

Bowdoin College

Bowdoin Digital Commons

Physics Faculty Publications

Faculty Scholarship and Creative Work

1-1-1992

Quantum kinks: Solitons at strong coupling

Stephen G. Naculich

Johns Hopkins University

Follow this and additional works at: <https://digitalcommons.bowdoin.edu/physics-faculty-publications>

Recommended Citation

Naculich, Stephen G., "Quantum kinks: Solitons at strong coupling" (1992). *Physics Faculty Publications*. 162.

<https://digitalcommons.bowdoin.edu/physics-faculty-publications/162>

This Article is brought to you for free and open access by the Faculty Scholarship and Creative Work at Bowdoin Digital Commons. It has been accepted for inclusion in Physics Faculty Publications by an authorized administrator of Bowdoin Digital Commons. For more information, please contact mdoyle@bowdoin.edu, a.sauer@bowdoin.edu.

Quantum kinks: Solitons at strong coupling

Stephen G. Naculich*

Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218

(Received 27 July 1992)

We examine solitons in theories with heavy fermions. These “quantum” solitons differ dramatically from semiclassical (perturbative) solitons because fermion loop effects are important when the Yukawa coupling is strong. We focus on kinks in a $(1+1)$ -dimensional ϕ^4 theory coupled to fermions; a large- N expansion is employed to treat the Yukawa coupling g nonperturbatively. A local expression for the fermion vacuum energy is derived using the WKB approximation for the Dirac eigenvalues. We find that fermion loop corrections increase the energy of the kink and (for large g) decrease its size. For large g , the energy of the quantum kink is proportional to g , and its size scales as $1/g$, unlike the classical kink; we argue that these features are generic to quantum solitons in theories with strong Yukawa couplings. We also discuss the possible instability of fermions to solitons.

PACS number(s): 11.10.Lm, 11.15.Kc, 11.15.Pg

I. INTRODUCTION

Topological solitons, despite their inherently nonperturbative character, are typically studied semiclassically, that is, in a perturbative expansion in the coupling constants [1]. The first term in this expansion, the classical soliton, is the solution to a nonlinear classical field equation. This solution is nonperturbative because its energy diverges as the coupling constants, which parametrize the nonlinearity, vanish. Perturbative corrections to the soliton are important: they split the degeneracies of the classical solution resulting from Poincaré and internal symmetries, and project the solitons onto eigenstates of momentum, angular momentum, and charge. If the coupling constants are small, however, corrections to the shape and energy of the soliton are small, and the classical description of the soliton is essentially accurate.

If the couplings are large, on the other hand, there is no reason to expect the quantum soliton states to resemble the classical solitons, at least quantitatively. In general, the strong coupling behavior of solitons in a quantum field theory is not well known. One notable exception is the sine-Gordon kink in $1+1$ dimensions; because of the equivalence of the sine-Gordon theory to the massive Thirring model [2], the sine-Gordon kink at strong coupling becomes a weakly coupled fermion in the Thirring model, which is well described by perturbation theory.

In this paper we study strongly coupled solitons more generally, when such a fortuitous equivalence does not arise. We focus in particular on solitons in theories with large Yukawa couplings. One motivation for doing so is the following. Fermions can acquire mass through a Yukawa coupling to a scalar field with nonvanishing vacuum expectation value. Solitons in such theories often carry (possibly fractional) fermion number. It has recurrent-

ly been suggested that when the Yukawa coupling is large such a soliton may have less energy than a fermion in a constant scalar field background; consequently, fermions may be unstable to the formation of solitons [3–11]. To determine whether this is so, however, one must know the form and energy of solitons in a strongly coupled theory, which may differ appreciably from classical solitons. Indeed, we expect fermion loop corrections to significantly affect the solitons when the Yukawa coupling is large.

One means of studying a strongly coupled Yukawa theory is through a large- N expansion [5,10–13]. To leading order in $1/N$, the theory can be solved for arbitrary values of the Yukawa coupling. This expansion captures some of the strong-coupling behavior of the theory, which one hopes is representative even when N is not large. To carry out this expansion, we introduce N fermion flavors and choose the N dependence of the couplings so that the theory has a sensible $N \rightarrow \infty$ limit, with only fermion loops contributing to Green's functions to leading order in $1/N$. The total contribution of the fermion loops can be summed in closed form to give the exact large- N effective action

$$S_{\text{eff}}[\phi] = S[\phi] - iN \ln \det(i\mathcal{D}), \quad (1.1)$$

where $S[\phi]$ is the classical scalar field action and \mathcal{D} is the Dirac operator in the presence of the field ϕ .

Solitons in this large- N theory are c -number configurations of the scalar fields; scalar field fluctuations are suppressed because scalar loops do not contribute to the effective action to leading order in $1/N$. The shape of the large- N soliton differs from the classical soliton, however, since it extremizes not the classical action but the effective action (1.1). The fermion loop contribution significantly alters the form of the soliton when the Yukawa coupling is large. In this regime, where quantum effects are so important, the large- N soliton is truly a “quantum soliton.”

To determine the form of the quantum soliton, we need to know $-iN \ln \det(i\mathcal{D})$ explicitly for an arbitrary scalar field configuration. One generally resorts to some local

*Electronic address: naculich@casa.pha.jhu.edu

approximation, such as the gradient expansion [5,14], accurate for slowly varying configurations. The gradient expansion, however, breaks down for topological solitons in the theories that we are considering. Another approach to computing the fermion loop contribution relies on the fact that for static solitons $(iN/T) \ln \det(i\mathcal{D})$ is just the energy of the “Dirac sea,” the sum of negative eigenvalues of the Dirac equation in the soliton background [15]. Unfortunately, the Dirac eigenvalues must be numerically computed [16] for each separate background considered, rendering this approach inconvenient for a variational problem.

