K}}\sum_{\mathfrak{a}\subset\mathcal{O}_K}\right)\frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^{1/2}}g\left(\frac{\log \mathrm{N}(\mathfrak{a})}{\log F}\right) + O\left(\frac{1}{h_{\mathfrak{f},K}}\right).
\end{aligned}$$

Finally, note that $L(s,\chi_0)$ is exactly $\zeta_K^{\mathfrak{f}}(s)$, where $\zeta_K^{\mathfrak{f}}(s)$ is the partial Dedekind zeta function away from primes dividing $\mathfrak{f}$. Writing $\rho_{\zeta_K} = \frac{1}{2} + \gamma_{\zeta_K}$ for the zeros, the difference between the left-hand side of the equation above and the statement of the proposition is

$$\frac{1}{h_{\mathfrak{f},K}}\sum_{\gamma_{\zeta_K}}\hat{g}\left(\gamma_{\zeta_K}\frac{\log F}{2\pi}\right).$$

This is seen to be $O\left(\frac{1}{h_{\mathfrak{f},K}}\right)$ because g is compactly supported and C^2, which implies $\hat{g}(y) \ll \frac{1}{y^2}$, so that the sum above converges absolutely. $\square$

4.4. **Future research directions.**

The main result of our thesis, namely the 1-level density as in Theorem 4.3.2, is a powerful formula which would be crucial in future investigations. Indeed, it is interesting to consider the averaged 1-level density, defined here by

$$D_{1;F/2,F}(\eta) \coloneqq \frac{1}{F/2}\sum_{F/2<\mathrm{N}(\mathfrak{f})\leq F} D_{1;\mathfrak{f}}(\eta),$$

which would allow us to obtain a result analogous to Theorem 3.2.2. This would involve a term-wise averaging of the terms in our obtained 1-level density $D_{1;\mathfrak{f}}(\eta)$. The main term obtained takes the form of an integral and must be split into four different regions with a careful analysis of each term. Promising partial results have been established, with some technical details to be completed.

Appendix A. Arithmetic Functions

Definition A.1. An *arithmetic function* is a function from the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ to the complex numbers $\mathbb{C}$.

We present five important examples of arithmetic functions. We will witness their power and uses later in this thesis.

Example A.2. Define $\epsilon(n) = 1$ if $n = 1$ and 0 otherwise. This is also called the *multiplication unit for Dirichlet convolution* or simply the *unit function.*

Example A.3. Fix $s \in \mathbb{C}$. Set $\sigma_s(0) = 0$ and for positive integers n

$$\sigma_s(n) = \sum_{d|n} d^s.$$

Note that $\sigma_0(n)$ counts the number of divisors of n and is known as the *divisor function*, generally denoted by $d(n)$ or $\tau(n)$.

Example A.4. Let $\phi(n)$ count the number of natural numbers co-prime to n and less than or equal to n. That is,

$$\phi(n) = \#\{a \leq n, a \in \mathbb{N} \,|\, (a, n) = 1\}.$$

This is *Euler's phi function*, or the *totient function.*

Example A.5. The *Möbius function* is given by $\mu(1) = 1$ and

$$\mu(u) = \begin{cases} (-1)^s & \text{if } n = p_1 p_2 \ldots p_k,\ p_i \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}.$$

Example A.6. Define the von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime,} \\ 0 & \text{otherwise.} \end{cases}.$$

Observe the Möbius function vanishes when n is not square-free, while the von Mangoldt function vanishes when n is not a prime power.

Definition A.7 (Multiplicative function)**.** An arithmetic function f is *multiplicative* if $f(1) = 1$ and

$$f(ab) = f(a)f(b)$$

whenever a and b are co-prime.
f is said to be *totally multiplicative* (or *completely multiplicative*) if $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Z}$.

It is easy to see that $\mu(n), \epsilon(n)$ and $\sigma_s(n)$ are all multiplicative. Indeed, $\epsilon(n)$ is totally multiplicative. It can also be shown that $\phi(n)$ is multiplicative. Note that $\Lambda(n)$ is not multiplicative.

Appendix B. Characters

We now conduct a general study of the theory of characters, in particular characters of a finite abelian groups. We follow closely the excellent lecture notes authored by Dave Platt [Pla].

Definition B.1. Let G be a finite abelian group. A *linear character* (or *multiplicative character*, or just *character*) of G is a group homomorphism

$$\chi : G \to \mathbb{C}^{\times},$$

where $\mathbb{C}^{\times}$ is the multiplicative group of non-zero complex numbers. The *set of characters* of G is denoted by $\hat{G}$.

