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Critical Phenomena in the Gravitational Collapse of Electromagnetic Dipole and Quadrupole Waves

An Honors Paper for the Department of Physics and Astronomy

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Bowdoin College, 2021

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Chapter 1

Introduction

Critical phenomena are not unique to gravitation, but rather they are common in many fields of physics. We understand that they occur close to phase transitions and that they tend to show power-law behavior and scaling laws. One well-studied example are the thermodynamic properties at the onset of magnetic ordering. For instance, there exists a critical temperature T_c , which is the Curie temperature for ferromagnets, above which magnetic ordering vanishes. We then identify *supercritical* temperatures as those that have zero magnetization. We also observe that the magnetization just below T_c , i.e. for *subcritical* temperatures, is well described by a power-law

$$M \propto (T_c - T)^\gamma \tag{1.1}$$

where the critical exponent γ is usually between 0.33 and 0.37. Fig. 1.1 illustrates this power-law behavior.

Matthew Choptuik was the first to report critical phenomena in gravitational collapse to black holes in his seminal 1993 article. More specifically, he studied massless scalar fields, minimally coupled to gravity, and in spherical symmetry. He explored different families of initial data characterized by an arbitrary parameter η (i.e. amplitude, length-scale, density, etc.) that determines the strength of the fields. Choptuik distin-

Figure 1.1: Ashcroft and Mermin (1976) plot the temperature dependence of the magnetic field for an antiferromagnet. The magnetic field vanishes at the critical temperature $T_c = 67.336K$.

guished *subcritical* data (i.e. those with weak initial data that disperse to infinity and leave behind flat space in dynamical evolutions) from *supercritical* data (i.e. those with strong initial data that form a black hole). Therefore, Choptuik observed critical phenomena in the vicinity of critical parameter η^* that separates data into subcritical and supercritical, and thus marks the threshold of black-hole formation.

While many aspects of critical phenomena are well understood in the case of spherical symmetry, as we later expand, the situation is much less clear when the solution is not spherically symmetric. The most important example is the gravitational collapse of vacuum gravitational waves, which cannot exist in spherical symmetry, as they showcase the properties of pure gravity in the absence of matter. In order to explore the absence of spherical symmetry, we study critical phenomena in the gravitational collapse of electromagnetic (EM) waves. EM waves are of particular interest because they share with vacuum gravitational waves the property that a critical solution cannot be spherically symmetric – and yet they are easier to handle numerically than gravitational waves. This

project expands the work of Baumgarte et al. (2019) (hereafter BGH) which focuses on dipolar EM waves. We generalize their findings by comparing dipolar and quadrupolar EM waves. Specifically, we use numerical simulations to investigate the behavior of the different waves and analyze how the different types of initial data affect the critical solution.

The outline of the thesis follows: In Chapter 2, we contextualize our motivation by giving a heuristic overview of critical phenomena in spherical symmetry. This, in turn, highlights the limits of our understanding when it comes to gravitational waves and the absence of said spherically symmetric solution. We then introduce the motivation behind our work with electromagnetic waves as a framework to investigate critical phenomena in the absence of a spherically symmetric solution. In Chapter 3, we derive analytical solutions in flat spacetimes for dipolar, quadrupolar, and octupolar EM waves. In Chapter 4, we present new contributions to the numerical framework used in BGH to explore different types of initial data. In Chapter 5 we showcase the results of our numerical evolutions, highlighting the qualitative differences resulting from the choice of initial data. We show evidence of non-uniqueness in the critical solution, at least globally. We also provide some evidence for non-uniqueness in a local scale. We conclude by arguing that in the case of non-spherical symmetry, there is no unique critical solution, at least globally.

Chapter 2

Critical Phenomena

2.1 Background

As previously mentioned, Choptuik (1993) was the first to report critical phenomena in gravitational collapse. He focused on one-parameter families of solutions that are generated by evolving initially in-going packets of scalar field in spherical symmetry. This parameter, η , characterizes the strength of the gravitational self-interaction of the scalar field. One of the families of initial data he explored is of the form

$$\phi(r) = \eta \tanh[(r - r_0)/\delta] \tag{2.1}$$

where the parameter η is the initial amplitude. He then evolved the initial data and observed the behavior. He noted that if η was “weak”, the evolutions dispersed to infinity, and if η was “strong” the end state of the evolution is a black hole. He then identified a critical parameter η^* that separates data into subcritical (leading to flat spacetime i.e. $\eta < \eta^*$) and supercritical (leading to black hole formation i.e. $\eta > \eta^*$). His paper also highlighted two main findings about gravitational collapse.

First, he noted that for supercritical evolutions there is a power law scaling for

the mass of the black hole formed,

$$M_{BH} \propto (\eta - \eta^*)^\gamma, \quad (2.2)$$

where γ is the “critical exponent”. This means that if we plot this on a log-log scale we get a straight line, as shown in Fig. 2.1, and the slope of the straight line is γ . Choptuik found that γ was universal for scalar fields i.e. no matter the choice of the parameter or the family of initial data they always had the same exponent $\gamma \simeq 0.37$. This is reminiscent of the magnetic ordering behavior discussed in Chap. 1 which is quite remarkable considering that these are completely different physical processes.

Second, he observed that in the vicinity of the critical parameter, the initial data evolve to approach a self-similar critical solution, i.e. one that contracts without changing shape towards an accumulation event. We will expand this further in the following section.

Choptuik’s initial discovery prompted a large body of research over the past three decades (see Gundlach and Martín-García (2007) for a review). As others continue to study critical phenomena, we now recognize that the self-similar critical solution can either be discretely self similar (DSS), which is what Choptuik found in his scalar fields, or continuously self similar (CSS), for example for perfect fluids studied by Evans and Coleman (1994). In addition, it has later been noted that the power-law scaling described in (2.2) is only true on “average” – i.e. for CSS solutions. For matter models that display a DSS critical solution the power law scaling is not a *true* straight line, but should, strictly speaking, showcase superimposed periodic “wiggles”. Although it is difficult to appreciate this when looking at Fig. 2.1, we want to highlight this property now as it is important for our later analysis.

Figure 2.1: Choptuik (1998) plots the mass of the black hole formed in supercritical data, M_{BH} as a function of the parameter ϕ_0 in a log-log plot. The inset denotes the data on a linear-scale pair of axis. This is reminiscent of Fig. 1.1 which showcases power-law behavior for magnetization.