In this paper, we propose a hybrid of the gradient expansion and eigenvalue sum methods. Following an idea of Wasson and Koonin [17], we use the WKB approximation to estimate the Dirac sea eigenvalues for an arbitrary static scalar field background. We then sum these to obtain a local expression for the fermion vacuum energy. Unlike the gradient expansion, this expression is finite for topologically nontrivial configurations. Using this WKB approximation, we extremize the effective action to find the form of the quantum soliton in the large- N theory.

We illustrate this method on a well-known example, the kink of the $(1+1)$ -dimensional ϕ^4 theory coupled to fermions. The classical kink is reviewed in Sec. II. In Sec. III, we derive the WKB approximation for the large- N effective action in this theory. This result is used in Sec. IV to find the form of the quantum kink, which is contrasted with the classical kink. The question of fermion stability is also discussed. In Sec. V, we present our conclusions and discuss the features of the model that we expect are generic to strongly coupled solitons.

II. CLASSICAL KINKS

We begin by recalling the form and quantum numbers of the classical kink [1]. The $(1+1)$ -dimensional ϕ^4 theory coupled to N flavors of fermion has the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4N}(\phi^2 - Nv^2)^2 + \sum_{i=1}^N \bar{\psi}^i \left[i\mathcal{D} - \frac{g}{\sqrt{N}} \phi \right] \psi^i. \quad (2.1)$$

The N dependence of the parameters has been chosen so that this theory has a sensible $N \rightarrow \infty$ limit. If we rewrite ϕ as $\sqrt{N}\varphi$, the parameter N becomes an overall scale:

$$\mathcal{L} = N \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{4}\lambda(\varphi^2 - v^2)^2 \right] + \sum_{i=1}^N \bar{\psi}^i (i\mathcal{D} - g\varphi) \psi^i. \quad (2.2)$$

In the vacuum state $|\varphi|=v$, the scalar field has mass $\sqrt{2\lambda}v$ and the fermion field mass gv . In two dimensions, v is dimensionless, the scalar self-coupling λ has dimension 2, and the Yukawa coupling g dimension 1. It is convenient to substitute for λ and g the parameters

$$x_{\text{cl}} = \left[\frac{2}{\lambda v^2} \right]^{1/2}, \quad y = g \left[\frac{2}{\lambda} \right]^{1/2}. \quad (2.3)$$

The parameter x_{cl} is proportional to the scalar field Compton wavelength (and, as we will see, the size of the classical kink), and will serve as the overall scale of length and energy in the theory. There are two dimensionless parameters v and y , the latter being proportional to the ratio of fermion and scalar masses.

The Lagrangian (2.2) gives rise to the field equations

$$\partial^2 \varphi + \lambda \varphi^3 - \lambda v^2 \varphi = -g \frac{1}{N} \sum_{i=1}^N \bar{\psi}^i \psi^i, \quad (2.4)$$

$$(i\mathcal{D} - g\varphi)\psi^i = 0. \quad (2.5)$$

The topologically nontrivial solutions of these equations give a “classical” description of the soliton states in the Hilbert space, which is accurate when the quantum corrections are small. If we neglect the fermion source term, the scalar field equation (2.4) has the well-known static kink solution

$$\varphi_{\text{cl}}(x) = v \tanh \left[\frac{x}{x_{\text{cl}}} \right], \quad (2.6)$$

which is the lowest energy state with topological charge $[\varphi(\infty) - \varphi(-\infty)]/2v = 1$. There is also an antikink solution which interpolates from v to $-v$, with topological charge -1 . The Dirac equation (2.5) in the kink background (2.6) has a self-conjugate zero-mode solution

$$\psi_0(x) = \begin{bmatrix} [\text{sech}(x/x_{\text{cl}})]^y \\ 0 \end{bmatrix}, \quad \begin{matrix} \gamma^0 = \sigma_1, \\ \gamma^1 = i\sigma_3. \end{matrix} \quad (2.7)$$

The state with the zero mode occupied has the same energy as that with the zero mode unoccupied. Since there is a zero mode for each flavor i , the kink is 2^N -fold degenerate. If n of the zero modes are occupied, the kink has fermion number $n - \frac{1}{2}N$, which ranges from $-\frac{1}{2}N$ to $\frac{1}{2}N$ [18]. The antikink too has degeneracy 2^N , and fermion number ranging from $-\frac{1}{2}N$ to $\frac{1}{2}N$. Although the fermion zero modes increase the degeneracy of the kink, their contribution to the source term in the scalar field equation (2.4) vanishes, so the kink (2.6) remains a solution even in the presence of fermions. The energy of the classical kink

$$E[\varphi_{\text{cl}}] = \frac{2\sqrt{2}}{3} N \sqrt{\lambda} v^3 = \frac{4}{3} v^2 \left[\frac{N}{x_{\text{cl}}} \right] \quad (2.8)$$

has no dependence on the Yukawa coupling g , even though the kink carries fermion number due to the zero modes.

This classical picture leads to the fascinating possibility that, even if fermion number is conserved, “ordinary” fermions may be unstable to the formation of solitons carrying fermion number. A configuration consisting of a widely separated kink and antikink, each carrying fermion number $\frac{1}{2}N$, has zero topological charge, fermion number N , and energy $\frac{8}{3}v^2(N/x_{\text{cl}})$. On the other hand, a set of N widely separated fermions in the vacuum background $\varphi=v$, a state which has the same quantum numbers, has energy $Ngv = y(N/x_{\text{cl}})$. Thus, when $y > \frac{8}{3}v^2$, it is energetically favorable for a state of N fermions to

coalesce onto a spontaneously created kink-antikink pair. Each kink acts as a kind of bound state of $\frac{1}{2}N$ fermions. Even more surprising, in a theory with one flavor of fermion ($N=1$), a single fermion could split into a kink-antikink pair, each with fermion number $\frac{1}{2}$.

This putative instability occurs only when y is large, however, where the quantum corrections from fermion loops are important and the semiclassical approximation breaks down. To determine whether fermions are truly unstable, one must compare their energy not with that of a classical kink, but of a “quantum kink,” which includes the effects of quantum corrections. The quantum kink extremizes not the action but rather the effective action. In the next section we will derive a local expression for the effective action suitable for finding the quantum kink.