Remark B.2. Historically, the first examples of characters occurred with target field $\mathbb{C}$ as above, but the concept was introduced in generality with target $F^{\times}$ by Dedekind (see [AM66]). Allowing multiplicative groups of other fields as target fields is very useful as well; for example, it finds applications in the general theory of Fourier Analysis on groups (see [Ter99]). We can also drop finiteness of the group and still retain linear independence of characters, which, in particular, finds great utility in Galois Theory (see [GQ09]) and is used in the proof of several important theorems including the normal basis theorem, Hilbert's Theorem 90 for cyclic Galois extensions, and also in Kummer theory and Artin-Schreier theory (see [Con]).

Remark B.3. The abelian restriction is in fact natural. Indeed, suppose $x, y \in G$ and $[x, y] := xyx^{-1}y^{-1}$ is the commutator of x, y. Then,

$$\chi([x, y]) = \chi(x)\chi(y)\chi(x)^{-1}\chi(y)^{-1} = 1,$$

since multiplicative groups of fields are of course abelian. So, we observe that χ annihilates the commutator subgroup $[G, G]$ (the subgroup generated by all commutators). Hence, χ in fact descends to a character on the abelianization of G, i.e., the quotient

$$\mathrm{Abel}(G) := G/[G, G].$$

Note that in the theory of Fourier Analysis on groups, this presents a problem since there are simply not enough characters to span the Hilbert space $L^2(G)$ (the inner product space of all complex-valued functions on G). Indeed, by Theorem B.7 there will be only $\#G/\#[G, G]$ linear characters; this gap is filled by relying on the representation theoretic approach to construct a Fourier Theory on G (see [Tao]).

It is clear that the linear characters form an abelian group under multiplication, which we will denote $\hat{G}$.

Remark B.4. Note that $\hat{G}$, the group of characters, is indeed the Pontryagin dual of G. For finite abelian groups, understanding Pontryagin duality is in fact equivalent to the theory of the discrete Fourier transform.

Now, it follows by homomorphy that

$$\chi(ab) = \chi(a)\chi(b)$$

and

$$\chi(e_G) = 1,$$

where e_G is the identity element of G.

We use $\chi_0 \in \hat{G}$ to denote the trivial character

$$\chi_0(a) = 1$$

for all $a \in G$. Oftentimes, χ_0 is said to be the *principal character.*

For n a positive integer, we will denote by μ_n the group of n-th roots of unity in $\mathbb{C}$.

Proposition B.5. *$\chi(G) \subseteq \mu_n$, where n is the exponent of G. In particular, note that $|\chi(g)| = 1$ for all g.*

Proof. By definition, $g^n = 1$, which implies $\chi(g)^n = 1$ by homomorphy. So, $\chi(g) \in \mu_n$. □

Finally, we state two key properties of characters.

Theorem B.6. *For G a finite abelian group, $G \cong \hat{G}$.*

Theorem B.7 (Orthogonality of Characters)**.** *For G a finite abelian group, the characters of G obey*

$$\sum_{g \in G} \chi(g) = \begin{cases} \#G & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0, \end{cases}$$

and further

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} \#\hat{G} & \text{if } g = e_G, \\ 0 & \text{if } g \neq e_G. \end{cases}$$

The proofs of these theorems are deferred to the end of the Appendix. Indeed, since we are dealing with finite abelian groups, a natural strategy is to first reduce to the case of cyclic groups, and then apply the Fundamental Theorem of Abelian Groups. This is precisely what we do.

Proposition B.8. *Let G (only for this proposition) be a finite cyclic group of order n, generated by a. Then,*

(1) *$\hat{G}$ has exactly n elements*

$$\chi_k(a^m) = \omega^{km}, k = 0, 1, \ldots, n-1,$$

where $\omega = e^{2\pi i/n}$.

(2) *G has orthogonality of characters.*

(3) *$\hat{G}$ is cyclic and generated by χ_1. So, $G \cong \hat{G}$.*

Proof. For (1), note that the image of a determines the entire image of χ. Since it must be the case that $\chi(a) = \omega^k$ for some $k = 0, 1, \ldots, n-1,$, there are exactly n distinct elements in $\hat{G}$. We label them $\chi_1, \chi_2, \ldots, \chi_n$.