2.2 Self-Similarity and Power-Law Scaling

As we have gathered a heuristic understanding of critical phenomena in spherical symmetry we have also understood that the two features, power-law scaling and self-similarity, are not independent, but rather the power law scaling follows as a direct consequence of the self-similar critical solution. In fact, the critical exponent γ is the inverse of the so called Lyapunov exponent λ of the critical solution (i.e. the growth rate of the perturbation). In this section we will develop a more in-depth description of just how the self-similarity and power-law scaling relate.

As a warm-up exercise, consider a small perturbation β of some fixed background solution. We will assume geometrized units, i.e. $c = 1 = G$, and further that β grows at a constant rate K , i.e.

$$\frac{d\beta}{d\tau} = K\beta. \quad (2.3)$$

Here we denote time with τ , which will later play the role of a proper time, and note that K has units of inverse length (recall that time and length have the same units when $c = 1$). Equation (2.3) is solved by

$$\beta = Ae^{K\tau}, \quad (2.4)$$

i.e. β grows exponentially in the time τ .

Our understanding of critical phenomena in gravitational collapse is based on the assumption that, close to the black-hole threshold, i.e. $\eta \simeq \eta^*$ (recall that η is a parameter characterizing our initial data), the initial data will evolve toward a solution that can be described as a self-similar solution plus a small perturbation. Specifically, what we mean by a self-similar solution is one that contracts without changing shape towards an accumulation event where $\tau = \tau^*$ as described in Fig. 2.2. By varying η and fine tuning to η^* , i.e. the smaller $(\eta - \eta^*)$, the evolution is close to a self-similar critical solution for longer. The reverse is also true: a greater difference means there will be less time during

Figure 2.2: Neilsen and Choptuik (2000) display a schematic diagram showing a continuously self-similar pulse at five different times as it moves toward the origin $r = 0$. The dotted lines are lines of constant $\zeta = r/t$, which is the similarity variable. These lines converge at the spacetime origin $(r, t) = (0, 0)$ in the upper left-hand corner of the plot. The inset shows the pulse as a function of ζ , like a snapshot of it at every point in time. As the pulse moves toward the origin it appears the same on smaller and smaller length scales. Rather than placing the accumulation event at the origin, we will allow it to be located at $(0, \tau^*)$.

which the solution will be self-similar. Accordingly, at any time τ , any length or time scale of the self-similar solution is proportional to $(\tau^* - \tau)$. Therefore, the growth rate of the perturbation of a self-similar solution will also not be constant, but rather scale according to

$$\bar{K} \propto 1/(\tau^* - \tau). \quad (2.5)$$

Rather than satisfying (2.3), a perturbation β will then satisfy

$$\frac{d\beta}{d\tau} = \frac{\bar{K}}{(\tau^* - \tau)}\beta. \quad (2.6)$$

To find a solution we use separation of variables and integration

$$\int \frac{1}{\beta} d\beta = \int \frac{\bar{K}}{(\tau^* - \tau)} d\tau \quad (2.7)$$

to find

$$\ln \beta = -\bar{K} \ln (\tau^* - \tau). \quad (2.8)$$

If we define “slow time” as

$$T = -\ln (\tau^* - \tau) \quad (2.9)$$

we can write our solution as

$$\beta = C e^{\bar{K}T} = C(\tau^* - \tau)^{-\bar{K}}. \quad (2.10)$$

β , which depends on the initial data, is the perturbation of our self-similar solution. The constant C also depends on the initial data, and it defines the distance from the critical solution $(\eta - \eta^*)$. For perfect fine tuning, i.e. $\eta = \eta^*$, we know that C must be zero. Therefore, we may Taylor expand our amplitude C around η^*

$$C = D(\eta - \eta^*) + \dots \quad (2.11)$$

and then replace C in (2.10) to leading order term to obtain

$$\beta = D(\eta - \eta^*)(\tau^* - \tau)^{-\bar{K}}. \quad (2.12)$$

The length scale that will be imprinted on the late time solution will be the length scale of the self-similar solution at the moment when the perturbation β takes a certain critical value β_c , say order unity, i.e. when it becomes non-linear. At that point, we can no longer treat our solution as a background plus a linear perturbation. Suppose that

happens when

$$\beta = \beta_c = D(\eta - \eta^*)(\tau^* - \tau_c)^{-\bar{K}}. \quad (2.13)$$

We can then solve for the length scale at the critical time r_c

$$r_c = \tau^* - \tau_c = \left(\frac{D}{\beta_c(\eta - \eta^*)} \right)^{-1/\bar{K}} = \left(\frac{\beta_c}{D} \right)^{1/\bar{K}} (\eta - \eta^*)^{1/\bar{K}}. \quad (2.14)$$

This shows that $r_c \propto (\eta - \eta^*)^{1/\bar{K}}$. Rewriting this relationship in terms of the “critical exponent” $\gamma \equiv 1/\bar{K}$ gives

$$r_c \propto (\eta - \eta^*)^\gamma. \quad (2.15)$$

This is now the length scale that will be imprinted on all quantities of unit length. For example, in our geometrized units, mass also has units of length, so we conclude that the mass of the black hole that forms should be

$$M_{BH} \propto (\eta - \eta^*)^\gamma. \quad (2.16)$$

Likewise, the energy density ρ has units

$$[\rho] = \frac{1}{length^2}, \quad (2.17)$$

therefore we expect

$$\rho \propto (\eta - \eta^*)^{-2\gamma}. \quad (2.18)$$

Lastly, it is also important to note the expected growth rate of the density in self-similar solutions e^{2T} . To check we can see

$$\rho \propto \frac{1}{r^2} \propto \frac{1}{(\tau^* - \tau)^2} \propto e^{2T} \quad (2.19)$$

where, again, T is the “slow time” (2.9).

2.3 Absence of Spherically Symmetric Solution

As we have pointed out earlier, critical phenomena are well understood in the presence of spherical symmetry. Even when looking at non-spherical evolutions of scalar fields or fluids we can see that since the matter model allows for spherically symmetric solutions, there still exist a limit to which such critical behavior is understood. In turn, our understanding becomes less clear when we encounter matter models that cannot allow for a spherically symmetric critical solution at all. The most important matter model with this property are vacuum gravitational waves, which display the properties of pure gravity. Gravitational waves carry energy, and according to Einstein, the energy curves spacetime, creating gravitational fields. Furthermore, if gravitational fields are strong enough they collapse creating a black hole. Abrahams and Evans (1993), (1994) were the first to report critical phenomena in the collapse of vacuum gravitational waves. Despite many attempts along the years, their findings have never been reproduced. Some of the most recent advances by Hilditch et al. (2017) and Ledvinka and Khirnov (2021) have found no convincing evidence of a strictly self-similar solution as well as non-universality when it comes to the critical exponent.