III. EFFECTIVE ACTION FOR KINKS

Quantum solitons are field configurations that extremize the effective action, which includes quantum corrections. To find the form of quantum solitons, one needs an explicit local expression for the effective action. The familiar gradient expansion, however, diverges for topologically nontrivial configurations in $(1+1)$ -dimensional ϕ^4 theory. In this section, we derive an alternative local approximation for the effective action that is finite for kinks.

Since we are interested in the properties of solitons for large Yukawa coupling g , the effective action must be calculated nonperturbatively in g . This can be done by taking the number N of fermion flavors to be large, holding λ , v , and g fixed, and calculating to leading order in $1/N$. Scalar field fluctuations are subleading in $1/N$, so only fermion loops contribute to the large- N effective action

$$\begin{aligned} S_{\text{eff}}[\varphi] &= \int d^2x \mathcal{L}_{\text{eff}}(\varphi) \\ &= N \int d^2x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{4}\lambda(\varphi^2 - v^2)^2 \right] \\ &\quad - iN \ln \det(i\partial - g\varphi) \\ &\quad + \int d^2x \delta\mathcal{L}(\varphi) + iN \ln \det(i\partial - gv). \end{aligned} \quad (3.1)$$

We have added the counterterm

$$\delta\mathcal{L}(\varphi) = AN(\varphi^2 - v^2) \quad (3.2)$$

to tame the divergent contributions of the fermion determinant to the one- and two-point functions, and the overall constant $iN \ln \det(i\partial - gv)$ to ensure that $S_{\text{eff}}[\varphi=v]=0$. The coefficient A is fixed by requiring the one-point function to vanish at $\varphi=v$,

$$0 = 2Av + ig \int \frac{d^2p}{(2\pi)^2} \text{tr} \left[\frac{i}{\not{p} - gv} \right], \quad (3.3)$$

so that v remains the minimum of the effective potential. With a cutoff Λ on the spatial momentum p_1 , Eq. (3.3) gives

$$\delta\mathcal{L}(\varphi) = -\frac{Ng^2}{2\pi}(\varphi^2 - v^2) \int_0^\Lambda \frac{dp_1}{\sqrt{p_1^2 + g^2v^2}}. \quad (3.4)$$

This counterterm also renders finite the two-point function

$$\Gamma_{\sigma\sigma}^{(2)}(p) \Big|_{p=0} = -2\lambda v^2 - \frac{g^2}{\pi}, \quad (3.5)$$

where $\sigma = \varphi - v$. Fermion loop contributions to all other Green's functions are finite.

We must write the effective action (3.1) in a more tractable form if we are to find the quantum kink explicitly. The gradient expansion [5,14]

$$\mathcal{L}_{\text{eff}}(\varphi) = -V_{\text{eff}}(\varphi) + \mathcal{L}_{\text{eff}}^{(2)}(\varphi) + \dots \quad (3.6)$$

is a useful approximation for slowly varying fields. The first term in this expansion is minus the effective potential:

$$\frac{V_{\text{eff}}(\varphi)}{N} = \frac{\lambda}{4}(\varphi^2 - v^2)^2 + \frac{g^2}{4\pi}\varphi^2 \ln \left[\frac{\varphi^2}{v^2} \right] - \frac{g^2}{4\pi}(\varphi^2 - v^2). \quad (3.7)$$

The term with two derivatives is

$$\frac{\mathcal{L}_{\text{eff}}^{(2)}(\varphi)}{N} = \frac{1}{2} \left[1 + \frac{1}{12\pi\varphi^2} \right] (\partial_\mu \varphi)^2. \quad (3.8)$$

At this point, we discover that the gradient expansion fails for topological solitons in this theory; any configuration $\varphi(x)$ with unit topological charge must pass through $\varphi=0$ somewhere, at which point $\mathcal{L}_{\text{eff}}^{(2)}(\varphi)$, as well as higher-order terms, diverges. This failure is quite general. For the gradient expansion to converge, field gradients must be small relative to $g\varphi$, the “local fermion mass.” Since the latter vanishes at the core of solitons with fermion zero modes, the gradient expansion necessarily breaks down there, no matter how slowly varying the field.

An alternative approach for a static scalar field background such as the kink is to express the effective action in terms of Dirac equation eigenvalues [15,4]. For time-independent $\varphi(x)$, the effective action equals $-E_{\text{eff}}[\varphi]T$, where $T = \int dt$ and $E_{\text{eff}}[\varphi]$ is the energy of the configuration,

$$E_{\text{eff}}[\varphi] = E_{\text{cl}}[\varphi] + Q[\varphi], \quad (3.9)$$

a sum of the classical energy,

$$E_{\text{cl}}[\varphi] = \frac{N}{x_{\text{cl}}} \int_{-\infty}^{\infty} dz \left[\frac{1}{2} \left(\frac{d\varphi}{dz} \right)^2 + \frac{1}{2v^2}(\varphi^2 - v^2)^2 \right], \quad (3.10)$$

$z = \frac{x}{x_{\text{cl}}}$,

and the quantum correction, the fermion vacuum energy,

$$Q[\varphi] = \frac{iN}{T} \ln \det(i\partial - g\varphi) - \frac{iN}{T} \ln \det(i\partial - gv) + \delta E[\varphi]. \quad (3.11)$$

The first term in Eq. (3.11) can be interpreted as the energy of the Dirac sea in the background $\varphi(x)$.