Clearly $(\chi_1)^k(a^m) = \omega^{mk} = \chi_k(a^m)$, i.e., $\chi_1^k = \chi_k$. So, $\hat{G}$ is cyclic and generated by χ_1, proving (3). Finally, consider for $\chi_k \neq \chi_0$ (the case for $\chi_k = \chi_0$ is trivial),

$$\sum_{g\in G} \chi_k(g) = \sum_{m=0}^{n-1} \chi_k(a^m) = \sum_{m=0}^{n-1} \omega^{km} = \frac{1-\omega^{kn}}{1-\omega^k} = 0.$$

Similarly, for $a^m = g \neq e_G$ (again, $g = e_G$ is trivial),

$$\sum_{\chi\in\hat{G}} \chi(g) = \sum_{k=0}^{n-1} \chi_k(a^m) = \sum_{k=0}^{n-1} \omega^{mk} = \frac{1-\omega^{mn}}{1-\omega^m} = 0.$$

□

Finally, we present a Lemma which allows us to extend the properties we just showed to hold in the case of cyclic groups, to groups built out of cyclic groups.

Lemma B.9. *Let G_1, G_2 be finite abelian groups and set $G = G_1 \times G_2$. Suppose $\chi_1 \in G_1, \chi_2 \in G_2$. Define $\chi : G \to \mathbb{C}^\times$ with $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$; χ is a character. Conversely, for any $\chi \in \hat{G}$, there is a unique choice of $\chi_1 \in G_1, \chi_2 \in G_2$ such that $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$. Indeed, $\hat{G} \cong \hat{G}_1 \times \hat{G}_2$. Furthermore, if G_1, G_2 have orthogonality of characters, then so does G.*

Proof. First, χ is clearly a character of G, since homomorphy follows from homomorphy of χ_1 and χ_2.

Next, suppose we have some $\chi \in \hat{G}$. Then define $\chi_1 : G_1 \to \mathbb{C}^\times$ by $\chi_1(g_1) = \chi(g_1, e_{G_2})$. Similarly, define $\chi_2 : G_2 \to \mathbb{C}^\times$ by $\chi_2(g_2) = \chi(e_{G_1}, g_2)$. By the homomorphy of χ, it follows that

$$\chi(g_1, g_2) = \chi(g_1, e_{G_2})\chi(e_{G_1}, g_2) = \chi_1(g_1)\chi_2(g_2).$$

Further, this choice of χ_1, χ_2 is unique. If instead we had $\chi = \chi_1'\chi_2'$, then

$$\chi_1(g_1) = \chi(g_1, e_{G_2}) = \chi_1'(g_1)\chi_2'(e_{G_2}) = \chi_1'(g_1),$$

for any g_1, which implies $\chi_1 = \chi_1'$. Similarly, $\chi_2 = \chi_2'$.

Clearly then, the isomorphism $\hat{G} \cong \hat{G}_1 \times \hat{G}_2$ is given by $\chi \leftrightarrow (\chi_1, \chi_2)$, with χ_1, χ_2 as above.

Finally, we can easily check for orthogonality of characters. We have

$$\sum_{g\in G} \chi(g) = \sum_{g_1\in G_1} \chi_1(g_1) \sum_{g_2\in G_2} \chi_2(g_2) = \begin{cases} \#G_1\#G_2 = \#G & \text{if } \chi_1 = \chi_0 \in \hat{G}_1 \text{ and } \chi_2 = \chi_0 \in \hat{G}_2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{\chi\in\hat{G}} \chi(g_1, g_2) = \sum_{\chi_1\in\hat{G}_1} \chi_1(g_1) \sum_{\chi_2\in\hat{G}_2} \chi_2(g_2) = \begin{cases} \#G_1\#G_2 = \#G & \text{if } g = (g_1, g_2) = (e_{G_1}, e_{G_2}) = e_G, \\ 0 & \text{otherwise.} \end{cases}$$

□

The proofs of Theorems B.6 and B.7 now follow as quick corollaries of our above work.

Proof of Theorems B.6 and B.7. Given G a finite abelian group, apply the Fundamental Theorem of Abelian groups to write it as a product of cyclic groups, as in $G = C_1 \times \cdots \times C_r$. Then, applying Lemma B.9 inductively, we obtain

$$\hat{G} = \hat{C}_1 \times \cdots \times \hat{C}_r,$$

and find that G has orthogonality of characters since by claim (2) in Proposition B.8 finite cyclic groups do.

Further, using Proposition B.8, we can write $C_i \cong \hat{C}_i$, so that

$$\hat{G} = \hat{C}_1 \times \cdots \times \hat{C}_r = C_1 \times \cdots \times C_r = G.$$

□

Appendix C. Basics of the Theory of the Fourier Transform

Here we recall some fundamental properties of the Fourier transform. The primary reference is [SS03].