Fig. 2.3 plots the maximum curvature invariant $I_K^{1/4}$, which denotes the strength of the gravitational fields, encountered during the entire evolutions for subcritical data. This is analogous to the maximum densities encountered for EM waves in our later analysis. Here the arbitrary parameter η is denoted by A , and therefore the power-law scaling follows

$$I_K^{1/4} \propto (A - A^*)^\gamma \quad (2.20)$$

where γ is the critical exponent.

If this were a strictly self-similar solution as seen in Fig. 2.1, we would expect to find straight lines with periodic wiggles. Here we do see the lines, yet the wiggles are not strictly periodic, a clear indication that the critical solutions are not strictly self-

Figure 2.3: Here Ledvinka and Khirnov (2021) plot extremes of the Kretschmann scalar I_K as a function of the parameter A for subcritical data, i.e. $A < A^*$, for varying initial data. They also show echo scale ratios on the lower right. In the case of a DSS solution, these echoes would approach the factor e^Δ , where Δ is the echoing period.

similar. Moreover, if we look at different sets of initial data, we can see that they have different slopes thus indicating different values of γ . This in turn shows that there does not appear to be one *unique* critical exponent for gravitational waves. This evidence suggests the lack of a universal and strictly self-similar solution for the critical collapse of vacuum gravitational waves. With this being said, we can then hypothesize reasons why this non-uniqueness appears for gravitational waves. Our suspicion is that this non-uniqueness is directly related to the absence of a spherically symmetric critical solution.

2.4 Electromagnetic Waves

In order to explore the absence of a spherically symmetric critically solution, we would like to introduce an easier framework. Electromagnetic (EM) waves are of particular interest because they share with gravitational vacuum waves the absence of a spherically symmetric solution as there is no monopole radiation in electromagnetism. In addition, they also are much easier to handle numerically than the vacuum case, as in axisymmetry the equations can be rewritten in terms of only the azimuthal angle component ϕ . We know that in flat spacetimes EM fields can carry energy to infinity. Coupling EM fields to gravity, we can then allow the energy that these EM fields carry to create gravitational fields. Therefore, for strong enough EM fields, they can create strong enough gravitational fields and therefore collapse and form a black hole. We can then parametrize these EM waves by the amplitude and observe critical phenomena near the critical amplitude. With this in mind, BGH studied critical phenomena in the gravitational collapse of electromagnetic waves, focusing on dipole waves. Their findings show an approximately DSS critical solution, but they explained that self-similarity is not exact, i.e. there is no strict periodicity in the “wiggles”. In addition, they found an approximate scaling law for the central energy densities ρ_c .

Figure 2.4: BGH show the energy density ρ evaluated at the center as a function of the slow time T for near-critical, both centered and off-centered, evolutions. They note that, although the amplitude of the oscillations approximately increases with $(\tau_\star - \tau)^{-2} = e^{2(T-T_0)}$, this DSS critical solution is not exact. For instance, the “wiggles” do not appear to be strictly periodic.

Inspired by their results, we now study the gravitational collapse of electromagnetic waves with varying multipole moments. Focusing on dipole and quadrupole waves as initial data, we study the behavior of the critical solution as we fine-tune the amplitude to the onset of black hole formation. We find qualitatively different behaviors for the two types of waves that we believe stem from the differences in symmetries of the multipoles. We can argue that these differences suggest the absence of a unique critical solution for electromagnetic waves, at least globally.

Chapter 3

Derivations of Analytical Solutions

In this section we will derive analytical solutions for Maxwell's equations in flat spacetimes for different multipole moments ℓ . We will then use these equations as our initial data for our numerical evolutions. Maxwell's equations in vacuum can be expressed as a pair of equations

$$\begin{aligned}\partial_t \mathbf{A} &= -\mathbf{E} - \nabla \Phi \\ \partial_t \mathbf{E} &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})\end{aligned}\tag{3.1}$$

with

$$\mathbf{B} = \nabla \times \mathbf{A},\tag{3.2}$$

where $\mathbf{A}$ is a vector potential. Without loss of generality, we may then choose a gauge in which $\Phi = 0$ so that $\mathbf{A}$ becomes purely spatial. Combining the two equations in (3.1) we then obtain

$$-\frac{\partial^2}{\partial t^2} \mathbf{A} + \nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = 0.\tag{3.3}$$

Adopting spherical polar coordinates, and assuming axisymmetry, we can write

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ A^{\hat{\phi}}(t, r, \theta) \end{pmatrix}, \quad (3.4)$$

where the hat on the $\hat{\phi}$ component denotes an orthonormal component. Note that the divergence of $\mathbf{A}$ is zero since $A^{\hat{\phi}}$ is not a function of ϕ .

Next, introducing the vector Laplacian, $\nabla^2 \mathbf{A}$, in spherical polar coordinates (3.3) becomes

$$-\frac{\partial^2}{\partial t^2} A^{\hat{\phi}} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial A^{\hat{\phi}}}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial A^{\hat{\phi}}}{\partial \theta} \right) - \frac{A^{\hat{\phi}}}{r^2 \sin^2 \theta} = 0. \quad (3.5)$$

Also note that

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial A^{\hat{\phi}}}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r A^{\hat{\phi}}). \quad (3.6)$$

We will now introduce a new variable, $\tilde{A} = r A^{\hat{\phi}}$, and insert into (3.5) to obtain

$$-\partial_t^2 \tilde{A} + \partial_r^2 \tilde{A} + \frac{1}{r^2 \sin \theta} \partial_\theta \left(\sin \theta \partial_\theta \tilde{A} \right) - \frac{\tilde{A}}{r^2 \sin^2 \theta} = 0. \quad (3.7)$$

Next, we will assume that we can find a solution by using a separable ansatz

$$\tilde{A}(t, r, \theta) = g_\ell(t, r) f_\ell(\theta), \quad (3.8)$$

with angular functions $f_\ell(\theta)$ so that

$$\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta f_\ell) - \frac{f_\ell}{\sin^2 \theta} = -\ell(\ell + 1) f_\ell. \quad (3.9)$$

Without the “extra” $f_\ell/\sin^2(\theta)$ term, (3.9) looks like the Legendre’s equation. However, because we are now working with vectors and have this additional term, we

do not obtain strict Legendre polynomials as solutions. Instead, we obtain a “vector version” of the Legendre polynomials as our solutions. For different values of ℓ we obtain different angular functions. For example, for $\ell = 1$ we have $f_\ell(\theta) = \sin \theta$, for $\ell = 2$ we have $f_2(\theta) = \sin \theta \cos \theta$, and for $\ell = 3$ we have $f_3(\theta) = (5 \cos^2 \theta - 1) \sin \theta$. Then (3.7) becomes

$$-\partial_t^2 g_\ell + \partial_r^2 g_\ell - \frac{\ell(\ell+1)}{r^2} g_\ell = 0. \quad (3.10)$$

This leads to the time-radial part of the solution. To solve we use the ansatz suggested by Rinne (2008)

$$g_\ell(t, r) = \sum_{j=0}^{\ell} c_j r^{j-\ell} F_{\pm}^{(j)}(x), \quad (3.11)$$

where c_j 's are constant coefficients and $F_{\pm}^{(j)}(x)$'s are functions of $x = r \pm t$. In the following section we derive recursion relationships for both.