To write Eq. (3.11) more explicitly, we observe that the

Dirac equation (2.5) implies that the spinor components

$$\psi^i = \begin{pmatrix} \psi_+^i \\ \psi_-^i \end{pmatrix}$$

obey the Schrödinger-type equations

$$\left[\frac{d^2}{dz^2} - y^2 V_{\pm}(z) - y^2 + x_{\text{cl}}^2 \epsilon_{\pm}^2 \right] \psi_{\pm} = 0 \quad (3.12)$$

in a static background $\varphi(z)$, where

$$V_{\pm}(z) = \left[\frac{\varphi^2}{v^2} - 1 \right] \mp \frac{1}{yv} \frac{d\varphi}{dz}. \quad (3.13)$$

We restrict $\varphi(z)$ to configurations of unit topological charge that obey $\varphi(-z) = -\varphi(z)$; the Schrödinger potentials $V_{\sigma}(z)$ are then even, and the solutions $\psi_{\sigma}(z)$ can be taken to be parity eigenstates. (Here $\sigma = \pm$ labels the upper and lower spinor components.) Since $|\varphi(\pm\infty)| = v$, the potentials $V_{\sigma}(z)$ vanish at $\pm\infty$, so Eq. (3.12) has a continuous spectrum of states classified by their asymptotic momentum, $k = \sqrt{x_{\text{cl}}^2 \epsilon^2 - y^2}$, and their parity. The asymptotic forms of the continuum wave functions

$$\begin{aligned} \psi_{\sigma, \text{even}}(k, z) &\sim \cos[kz \pm \tfrac{1}{2} \delta_{\sigma, \text{even}}(k)], \\ \psi_{\sigma, \text{odd}}(k, z) &\sim \sin[kz \pm \tfrac{1}{2} \delta_{\sigma, \text{odd}}(k)], \end{aligned} \quad (3.14)$$

serve to define the phase shifts $\delta_{\sigma, \text{even}}(k)$ [$\delta_{\sigma, \text{odd}}(k)$] for the even- (odd-) parity states. If we put the system into a box, $|z| \leq \frac{1}{2}L$, with periodic boundary conditions, Eq. (3.14) implies that the allowed momenta satisfy $k_{\sigma n} L + \delta_{\sigma}(k) = 2\pi n$. Equation (3.12) may also have a series of discrete bound states with eigenvalues $\epsilon_{\sigma i}^2 < y^2/x_{\text{cl}}^2$. Because the configuration has topological charge 1, the upper spinor component is guaranteed [18] to have a zero mode, $\epsilon_+ = 0$. Thus, any configuration with unit topological charge can carry fermion quantum numbers.

The difference of fermion loop contributions can be written as the shift of the Dirac sea energy [15]:

$$\begin{aligned} \frac{iN}{T} \ln \det(i\partial - g\varphi) - \frac{iN}{T} \ln \det(i\partial - gv) \\ = -\frac{1}{2} N \sum_{\sigma} \sum_{\lambda} \left[\epsilon_{\sigma\lambda} - \epsilon_{\sigma\lambda}^{(0)} \right], \end{aligned} \quad (3.15)$$

where $\epsilon_{\sigma\lambda}$ denotes the positive root of $\epsilon_{\sigma\lambda}^2$, and $\epsilon_{\sigma\lambda}^{(0)}$ are the Dirac eigenvalues in the constant configuration $\varphi(x) = v$. Equation (3.15) may be separated into the sum over discrete eigenvalues,

$$E_{\text{disc}}[\varphi] = -\frac{1}{2} N \sum_{\sigma} \sum_i \left[\epsilon_{\sigma i} - \frac{y}{x_{\text{cl}}} \right], \quad (3.16)$$

and the sum over continuum eigenvalues,

$$\begin{aligned} E_{\text{cont}}[\varphi] = -\frac{1}{2} N \sum_{\sigma} \sum_{n>0} \sum_{\text{parity}} \left[\epsilon(k_{\sigma n}) - \epsilon(k_{\sigma n}^{(0)}) \right], \\ \epsilon(k) = \frac{\sqrt{k^2 + y^2}}{x_{\text{cl}}}. \end{aligned} \quad (3.17)$$

Using $k_{\sigma n} L + \delta_{\sigma}(k) = k_{\sigma n}^{(0)} L = 2\pi n$, and letting $L \rightarrow \infty$, we can write Eq. (3.17) as [15]

$$\begin{aligned} E_{\text{cont}}[\varphi] &= N \sum_{\sigma} \int_0^{\Lambda} \frac{dk}{4\pi} \frac{d\epsilon}{dk} [\delta_{\sigma, \text{even}}(k) + \delta_{\sigma, \text{odd}}(k)] \\ &= N \sum_{\sigma} \int_0^{\Lambda} \frac{dk}{2\pi} \frac{d\epsilon}{dk} \delta_{\sigma}(k), \end{aligned} \quad (3.18)$$

where $\delta = \frac{1}{2}(\delta_{\text{even}} + \delta_{\text{odd}})$. The integral over k diverges as the momentum cutoff Λ is removed, but this divergence is canceled by the counterterm energy:

$$\begin{aligned} \delta E[\varphi] &= - \int_{-\infty}^{\infty} dz \delta \mathcal{L}(\varphi) \\ &= \frac{y^2}{2\pi} \left[\frac{N}{x_{\text{cl}}} \right] \int_0^{\Lambda} \frac{dk}{\sqrt{k^2 + y^2}} \int_{-\infty}^{\infty} dz \left[\frac{\varphi^2}{v^2} - 1 \right]. \end{aligned} \quad (3.19)$$

The sum of Eqs. (3.16), (3.18), and (3.19),

$$Q[\varphi] = E_{\text{disc}}[\varphi] + E_{\text{cont}}[\varphi] + \delta E[\varphi] \quad (3.20)$$

is precisely the fermion vacuum energy (3.11).

The expression (3.20) for the fermion vacuum energy is much more explicit than Eq. (3.1), and can even be computed analytically for certain scalar field configurations [19]. For an arbitrary background, however, $\epsilon_{\sigma i}$ and $\delta_{\sigma}(k)$ must be computed numerically [16]. Wasson and Koonin [17] showed how to speed up the convergence of these “brute force” numerical calculations by employing the WKB approximation for the high momentum phase shifts, but the discrete eigenvalues and low momentum phase shifts must still be computed numerically for each separate field configuration. Thus, Eq. (3.20) is still not very convenient for extremizing the effective action.¹

Taking our cue from Ref. [17], we adopt the WKB approximation for all the Dirac eigenvalues, both continuous and discrete, and use them in Eq. (3.20) to obtain a local expression for the energy of an arbitrary scalar field configuration. The resulting expression will be accurate for field configurations slowly varying on the scale of the fermion Compton wavelength, but unlike the gradient expansion, does not diverge for solitons. We will then use this approximation expression to find the form of the quantum kink in Sec. IV.