Definition C.1. Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function, i.e., $f \in L^1(\mathbb{R})$. The *Fourier transform* of f is a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx..$$

The critical case for the Fourier transform is the Gaussian, whereby the Fourier transform a Gaussian is another Gaussian.

Proposition C.2. *The Gaussian $e^{-\pi t^2}$ is its own Fourier transform. More precisely, for*

$$g(x) = e^{-\pi x^2},$$

we have

$$\hat{g}(y) = e^{-\pi y^2} = g(y).$$

Proof. We have

$$\begin{aligned}
\hat{g}(y) &= \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y}\, dx \\
&= \int_{\mathbb{R}} e^{-\pi (x+iy)^2 - \pi y^2}\, dx \\
&= e^{-\pi y^2} \int_{\mathbb{R}} e^{-\pi (x+iy)^2}\, dx \\
&= e^{-\pi y^2} \int_{\mathbb{R}+iy} e^{-\pi z^2}\, dz. \qquad \text{(C.1)}
\end{aligned}$$

We can now shift contours. In particular, we would like to shift the integral down from $\mathbb{R} + iy$ to just $\mathbb{R}$ (if $y > 0$, but the argument for shifting up when $y < 0$ will also be clear). By Cauchy's theorem, the integral over the rectangular contour determined by the four points $(\pm R, 0), (\pm R, iy)$ will be zero, so that

$$\int_{(-R,0)}^{(R,0)} e^{-\pi z^2}\, dz + \int_{(R,0)}^{(R,iy)} e^{-\pi z^2}\, dz + \int_{(R,iy)}^{(-R,iy)} e^{-\pi z^2}\, dz + \int_{(-R,iy)}^{(-R,0)} e^{-\pi z^2}\, dz = 0. \tag{C.2}$$

The second and fourth integrals vanish in the limit as $R \to \infty$, which is made clear by using the estimation lemma (also known as the ML equality), so that

$$\begin{aligned} \left| \int_{(\pm R,0)}^{(\pm R,iy)} e^{-\pi z^2}\, dz \right| &\leq y \sup_{0\leq t\leq y} \left| e^{-\pi (R+it)^2} \right| \\ &\leq y\, e^{-\pi R^2} e^{\pi y^2}. \end{aligned}$$

Hence, taking the limit as $R \to \infty$ in (C.2), we obtain

$$\int_{\mathbb{R}+iy} e^{-\pi z^2}\, dz = \int_{\mathbb{R}} e^{-\pi z^2}\, dz.$$

Returning to (C.1), we have

$$\begin{aligned} \hat{g}(y) &= e^{-\pi y^2} \int_{\mathbb{R}+iy} e^{-\pi z^2}\, dz \\ &= e^{-\pi y^2} \int_{\mathbb{R}} e^{-\pi z^2}\, dz \\ &= e^{-\pi y^2}, \end{aligned}$$

where the last equality follows from the well-known fact that the standard normal distribution integrates to 1, that is,

$$\int_{\mathbb{R}} e^{-\pi z^2}\, dz = 1.$$

□

Remark C.3. The Gaussian also plays a crucial role in proving the Fourier inversion theorem through Dirac kernel methods. Indeed, the Gaussian function is a natural choice for approximating the Dirac delta function because it is smooth, well-behaved, and its Fourier transform as shown above is also a Gaussian, making calculations tractable. The key property used here is that as the standard deviation of a Gaussian function approaches zero, the Gaussian function approaches the Dirac delta function in a distributional sense. This property is utilized in proving the Fourier inversion theorem using kernel methods. By convolving a function with an approximating sequence of Gaussians (narrowing Gaussians), one can show that in the limit, the operation becomes equivalent to convolving with the Dirac delta function, which essentially recovers the original function - demonstrating the Fourier inversion.

Next is a useful result to quickly compute the Fourier transform of a scaled function when it is easier to compute the transform of the unscaled version.