3.1 Dipole Waves

We now derive the solution for the dipole waves. Inserting $\ell = 1$ into (3.11) we have

$$g_1 = \frac{c_0}{r} F_{\pm}^{(0)} + c_1 F_{\pm}^{(1)}. \quad (3.12)$$

Then

$$\partial_t^2 g_1 = \frac{c_0}{r} F_{\pm}^{(0)''} + c_1 F_{\pm}^{(1)''}, \quad (3.13)$$

and

$$\partial_r^2 g_1 = \frac{2c_0 F_{\pm}^{(0)}}{r^3} - \frac{2c_0 F_{\pm}^{(0)'}}{r^2} + \frac{c_0 F_{\pm}^{(0)''}}{r} + c_1 F_{\pm}^{(1)''}. \quad (3.14)$$

So, (3.10) now becomes

$$0 = -\frac{c_0}{r}F_{\pm}^{(0)''} - c_1F_{\pm}^{(1)''} + \frac{2c_0F_{\pm}^{(0)'}}{r^3} - \frac{2c_0F_{\pm}^{(0)'}}{r^2} + \frac{c_0F_{\pm}^{(0)'}}{r} + c_1F_{\pm}^{(1)''} - \frac{2}{r^2}\left(\frac{c_0}{r}F_{\pm}^{(0)} + c_1F_{\pm}^{(1)}\right). \quad (3.15)$$

Simplifying further, we see that many of the terms cancel neatly, leaving us only with

$$c_0F_{\pm}^{(0)'} = -c_1F_{\pm}^{(1)}, \quad (3.16)$$

where c_0 is undetermined and thus arbitrary. We can then write the recursion relationship for the coefficients

$$c_1 = -c_0, \quad (3.17)$$

as well as the recursion relationship between derivatives

$$F_{\pm}^{(1)} = F_{\pm}^{(0)'}. \quad (3.18)$$

Next, we choose $c_0 = 1$, and $F_{\pm}^{(0)} = G_{\pm}(x)$, then

$$g_1 = \frac{G_{\pm}}{r} - G'_{\pm}. \quad (3.19)$$

Also, recall that

$$\tilde{A}_1 = g_1 \sin \theta. \quad (3.20)$$

So, for a dipole wave,

$$A^{\hat{\phi}} = \frac{\tilde{A}}{r} = \left(\frac{G_{\pm}}{r^2} - \frac{G'_{\pm}}{r}\right) \sin \theta. \quad (3.21)$$

Now, we will choose a superposition of incoming and outgoing waves with a Gaussian

$$G_{\pm} = G_- - G_+ = \mathcal{A}\sigma^2(e^{-(r-t)^2/\sigma^2} - e^{-(r+t)^2/\sigma^2}) \quad (3.22)$$

where $\mathcal{A}$ is the dimensionless amplitude of the wave, and σ is a constant with units of length in order to construct a dimensionless $A^{\hat{\phi}}$. We then insert into (3.12) and have our solution for the dipolar wave

$$A^{\hat{\phi}} = \mathcal{A}\sigma^2 \sin \theta \left\{ \frac{e^{-(r-t)^2/\sigma^2} - e^{-(r+t)^2/\sigma^2}}{r^2} + \frac{2(r-t)e^{-(r-t)^2/\sigma^2} - 2(r+t)e^{-(r+t)^2/\sigma^2}}{\sigma^2 r} \right\}. \quad (3.23)$$

More simply, by introducing the dimensionless combinations $u = (r-t)/\sigma$ and $v = (r+t)/\sigma$, we may rewrite this as

$$A^{\hat{\phi}} = \mathcal{A} \sin \theta \left(\frac{e^{-u^2} - e^{-v^2}}{(r/\sigma)^2} + \frac{2ue^{-u^2} - 2ve^{-v^2}}{r/\sigma} \right). \quad (3.24)$$

We can compute the electric field $E^{\hat{\phi}}$ from (3.1) and evaluate the result at the initial time $t = 0$ to obtain

$$E^{\hat{\phi}} = -8\mathcal{A} \frac{r \sin \theta}{\sigma^2} e^{-(r/\sigma)^2} \quad (t = 0). \quad (3.25)$$

In addition, we can compute the magnetic field $\mathbf{B}$ from (3.2). With those two, we can then also compute the energy density ρ , where

$$\rho = \frac{1}{8\pi} (E_i E^i + B_i B^i), \quad (3.26)$$

to obtain

$$\rho = \frac{32\mathcal{A}^2}{9\pi} \frac{t^2(3\sigma^2 - 2t^2)^2}{\sigma^8} e^{-2(t/\sigma)^2} \quad (r = 0). \quad (3.27)$$

From the analytical solutions alone we can note a few things regarding dipole waves. First, the solution (3.24) is symmetric across the equator established by the axisymmetric ansatz thanks to the $\sin \theta$ term. In addition, since $\mathbf{B}$ is non-zero at the

origin, ρ is also non-zero at the origin. In fact, as we will showcase later in our numerical simulations, we encounter the largest densities at the center for dipole waves. We would also like to highlight that we consider only vacuum waves produced by a magnetic dipole. These are similar in structure to the electric dipole, however, in the magnetic case we have $E^{\hat{\phi}}$ and $B^{\hat{\theta}}$, whereas for electric dipoles it is the other way around (see Griffiths (2013) section 11.1).