In the WKB approximation, the continuum eigenfunctions of Eq. (3.12) are

¹Campbell and Liao [4] were able to extremize (3.9) using powerful inverse scattering methods, but only for the special case $y = 1$.

$$\begin{aligned}\psi_{\sigma,\text{even}}^{\text{WKB}}(k,z) &= \frac{1}{\sqrt{k_\sigma(z)}} \cos \left[\int_0^z k_\sigma(z') dz' \right], \\ \psi_{\sigma,\text{odd}}^{\text{WKB}}(k,z) &= \frac{1}{\sqrt{k_\sigma(z)}} \sin \left[\int_0^z k_\sigma(z') dz' \right], \\ k_\sigma(z) &= \sqrt{k^2 - y^2 V_\sigma(z)},\end{aligned}\quad (3.21)$$

whence the phase shift defined through Eq. (3.14) is given

by

$$\delta_\sigma^{\text{WKB}}(k) = \int_{-\infty}^{\infty} dz [k_\sigma(z) - k], \quad (3.22)$$

independent of parity. [We assume $V_\sigma(z) \leq 0$ everywhere; this will be true if $\varphi(z)$ does not vary too rapidly.] Using the WKB phase shifts (3.22) in the integral (3.18) and adding the counterterm energy (3.19), we find

$$\begin{aligned}E_{\text{cont}}^{\text{WKB}}[\varphi] + \delta E[\varphi] &= \frac{y^2}{4\pi} \left[\frac{N}{x_{\text{cl}}} \right] \int_{-\infty}^{\infty} dz \left[\left[1 - \frac{\varphi^2}{v^2} \right] - \left[\sqrt{-V_+} + \sqrt{-V_-} \right] \right. \\ &\quad \left. + (1+V_+) \ln \left[1 + \sqrt{-V_+} \right] + (1+V_-) \ln \left[1 + \sqrt{-V_-} \right] \right].\end{aligned}\quad (3.23)$$

We also need to approximate the sum over discrete eigenvalues (3.16). In the WKB approximation, the Schrödinger equation (3.12) has discrete eigenvalues ϵ whenever $w_\sigma(\epsilon)$, defined by

$$w_\sigma(\epsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz k_\sigma(z) \Theta(k_\sigma^2(z)), \quad k_\sigma(z) = \sqrt{x_{\text{cl}}^2 \epsilon^2 - y^2 - y^2 V_\sigma(z)}, \quad (3.24)$$

equals half an odd integer, $w \in \mathbb{Z} + \frac{1}{2}$. The number of discrete eigenstates is given by the integer closest to $w_\sigma(y/x_{\text{cl}})$. We define $\epsilon_\sigma(w)$ by inverting Eq. (3.24) and setting $\epsilon_\sigma(w) = 0$ for $0 \leq w \leq w_\sigma(0)$. The sum over discrete eigenvalues (3.16) in the WKB approximation is then written

$$E_{\text{disc}}^{\text{WKB}}[\varphi] = -\frac{1}{2} \sum_{\sigma} \sum_{\substack{w \in \mathbb{Z} + 1/2 \\ 0 < w \leq w_\sigma(y/x_{\text{cl}})}} \left[\epsilon_\sigma(w) - \frac{y}{x_{\text{cl}}} \right]. \quad (3.25)$$

We separate this into two terms,

$$E_{\text{disc}}^{\text{WKB}}[\varphi] = E_{\text{disc}}^{(1)}[\varphi] + E_{\text{disc}}^{(2)}[\varphi], \quad (3.26)$$

where $E_{\text{disc}}^{(1)}[\varphi]$ is the integral approximation of the sum (3.25),

$$E_{\text{disc}}^{(1)}[\varphi] = -\frac{1}{2} \sum_{\sigma} \int_0^{w_\sigma(y/x_{\text{cl}})} dw \left[\epsilon_\sigma(w) - \frac{y}{x_{\text{cl}}} \right], \quad (3.27)$$

and $E_{\text{disc}}^{(2)}[\varphi]$ is the remainder. The integral (3.27) may be rewritten

$$\begin{aligned}E_{\text{disc}}^{(1)}[\varphi] &= -\frac{1}{2} \sum_{\sigma} \int_0^{y/x_{\text{cl}}} d\epsilon w_\sigma(\epsilon) \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} dz \sum_{\sigma} \int_0^{y/x_{\text{cl}}} d\epsilon k_\sigma(z) \Theta(k_\sigma^2(z)) \\ &= \frac{y^2}{4\pi} \left[\frac{N}{x_{\text{cl}}} \right] \int_{-\infty}^{\infty} dz \left[\sqrt{-V_+} + \sqrt{-V_-} + (1+V_+) \ln \left[\frac{\sqrt{|1+V_+|}}{1+\sqrt{-V_+}} \right] \right. \\ &\quad \left. + (1+V_-) \ln \left[\frac{\sqrt{|1+V_-|}}{1+\sqrt{-V_-}} \right] \right].\end{aligned}\quad (3.28)$$

Adding the contributions from the continuum (3.23) and discrete (3.26) states, we obtain

$$\begin{aligned}Q^{\text{WKB}}[\varphi] &= \frac{y^2}{4\pi} \left[\frac{N}{x_{\text{cl}}} \right] \int_{-\infty}^{\infty} dz \left[\left[1 - \frac{\varphi^2}{v^2} \right] + \frac{1}{2} \left[\frac{\varphi^2}{v^2} + \frac{1}{yv} \frac{d\varphi}{dz} \right] \ln \left[\frac{\varphi^2}{v^2} + \frac{1}{yv} \frac{d\varphi}{dz} \right] \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{\varphi^2}{v^2} - \frac{1}{yv} \frac{d\varphi}{dz} \right] \ln \left[\frac{\varphi^2}{v^2} - \frac{1}{yv} \frac{d\varphi}{dz} \right] \right] + E_{\text{disc}}^{(2)}[\varphi]\end{aligned}\quad (3.29)$$

for the fermion vacuum energy in the WKB approximation.