Lemma C.4 (Scaled Fourier transform)**.** *For $h, f : \mathbb{R} \to \mathbb{C}$ integrable, and $\lambda \in \mathbb{R} \setminus \{0\}$, such that $h(x) = f(\lambda x)$, the Fourier transform of h is given by*

$$\hat{h}(y) = \frac{1}{|\lambda|}\hat{f}\Big(\frac{y}{\lambda}\Big).$$

Proof. This is a simple application of definitions and change of variables.

$$\begin{aligned}\hat{h}(y) &= \int_{-\infty}^{\infty} h(x)e^{-2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} f(\lambda x)e^{-2\pi ixy}\,dx \\ &= \frac{1}{|\lambda|}\int_{-\infty}^{\infty} f(z)e^{-2\pi iz\frac{y}{\lambda}}\,dz \\ &= \frac{1}{|\lambda|}\hat{f}\Big(\frac{y}{\lambda}\Big).\end{aligned}$$

□

We now consider the Schwartz space $\mathcal{S}$, which is the function space of all functions whose derivatives are rapidly decreasing. This space has the important property that the Fourier transform is an automorphism on this space (see [SS03]). A function in the Schwartz space is oftentimes called a *Schwartz function.*

Definition C.5. The *Schwartz space*, or *space of rapidly decreasing functions on* $\mathbb{R}$, is the function space

$$\mathcal{S}(\mathbb{R}, \mathbb{C}) \coloneqq \{f \in C^{\infty}(\mathbb{R}, \mathbb{C}) \,|\, \forall \alpha, \beta \in \mathbb{N}, \|f\|_{\alpha,\beta} < \infty\},$$

where $C^{\infty}(\mathbb{R}, \mathbb{C})$ is the function space of smooth functions from $\mathbb{R}$ into $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} \coloneqq \sup_{x \in R} |x^{\alpha} f^{(\beta)}(x)|.$$

Here, $f^{(\beta)}(x)$ denotes the β-th derivative of $f(x)$ with respect to x.

In other words, a rapidly decreasing function, or Schwartz function, is a function such that $f(x), f'(x), f''(x), \ldots$ all exist everywhere on $\mathbb{R}$ and decay to zero faster than any reciprocal power of x. Note that since bounds on $|x^{\alpha} f^{(\beta)}(x)|$ are useless at 0, we sometimes consider $|(1+x)^{\alpha} f^{(\beta)}(x)|$ instead. Simple examples of such functions include $x^m e^{-ax^2}$ for any m and $a > 0$.

When working in Schwartz space (though this restriction can be considerably weakened as the proof will illustrate), it is particularly easy to compute the Fourier transform of a derivative in terms of the transform of the anti-derivative.

Lemma C.6 (Fourier transform of derivative)**.** *Suppose $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ we have*

$$\hat{f'}(y) = 2\pi iy\hat{f}(y),$$

where f' is the derivative of f.

Proof. We apply definitions and integrate by parts.

$$\begin{aligned}\hat{f'}(y) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi ixy}\,dx \\ &= \left[e^{-2\pi ixy}f(x)\right]_{-\infty}^{\infty} - (-2\pi iy)\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= 2\pi iy\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= 2\pi iy\hat{f}(y),\end{aligned}$$

where in the third equality the first term vanished precisely because f is Schwartz. □

The final fact we must recall is the Poisson summation formula.

Theorem C.7 (Poisson summation)**.** *Let* $f \in \mathcal{S}(\mathbb{R},\mathbb{C})$*. Then,*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. Consider the periodic summation of the function f defined by

$$F(x) = \sum_{m=-\infty}^{\infty} f(x+m).$$

This function is well-defined and safe to work with since the sum converges absolutely, precisely because f is Schwartz, and so the terms being summed decay rapidly.

We observe that F is periodic with period 1. This means the function has a Fourier series expansion given by

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi ix},$$

with

$$C_n = \int_0^1 F(x)e^{-2\pi inx}\,dx.$$

We now compute this coefficient.

$$\begin{aligned}C_n &= \int_0^1 F(x)e^{-2\pi inx}\,dx \\ &= \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-2\pi inx}\,dx \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m)e^{-2\pi inx}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m)e^{-2\pi inx}\,dx \qquad\qquad (\text{change variables } x+m \to \tau)\\
&= \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(\tau)e^{-2\pi in(\tau-m)}\,dx\\
&= \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(\tau)e^{-2\pi in\tau}\,d\tau \qquad\qquad (n,m \in \mathbb{Z}, \text{ so } e^{2\pi imn} = 1)\\
&= \int_{-\infty}^{\infty} f(\tau)e^{-2\pi in\tau}\,d\tau\\
&= \hat{f}(n).
\end{aligned}
$$

We have shown that

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi ix},$$

which means

$$\sum_{m=-\infty}^{\infty} f(x+m) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi ix}.$$

Setting $x = 0$ we obtain

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

as desired. $\square$

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