3.2 Quadrupole Waves

We will now derive the solution for $\ell = 2$. Inserting $\ell = 2$ into (3.11) we have

$$g_2 = \frac{c_0}{r^2} F_{\pm}^{(0)} + \frac{c_1}{r} F_{\pm}^{(1)}. \quad (3.28)$$

We insert $\partial_t^2 g_2$ and $\partial_t^2 g_2$ into (3.10) and find the recursion relation for coefficients. We again notice that c_0 is arbitrary while

$$\begin{aligned} c_1 &= -c_0, \\ c_2 &= -\frac{1}{3}c_1, \end{aligned} \quad (3.29)$$

as well as the relationship between derivatives

$$\begin{aligned} F_{\pm}^{(1)} &= F_{\pm}^{(0)'} \\ F_{\pm}^{(2)} &= F_{\pm}^{(1)'}. \end{aligned} \quad (3.30)$$

If we choose $c_0 = 1$, and $F_{\pm}^{(0)} = G_{\pm}(x)$, then

$$g_2 = \frac{G_{\pm}}{r^2} - \frac{G'_{\pm}}{r} + \frac{G''_{\pm}}{3} \quad (3.31)$$

Also, recall that

$$\tilde{A}_2 = g_2 \sin \theta \cos \theta. \quad (3.32)$$

So, for a quadrupole wave,

$$A^{\hat{\phi}} = \frac{\tilde{A}}{r} = \left(\frac{G_{\pm}}{r^3} - \frac{G'_{\pm}}{r^2} + \frac{G''_{\pm}}{3r} \right) \sin \theta \cos \theta. \quad (3.33)$$

Now, we will choose (3.22) to insert into (3.28) and our solution for the quadrupole wave is

$$A^{\hat{\phi}} = \mathcal{A} \sin \theta \cos \theta \left\{ \frac{e^{-u^2} - e^{-v^2}}{(r/\sigma)^3} + \frac{2ue^{-u^2} - 2ve^{-v^2}}{(r/\sigma)^2} + \frac{4u^2e^{-u^2} - 4v^2e^{-v^2} - 2e^{-u^2} + 2e^{-v^2}}{3r/\sigma} \right\}. \quad (3.34)$$

We can compute the electric field $E^{\hat{\phi}}$ from (3.1) and evaluate the result at the initial time $t = 0$ to obtain

$$E^{\hat{\phi}} = -\frac{16\mathcal{A}}{3} \frac{r^2 \sin \theta \cos \theta}{\sigma^3} e^{-(r/\sigma)^2} \quad (t = 0). \quad (3.35)$$

We can also compute the energy density ρ from (3.26) for quadrupole waves and we note that expanding $E^{\hat{\phi}}$ and $A^{\hat{\phi}}$ around the center we get a quadratic relationship in r . Therefore, the energy density for the quadrupole waves vanishes identically at the center, which is different from dipole waves. This is again consistent with the numerical results we will present later as the maximum densities occur along the symmetry axis but *away* from the center.

In contrast to dipole waves, we note that quadrupole waves are antisymmetric across the equator, as we can see by the extra $\cos \theta$ term in (3.34). Since these symmetries are maintained even when the solutions are coupled to gravity, this finding alone indicates that the critical solution for quadrupole waves cannot be the same as that for dipole

waves. This argument alone demonstrates that the critical solution for the gravitational collapse of electromagnetic waves cannot be unique, at least not globally.

3.3 Octupole Waves

In this section we will now generalize the solution for $\ell = 3$. Although the heart of our work focuses on dipolar and quadrupolar waves, we would like to highlight some qualities of octupolar waves as a reference for possible behavior of higher multipoles. Therefore, it is necessary to derive the solution for these waves as well. We again begin by inserting $\ell = 3$ into (3.11) which gives

$$g_3 = \frac{c_0}{r^3} F_{\pm}^{(0)} + \frac{c_1}{r^2} F_{\pm}^{(1)} + \frac{c_2}{r} F_{\pm}^{(2)} + c_3 F_{\pm}^{(3)}. \quad (3.36)$$

We now insert $\partial_t^2 g_3$ and $\partial_t^2 g_3$ into (3.10) and find the recursion relation for coefficients. Again, we find that c_0 is arbitrary while

$$\begin{aligned} c_1 &= -c_0, \\ c_2 &= -\frac{2}{5}c_1, \\ c_3 &= -\frac{c_2}{6}. \end{aligned} \quad (3.37)$$

The relationship between derivatives gives

$$\begin{aligned} F_{\pm}^{(1)} &= F_{\pm}^{(0)'}, \\ F_{\pm}^{(2)} &= F_{\pm}^{(1)'}, \\ F_{\pm}^{(3)} &= F_{\pm}^{(2)'}. \end{aligned} \quad (3.38)$$

Now, say we choose $c_0 = 1$, and $F_{\pm}^{(0)} = G_{\pm}(x)$, then

$$g_3 = \frac{G_{\pm}}{r^3} - \frac{G'_{\pm}}{r^2} + \frac{2G''_{\pm}}{5r} - \frac{G'''_{\pm}}{15} \quad (3.39)$$

Also, recall that

$$\tilde{A}_3 = g_3 (5 \cos^2 \theta - 1) \sin \theta. \quad (3.40)$$

So,

$$A^{\hat{\phi}} = \frac{\tilde{A}}{r} = \left(\frac{G_{\pm}}{r^4} - \frac{G'_{\pm}}{r^3} + \frac{2G''_{\pm}}{5r^2} - \frac{G'''_{\pm}}{15r} \right) (5 \cos^2 \theta - 1) \sin \theta. \quad (3.41)$$

Now, we will choose (3.22) to insert into (3.36) and have our solution for the octupole wave be

$$\begin{aligned} A^{\hat{\phi}} = \mathcal{A} (5 \cos^2 \theta - 1) \sin \theta & \left\{ \frac{e^{-u^2} - e^{-v^2}}{(r/\sigma)^4} + \frac{2ue^{-u^2} - 2ve^{-v^2}}{(r/\sigma)^3} \right. \\ & + \frac{8u^2e^{-u^2} - 8v^2e^{-v^2} - 4e^{-u^2} + 4e^{-v^2}}{5(r/\sigma)^2} \\ & \left. + \frac{8u^3e^{-u^2} - 8v^3e^{-v^2} - 12ue^{-u^2} + 12ve^{-v^2}}{15r/\sigma} \right\}. \end{aligned} \quad (3.42)$$

As before we compute the electric field from (3.1); evaluating the result for the initial time $t = 0$ yields

$$E^{\hat{\phi}} = -\frac{32\mathcal{A}}{15} \frac{r^3(\cos^2 \theta - 1) \sin \theta}{\sigma^5} e^{-(r/\sigma)^2} \quad (t = 0). \quad (3.43)$$

Expanding the fields about the center shows that A^{ϕ} and E^{ϕ} now scale with r^3 there, and B^{ϕ} now scales with r^2 . Similar to quadrupole waves, the energy density of the fields therefore vanishes at the center, and takes maxima along the axis of symmetry but away from the origin. Now, similar to dipole waves, however, the octupole fields are again *symmetric* across the equator. Therefore, we expect the critical solution for octupole waves to be different from that for both the dipole and the quadrupole waves, supporting our conjecture of non-uniqueness for electromagnetic waves.