When $\varphi(z)$ is slowly varying on the scale of the fermion Compton wavelength, the number of discrete states $w_\sigma(y/x_{\text{cl}})$ is large, the sum (3.25) is well approximated by the integral (3.27), and $E_{\text{disc}}^{(2)}[\varphi]$ is much smaller than $E_{\text{disc}}^{(1)}[\varphi]$. If we therefore neglect $E_{\text{disc}}^{(2)}[\varphi]$, Eq. (3.29) provides a completely explicit local expression for the energy of a static configuration:

$$E_{\text{eff}}^{\text{WKB}}[\varphi] = \frac{N}{x_{\text{cl}}} \int_{-\infty}^{\infty} dz \left[\frac{1}{2} \left(\frac{d\varphi}{dz} \right)^2 + \frac{1}{2v^2} (\varphi^2 - v^2)^2 \right] \\ + \frac{y^2}{4\pi} \left[\frac{N}{x_{\text{cl}}} \int_{-\infty}^{\infty} dz \left[\left(1 - \frac{\varphi^2}{v^2} \right) + \frac{1}{2} \left(\frac{\varphi^2}{v^2} + \frac{1}{yv} \frac{d\varphi}{dz} \right) \ln \left| \frac{\varphi^2}{v^2} + \frac{1}{yv} \frac{d\varphi}{dz} \right| \right. \right. \\ \left. \left. + \frac{1}{2} \left(\frac{\varphi^2}{v^2} - \frac{1}{yv} \frac{d\varphi}{dz} \right) \ln \left| \frac{\varphi^2}{v^2} - \frac{1}{yv} \frac{d\varphi}{dz} \right| \right] \right]. \quad (3.30)$$

For $\varphi(z)$ constant, Eq. (3.30) reduces to the effective potential (3.7). When $\varphi(z)$ is not constant, Eq. (3.30) yields a correction to the effective potential which, unlike the gradient expansion, does not diverge for configurations going through $\varphi=0$.

We conclude this section by comparing the WKB approximation of the fermion vacuum energy of the classical kink, $\varphi_{\text{cl}}(z) = v \tanh(z)$, with the known exact result. The WKB approximation should be accurate for $y \gg 1$, when $\varphi_{\text{cl}}(z)$ is slowly varying relative to the fermion Compton wavelength. The Dirac equation can be solved analytically in the classical kink background. Using the resulting eigenvalues, Chang and Yan [19] computed the exact fermion loop correction (3.20) to the energy of the classical kink:

$$Q[\varphi_{\text{cl}}] = \left[\frac{N}{x_{\text{cl}}} \right] \Delta(y). \quad (3.31)$$

The function $\Delta(y)$ is given by a complicated integral, but for integer y it simplifies to [19]

$$\Delta(y) = \frac{y^2}{\pi} + \sum_{n=1}^{y-1} \left[-\sqrt{2yn-n^2} + \frac{2}{\pi} \sqrt{y^2-n^2} \arctan \left(\frac{y^2-n^2}{n^2} \right)^{1/2} \right], \quad y \in \mathbb{Z}. \quad (3.32)$$

Using the Euler-Maclaurin formula, we obtain the large y behavior of Eq. (3.32):

$$\Delta(y) = \left[\frac{3}{2\pi} - \frac{\pi}{8} \right] y^2 + \left[2\sqrt{2}\beta + \frac{1}{3\sqrt{2}} \right] \sqrt{y} + \text{const}, \quad \beta = \sum_{k=1}^{\infty} \frac{(4k-5)!!}{2^{2k}(2k)!} B_{2k} \approx 0.0206 \dots \quad (3.33)$$

where B_{2k} are the Bernoulli numbers. The series defining β is asymptotic, so we only keep 4 or 5 terms in the sum.

The WKB approximation is obtained by substituting $\varphi_{\text{cl}}(z)$ into Eq. (3.29) and expanding for large y :

$$Q^{\text{WKB}}[\varphi_{\text{cl}}] = \frac{N}{x_{\text{cl}}} \left[\left[\frac{3}{2\pi} - \frac{\pi}{8} \right] y^2 + \frac{1}{6} \sqrt{y} + \text{const} \right] \\ + E_{\text{disc}}^{(2)}[\varphi_{\text{cl}}]. \quad (3.34)$$

Using the WKB approximation for the discrete eigenvalues $\epsilon_{\sigma i}$ together with the Euler-Maclaurin formula, we find the leading behavior of the remainder term:

$$E_{\text{disc}}^{(2)}[\varphi_{\text{cl}}] = \frac{N}{x_{\text{cl}}} \left[\left[2\sqrt{2}\beta + \frac{1}{3\sqrt{2}} - \frac{1}{6} \right] \sqrt{y} + \text{const} \right]. \quad (3.35)$$

Thus the WKB approximation of the fermion vacuum energy is

$$Q^{\text{WKB}}[\varphi_{\text{cl}}] = \frac{N}{x_{\text{cl}}} \left[\left[\frac{3}{2\pi} - \frac{\pi}{8} \right] y^2 \right. \\ \left. + \left[2\sqrt{2}\beta + \frac{1}{3\sqrt{2}} \right] \sqrt{y} + \text{const} \right], \quad (3.36)$$

in agreement with Eq. (3.33) to this accuracy. The y^2 term, of course, is just the contribution from the effective potential (3.7). The nonanalytic subleading \sqrt{y} dependence cannot be seen in the gradient expansion, but is correctly given by the WKB approximation (3.29).

Obviously, the coefficient obtained for the subleading \sqrt{y} dependence would be incorrect if we made the further approximation of dropping $E_{\text{disc}}^{(2)}[\varphi]$, as was done in obtaining the local expression (3.30). Nonetheless, Eq. (3.30) correctly gives the order of the subleading dependence. In general, it provides a useful estimate of the correction to the effective potential for a spatially varying field.