Chapter 4

Numerical Evolutions

4.1 Initial Data

We construct initial data that are time symmetric (i.e. $K_{ij} = 0$) and conformally flat (i.e. $\gamma_{ij} = \psi^4 \eta_{ij}$, where ψ is the conformal factor and η_{ij} the flat metric). As our initial data for the electromagnetic fields we adopt expressions that reduce to those of Chap. 3 at $t = 0$ in the limit of weak fields. Specifically, we choose $A^i = 0$ initially, so that $B^i = 0$ also, but E^i is nonzero. This means that the momentum density of the electromagnetic fields vanishes initially, and that the momentum constraint is satisfied identically.

This leaves us with having to solve the Hamiltonian constraint

$$\bar{D}^2 \psi = 2\pi \psi^5 \rho \tag{4.1}$$

only, where $\bar{D}^2$ is the flat Laplace operator and ρ the energy density (3.26). We solve this equation iteratively to obtain nonlinear solutions to Einstein's equations as follows. In order to help with the convergence of this iteration, we adopt as the initial electric fields not the expressions (3.25), (3.35) or (3.43) themselves, but rather those expressions divided by ψ^6 . In practice, we start with an initial guess for ψ , then compute the electric field given our choice of the amplitude $\mathcal{A}$, evaluate the density ρ from (3.26), and

then solve the Hamiltonian constraint (4.1) for a new conformal factor ψ . We repeat the process until convergence to within a desired tolerance has been achieved. In the weak-field limit we have $\psi \rightarrow 1$, so that our numerical solutions approach the analytical solutions of Chap. 3 in this regime.

We also note that in the absence of gravity, electrodynamics becomes linear, which allows us to identify a well-defined multipole moment as described in Chapter 3. However, when we introduce gravity, varying multipole moments will couple to each other through the nonlinearities present in Einstein’s equations. Furthermore, because of the equatorial symmetry in Einstein’s equations, we then expect that modes of even and odd ℓ will be coupled to modes of the respective parity. In the rest of the thesis we will continue referring to “dipole” and “quadrupole waves”, as we expect that the data will be dominated by its corresponding multipole, but we acknowledge that nonlinear coupling introduces other multipoles.

4.2 Evolutions with New Grid

We evolve the initial data using the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism in spherical polar coordinates as described in BGH. One key difference in our approach here from BGH is the allocation of our radial grid points because of where the collapse happens for the different sets of initial data. Using methods delineated in Ruchlin et al. (2018), we know that the radial grid can be constructed by mapping a uniform grid in a variable x , in the interval $[0, 1]$, to our radial variable so that $r = r(x)$, in the interval $[0, r_{max}]$. We then adopt the function

$$r = \frac{r_{max}}{1 + A} \left(\frac{\sinh(s_p x)}{\sinh s_p} + A \frac{\tanh(t_p x)}{\tanh t_p} \right) \quad (4.2)$$

for this map where A , s_p , and t_p are dimensionless parameters. For dipole waves, we have chosen $A = 0$ and $s_p = 6.57$, which results in the same “sinh” grid setup as in BGH. This

grid allows for a high and nearly uniform resolution at the origin, but an increasingly coarse and logarithmic resolution at large separations from the origin. While optimal for the dipole waves, this grid was “wasteful” for higher multipoles where we observed the collapse away from the center. Instead, the additional “tanh” term in (4.2) makes it possible to construct a relatively coarse grid at the origin that can become finer at some distance away, but then also becomes logarithmic at large simulations. Therefore, for our quadrupole waves we chose $A = 0.0015$, $s_p = 6.0$, and $t_p = 50.0$.

We evolve the fields using the “1+log” slicing condition described by Bona et al. (1995). In Baumgarte (2018) and Celestino and Baumgarte (2018) this 1+log slicing resulted in spatial slices that reflect the symmetry in the self-similar critical solutions. In these preferred slices, slicing-dependent quantities take on invariant meanings. Therefore, we will now assume that the energy density ρ (see equation 2.18) provides a suitable diagnostic for our simulations. Furthermore, we will be focusing on subcritical evolutions.

4.3 Convergence Tests

The majority of the numerical calculations performed depend on the pre-existing code used in BGH. Our new main contribution is to generalize the existing EM routine for dipolar waves to also incorporate initial data for quadrupolar waves. To verify that our new numerical code is producing correct results, we perform error analysis through convergence tests in flat spacetimes. We show results of a specific convergence test in Figs. 4.1, 4.2, and 4.3. For these tests, we use the parameters $A = 0.1$, $s_p = 3.0$, and $t_p = 10.0$. We have also chosen snapshots of time at $t = 4.9$ and $\theta = \pi/4$. The low resolution used 64 radial grid points and 12 angular grid points. The medium resolution used 128 radial grid points and 24 angular grid points. Finally, the high resolution used 256 and 48 angular grid points. We begin by showcasing the grid setup for the varying resolutions against the analytical solution for the quadrupole waves given by

equation (3.34) in Fig. 4.1. Since the analytical solution is known in flat spacetimes, we

Figure 4.1: $A^{\hat{\phi}}$ as a function of r at coordinate time $t = 4.9$ and with $\theta = \pi/4$. For the different resolutions, we can compare the distribution of the grid points.

can perform error analysis by computing the difference between our expected analytical result and our numerical result. We define the error to be

$$\Delta A^{\hat{\phi}} = A^{\hat{\phi}}_{analytical} - A^{\hat{\phi}}_{numerical}. \quad (4.3)$$

As hoped, in Fig. 4.2 we see that increasing resolution results in a smaller error. When we write our code, we acknowledge that our methods are only accurate to a certain order. We can see that even between the medium and high resolution, there exists a sizeable difference in the error. This simply indicates that the higher order terms still play a role. However, as $\Delta A^{\hat{\phi}}$ decreases, the higher order terms play a smaller role, and $\Delta A^{\hat{\phi}}$ is ultimately dominated by the leading order term.

Figure 4.2: Plotting $\Delta A^{\hat{\phi}}$ as a function of the coordinate r . Note that the error gets smaller with increasing resolution.