IV. QUANTUM KINKS

The quantum kink extremizes the effective action of the (1+1)-dimensional ϕ^4 theory. For small Yukawa coupling y , the effective action (3.1) differs only slightly from the classical action, so the quantum kink nearly coincides with the classical kink. When y is large, however, fermion loop corrections are important, and the quantum kink differs significantly from the classical kink.

To find the explicit form of the quantum kink, we use the local approximation (3.30) for the energy $E_{\text{eff}}[\varphi]$ of a static scalar field configuration derived in Sec. III. The equation of motion for the quantum kink follows from extremizing Eq. (3.30):

$$\left[1 + \frac{1}{4\pi v^4} \frac{\varphi^2}{f(\varphi)}\right] \frac{d^2\varphi}{dz^2} = 2\varphi \left[\frac{\varphi^2}{v^2} - 1 \right] + \frac{y^2}{4\pi v^2} \varphi \ln|f(\varphi)| + \frac{1}{2\pi v^4} \frac{\varphi}{f(\varphi)} \left[\frac{d\varphi}{dz} \right]^2, \quad (4.1)$$

$$f(\varphi) = \frac{\varphi^4}{v^4} - \frac{1}{y^2 v^2} \left[\frac{d\varphi}{dz} \right]^2.$$

Using the program COLSYS [20], we have solved this equation numerically for various values of the parameters subject to the boundary condition $\varphi(\pm\infty) = \pm v$. The solutions obtained interpolate smoothly between $-v$ and v . Indeed, their profiles are almost indistinguishable from the hyperbolic tangent shape of the classical kink (see Fig. 1). The slope of the quantum kink differs from that of the classical kink, however, being much steeper for certain values of the parameters.

We can more easily see how the slope of the quantum kink depends on the parameters of the theory by restricting $\varphi(x)$ to the one-parameter family of functions:

$$\varphi_{x_0}(x) = v \tanh \left[\frac{x}{x_0} \right], \quad (4.2)$$

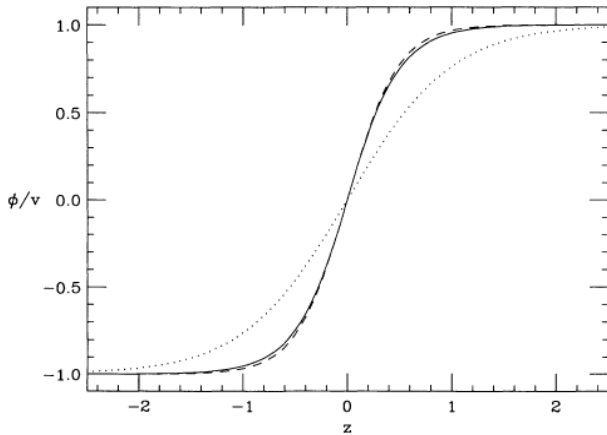


FIG. 1. A representative quantum kink. The solid line shows the solution of Eq. (4.1) for the parameters $y=20$ and $v=4$. The dashed line shows the ansatz $\varphi(z) = v \tanh(z/z_0)$, with z_0 given by Eq. (4.6). The dotted line shows the classical kink, $z_0=1$.

where x_0 is the “size” of the ansatz. We write the energy of the ansatz,

$$E_{\text{eff}}(z_0) = E_{\text{cl}}(z_0) + Q(z_0), \quad z_0 = \frac{x_0}{x_{\text{cl}}}, \quad (4.3)$$

where z_0 is the ratio of the size of the ansatz to that of the classical kink. The classical contribution

$$E_{\text{cl}}(z_0) = \frac{N}{x_{\text{cl}}} \left[\frac{2}{3} v^2 \left[z_0 + \frac{1}{z_0} \right] \right] \quad (4.4)$$

has a minimum at $z_0=1$, of course. The quantum contribution is obtained by substituting the ansatz (4.2) into the WKB approximation (3.29) and retaining the leading power of y

$$Q^{\text{WKB}}(z_0) = \frac{N}{x_{\text{cl}}} \left[\left[\frac{3}{2\pi} - \frac{\pi}{8} \right] y^2 z_0 + O \left[\left[\frac{y}{z_0} \right]^{1/2} \right] \right]. \quad (4.5)$$

The WKB approximation is accurate when the neglected terms are small, which requires $z_0 \gg 1/y$. That is, the size of the ansatz must be much larger than the fermion Compton wavelength ($x_0 \gg 1/gv$).

In the following discussion, we assume large Yukawa coupling $y \gg 1$. The size of the quantum kink is found by minimizing (4.3),

$$z_0 = \left[1 + \left[\frac{9}{4\pi} - \frac{3\pi}{16} \right] \frac{y^2}{v^2} \right]^{-1/2}, \quad (4.6)$$

and depends on the values of both dimensionless parameters y and v . When $v \gg y$, the kink size $z_0 \approx 1$ and the quantum kink reduces to the classical kink, because the classical contribution to the energy is dominant in this regime. On the other hand, when $v \ll y$ (but $v \gg 1$), the kink size $z_0 \approx (9/4\pi - 3\pi/16)^{-1/2} (v/y) \approx 2.8(v/y)$; the quantum kink is much smaller than the classical kink. The energy of the quantum kink in this regime, $E_{\text{eff}} \approx (4/\pi - \pi/3)^{1/2} v y (N/x_{\text{cl}}) \approx 0.48 v y (N/x_{\text{cl}})$, is larg-

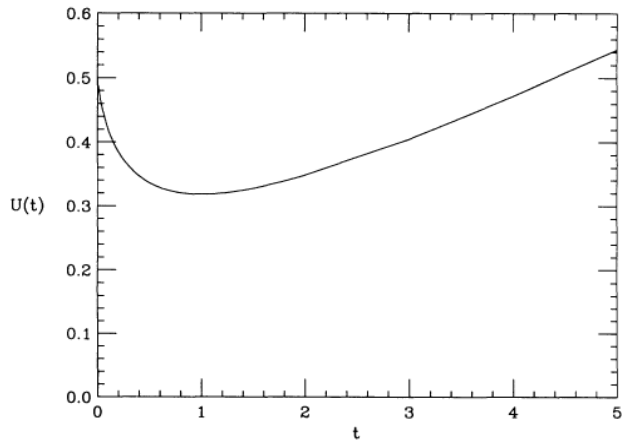


FIG. 2. The function $U(t)$. The exact fermion vacuum energy for the ansatz $\varphi(z) = v \tanh(z/z_0)$ is given by $(N/x_{\text{cl}}) y U(yz_0)$.

er than the classical kink energy (2.8) due to the positive fermion vacuum energy (4.5).