In a convergence test, we not only check that the errors decrease, but we verify that they decrease at the expected rate. Because this is a fourth-order code, if we double the resolution we expect the error to decrease by $2^4 = 16$. If we double the resolution again, we then expect the error to decrease by a factor of $2^8 = 256$. When we take a higher order error term, for example the medium resolution, and we multiply it by the rescaling factor, in this case 16, we can check to see if we obtain the same error as the previous order term. However, in Fig. 4.3 we see that the errors deviate slightly, i.e. they are not identical due to these higher order error terms. As $\Delta A^{\hat{\phi}}$ gets increasingly smaller, for example between the medium and high resolutions, we know that the higher order terms play an increasingly smaller role. This means that we are approaching the leading order term. Therefore, what we really look for in a convergence tests is not the alignment of the lines, but rather that, with increasing resolution, they get closer and closer to each other, converging into a line, which is exactly what we see here.

Figure 4.3: Here we rescale the error for the Medium and High resolutions by 2^4 and 2^8 respectively. We see that the rescaled errors converge towards each other.

Chapter 5

Results

5.1 Minimum Lapse and Maximum Density

The lapse function α indicates the ratio between proper time τ and coordinate time t . For subcritical data, α tends towards unity at late times as the wave disperses and leaves behind flat spacetime. On the other hand, for supercritical data the lapse tends towards zero, indicating the “collapse of the lapse”. While α is a coordinate-dependent quantity, simulations of critical collapse with 1+log slicing by Hilditch et al. (2013), Baumgarte and Montero (2015), and Baumgarte (2018) have shown that a “collapse of the lapse” is indicative of black-hole formation. By fine-tuning the amplitude $\mathcal{A}$ to its critical value $\mathcal{A}_*$ and seeing whether α collapses we can identify the critical solution.

An example of this fine-tuning to critical solution is presented in Fig. 5.1. We plot varying amplitudes fine-tuned to the third digit. Examining the behavior of α , we distinguish subcritical data ($\mathcal{A} \leq 3.53$) from supercritical data ($\mathcal{A} \geq 3.54$). We then “zoom-in” between $\mathcal{A} = 3.53$ and $\mathcal{A} = 3.54$ by adding the next digit, and running the simulations with the new varying amplitudes. We examine again whether α collapses or not, and repeat the procedure until we reach a significant number of digits.

Fig. 5.2 plots the results of the lapse function α as a function of proper time τ for

Figure 5.1: The lapse function α as a function of coordinate time for a quadrupolar wave ($\ell = 2$) evolution fined tuned to 3 digits. The lines that trend towards unity display subcritical data, while the ones that trends towards zero are supercritical, and thus indicate the formation of a black hole. The collapse happens between $\mathcal{A} = 3.53$ and $\mathcal{A} = 3.54$.

pairs of data identifying the critical solutions of dipole and quadrupole waves. Starting here we will refer to proper time as that measured by an observer at the center. From Fig. 5.2 we notice that, for dipole waves, the dark and faded lines overlap for most of the evolution. This indicates that the lapse takes its minimum value at the center, which is consistent with the results from BGH. The quadrupole case, however, is different in that the lapse takes a minimum *away* from the center for most of the evolution, including the “collapse of the lapse” for supercritical evolutions. This difference is a first suggestion that for higher multipole moments, $\ell > 1$, the centers of collapse form away from the center. This phenomenon is similar to the “bifurcations” reported by Choptuik et al. (2003), Hilditch et al. (2017), Baumgarte (2018), and Ledvinka and Khirnov (2021). This result, however, may not be as surprising since we have seen in Section 3.2 and

Section 3.3 that the energy density ρ vanishes at the center.

Figure 5.2: The lapse function α as a function of proper time τ as observed by an observer at the center, for dipole waves ($\ell = 1$) in the top panel and quadrupole waves ($\ell = 2$) in the bottom panel. The dark lines represent the minimum values of the lapse on spatial slices with the same coordinate time as that of the central observer, while the faint lines represent values of the lapse at the center, both for subcritical solutions (the solid red lines) and supercritical solutions (the dashed green lines). Note that, for most of the evolution, the minimum values of the lapse are found at the center for the dipole waves, but away from the center for quadrupole waves. The vertical (orange) lines mark the times of the snapshots shown in Figs. 5.4, 5.5, and 5.6.

Further comparing dipole and quadrupole data in Fig. 5.2, we can see that fine-tuning dipole data to 11 digits results in very short oscillation periods in the late evolution time. This indicates that the evolution follows the critical solution relatively close to the accumulation event. In contrast, we see that for quadrupole data, when fine-tuning to the same number of digits, the oscillation periods are not quite as short, meaning that the critical solution remains farther away from the accumulation event. Because of this, we can make a quite accurate estimate of the proper time of the accumulation for the dipole waves, $\tau_{\star}^{\text{dip}} \simeq 5.66$, but we are left with a crude estimate for the quadrupole waves,

$$\tau_{\star}^{\text{quad}} \simeq 29.5.$$

Figure 5.3: The density ρ (see Eq. 3.26) as a function of the “slow time” (2.9) for the subcritical solutions shown in Fig. 5.2. We show results for dipole waves ($\ell = 1$) in the top panel and quadrupole waves ($\ell = 2$) in the bottom panel. For dipole data we have included both the maximum values on a given slice of constant coordinate time (the dark lines) and values at the center (the faint lines) while, for quadrupole waves, we have included the former only, since the density vanishes identically at the center (see the discussion in Section 3.2). The dotted (blue) lines show the exponential growth e^{2T} expected for the density in a self-similar contraction, while the vertical (orange) lines indicate the times of the snapshots shown in Figs. 5.4, 5.5, and 5.6.

In Fig. 5.3 we plot the energy density ρ as a function of the “slow time”

$$T \equiv -\log(\tau^* - \tau) + T_0. \quad (5.1)$$

Again, τ is the proper time of an observer at the origin, but now, in comparison with (2.9), we have chosen the arbitrary offset T_0 to vanish for dipole data and $T_0 = 2$ for quadrupole data. We would like to note that there exists some ambiguity in how best to define T for the quadrupolar case, i.e. when the centers of collapse are not at

the origin. An alternative to considering the proper time of an observer at the origin would be to instead consider an observer whose worldline passes through those centers, as discussed in Ledvinka and Khirnov (2021). The blue dotted lines in Fig. 5.3 represent curves proportional to the expected growth rate of the density, (2.19), in a contracting self-similar solution.

As previously observed by BGH, we can see that the dipolar evolutions are consistent with an approximate DSS solution. While we do not see strict periodicity, the maxima grow at the expected rate. Therefore, we can identify a dominant periodicity and thus compute an approximate echoing period of $\Delta^{dip} \simeq 0.55$. For quadrupolar data, we again notice that there is an overall consistent growth rate, and while there is no strict periodicity, we can also estimate an approximate echoing period of $\Delta^{quad} \simeq 0.3$. Despite of this absence of strict periodicity, as well as ambiguities of slow time, our findings suggest that the echoing periods differ.

5.2 Profiles

In the next few figures we will compare characteristic functions for near critical evolutions of dipole and quadrupole data. In the panels, the dipole data is presented on the left while the quadrupole data is presented on the right. We begin by showing the quantity

$$A_\xi \equiv \frac{\xi^a A_a}{(\xi^a \xi_a)^{1/2}} = \frac{A_\phi}{g_{\phi\phi}}, \quad (5.2)$$

which is formed from the vector potential A_a and the Killing vector generating axisymmetry, ξ^a , and represents a gauge invariant measure of $A^{\hat{\phi}}$. We can see from Fig. 5.4 the symmetries discussed in Sections 3.1 and 3.2. For instance, we can note that dipole data are symmetric across the equator, represented as the x-axis, while quadrupole data are antisymmetric. In addition, we can note that, for quadrupole data, A_ξ vanishes on both the equator and the symmetry axis (represented as the z-axis). Next, in Fig. 5.5 we

display profiles for the lapse function α . In agreement with Fig. 5.2, we can see that, in the left column, α takes a minimum value at the center for dipole data while, in the right column, it takes a minimum away from the center for quadrupole data. We can also note that the increase in sharpness of the minima over time is consistent with a self-similar contraction.

Lastly, we compare profiles of the energy density ρ in Fig. 5.6. Again, we notice that for dipole data, the maximum in ρ occurs at the center, while for quadrupole data the maximum occurs away from the center but along the symmetry axis. Also, we notice that the peaks become sharper as time advances, which agrees with what we would expect for the growth of ρ . We would also like to acknowledge numerical limitations in our code. Spherical polar coordinates in our code are optimal to resolve the density peaks when they occur in the center, i.e. for dipole data. However, this is not the case for the peaks away from the center encountered in quadrupole data. Therefore, the numerical resolution becomes poorer when trying to resolve the peaks for quadrupole data.

Figure 5.4: Snapshots of A_ξ (see Eq. 5.2) for a near-critical evolution at the instants marked by the solid vertical lines in Figs. 5.2 and 5.3. We show results for dipole data in the left column, and quadrupole data in the right column. Note that the dipole data are symmetric across the equator, while the quadrupole data are antisymmetric (see also the discussion in Sections 3.1 and 3.2).

Figure 5.5: Same as Fig. 5.4, but for the lapse function α .

Figure 5.6: Same as Figs. 5.4 and 5.5, but for the energy density ρ .

5.3 Scaling

We can record the maximum densities encountered over all time and plot them as a function of the amplitudes, as done in Fig. 5.7. The amplitudes are fine tuned to the critical value up to 11 digits on our numerical grid. We have adopted the approximate critical values of $\mathcal{A}_\star^{dip} \simeq 0.91295765109$ and $\mathcal{A}_\star^{quad} \simeq 3.533437407467$. We also included

the dotted lines for the expected power-law scaling (2.18). Here we can also notice that the slopes of both lines appear different, and we have introduced the fitted values $\gamma^{dip} = 0.145$ and $\gamma^{quad} = 0.11$. If we were to expect a strictly DSS solution, in Fig. 5.7 we would expect periodic “wiggles” superimposed on the scaling. However, we see that there is no strict periodicity in the wiggle. This is similar to the findings of Ledvinka and Khirnov (2021) for gravitational waves where different sets of initial data yield different critical exponents.

Figure 5.7: The maximum densities encountered for dipole (top panel) and quadrupole waves (bottom panel) as a function of $\mathcal{A}_* - \mathcal{A}$. The dotted lines are fits $\rho_{\max} \simeq (\mathcal{A}_* - \mathcal{A})^{2\gamma}$ with $\gamma^{dip} = 0.145$ for the dipole waves (see BGH) and $\gamma^{quad} = 0.11$ for the quadrupole waves.

5.4 Uniqueness of the Critical Solution

As discussed in previous sections, it is clear that the solution cannot be unique globally. However, there is the possibility that perhaps the solution is unique locally. For instance, one could argue that, while we encounter globally distinct characteristics for the quadrupole solution, perhaps each individual center of collapse could behave like that for the dipole data. Calling the quadrupolar solution, with two centers of collapse, a “bifurcation” might imply that in fact these new off-centered collapses are nothing more than displaced “copies” of the dipole center of collapse. However, we believe that the evidence shows that these quadrupolar centers exhibit their own distinct properties. For instance, the difference in echoing periods discussed in Sec. 5.1 suggest that $\Delta^{dip} > \Delta^{quad}$. This might agree with our intuition regarding higher-order modes and their respective oscillations.

In a similar way, if the quadrupole centers of collapse were indeed “copies” of the dipole centers, we would also expect to see the same critical exponent γ . Yet, Fig. 5.7 suggests that $\gamma^{dip} \neq \gamma^{quad}$. Together, these findings suggest that the two centers of collapse found for quadrupole waves might be features of a global critical solution for quadrupole waves, rather than two distinct local copies of the dipole critical solution.

Chapter 6

Conclusion

In this thesis we compared critical phenomena in the gravitational collapse of dipole and quadrupole EM waves in hopes of better understanding these critical phenomena in the absence of a spherically symmetric critical solution. Generalizing the findings of BGH, we derived analytical solutions for dipole and quadrupole EM waves and utilized those as initial data for numerical evolutions. When exploring how this choice of initial data affects the critical solution, we found that the critical solution is not unique, at least not globally. For instance, dipole and quadrupole waves have different symmetries in the analytical solution, already indicating non-uniqueness. Furthermore, when we evolved our data we discovered that, in contrast to dipole waves, quadrupole waves collapse away from the center. In addition we present some evidence that suggest that the critical solutions differ even locally. Specifically, we highlight that the difference in echoing periods as well as the critical exponents denote unique properties in the critical solution for the respective type of wave. We ultimately believe this non-uniqueness in the critical solution is related to absence of spherically symmetric critical solution. Therefore, we believe our findings also apply to the critical collapse of gravitational waves, where there too exists this absence of a spherically symmetric critical solution.

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¡Sí se pudo!

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