When $v \lesssim 1$, the WKB approximation (4.5) breaks down because the kink size is no longer much larger than the fermion Compton wavelength. By using the exact Dirac eigenvalues for the background (4.2) rather than the WKB eigenvalues, however, we can calculate the fermion vacuum energy $Q(z_0)$ without approximation, just as for the classical kink. We find

$$Q(z_0) = \left[\frac{N}{x_{cl}} \right] y U(yz_0), \quad U(t) = \frac{\Delta(t)}{t}, \quad (4.7)$$

where $\Delta(y)$ is the fermion vacuum energy of the classical kink defined in Sec. III. The function $U(t)$ is shown in Fig. 2, and equals $1/\pi$ at its minimum $t=1$. (That its minimum is at $t=1$ can be seen from Eq. (3.32) and from

$$\frac{d\Delta}{dt} = \frac{1}{2} \delta_{t0} + \frac{t}{\pi} + \sum_{n=1}^{t-1} \left[- \left[\frac{n}{2t-n} \right]^{1/2} + \frac{2}{\pi} \left[\frac{t-n}{t+n} \right]^{1/2} \arctan \left[\frac{t^2}{n^2} - 1 \right]^{1/2} \right], \quad t \in \mathbb{Z}, \quad (4.8)$$

obtained by a calculation similar to that in Ref. [19].) When $v \ll 1$ (and $y \gg 1$), the fermion vacuum contribution (4.7) dominates the energy, so the size of the quantum kink is determined by the minimum of $Q(z_0)$, that is, $z_0 \approx 1/y$. The quantum kink energy, $E_{\text{eff}} \approx (y/\pi)(N/x_{cl})$, is much larger than that of the classical kink (2.8).

The classical contribution to the energy $E_{cl}(z_0)$ is minimized for $z_0=1$, when the ansatz size equals the scalar field Compton wavelength, $x_0=x_{cl}$. The quantum contribution $Q(z_0)$ is minimized for $z_0=1/y$, when the ansatz size equals the fermion Compton wavelength, $x_0=x_{cl}/y=1/gv$. The size of the quantum kink always lies somewhere between these two values. The three limits we considered above,

$$\begin{aligned} 1 \ll y \ll v &\Rightarrow z_0 \approx 1, \quad E_{\text{eff}} \approx \frac{4v^2}{3} \left[\frac{N}{x_{cl}} \right], \\ 1 \ll v \ll y &\Rightarrow z_0 \approx \left[\frac{9}{4\pi} - \frac{3\pi}{16} \right]^{-1/2} \frac{v}{y}, \quad E_{\text{eff}} \approx \left[\frac{4}{\pi} - \frac{\pi}{3} \right]^{1/2} v y \left[\frac{N}{x_{cl}} \right], \\ v \ll 1 \ll y &\Rightarrow z_0 \approx \frac{1}{y}, \quad E_{\text{eff}} \approx \frac{y}{\pi} \left[\frac{N}{x_{cl}} \right], \end{aligned} \quad (4.9)$$

correspond to the regime in which the classical energy is dominant ($v \gg y$), the regime in which the fermion vacuum energy is dominant ($v \ll 1$), and the regime in which both contributions are important ($1 \ll v \ll y$). For $v \gtrsim y$, the classical and quantum kink nearly coincide, while for $v \lesssim y$, the quantum kink is smaller and has greater energy than the classical kink. Note that due to the fermion vacuum contribution (4.7), the energy of the kink is bounded below by $(y/\pi)(N/x_{cl}) = Ngv/\pi$, that is, $1/\pi$ times the mass of N fermions.

We now turn to the question of fermion stability. In Sec. II, we saw that for sufficiently strong Yukawa coupling, $y > \frac{8}{3}v^2$, a state of N widely separated fermions has greater energy than a kink-antikink pair, computed in the classical approximation, so one might expect a kink-antikink pair to appear spontaneously, with the fermions coalescing to occupy the zero modes. Since the zero modes do not increase the kink energy, the energy of the fermions on the kinks is independent of y in the classical approximation, and would be much less than the energy of the fermions in a constant scalar field background for $y \gg v^2$. The kink binding energy could approach 100% for very large Yukawa coupling.

Instead we have found that, for large y , quantum corrections significantly increase the energy of the kink. For $v \gtrsim 1$ (but $v \ll y$), a kink-antikink pair has energy $\sim yv(N/x_{cl})$, greater than the energy of N fermions, so the fermions are stable. For $v \lesssim 1$, the energy of a kink-antikink pair may be less than $y(N/x_{cl}) = Ngv$, in which

case a state of N fermions may be unstable to the formation of a kink and antikink, each carrying fermion number $\frac{1}{2}N$. Since the kink energy is never less than $(y/\pi)(N/x_{cl})$, however, the energy of a widely separated kink-antikink pair is not significantly less than that of the original fermions; the binding energy per fermion cannot exceed $1 - 2/\pi \sim 36\%$.

Up to this point, we have been chiefly concerned with a large Yukawa coupling, $y \gg 1$; we conclude this section by briefly considering $y \lesssim 1$. When y is not large, the WKB approximation is no longer useful, but we can use the exact solution (4.7) for the ansatz (4.2). The case $y=1$ is interesting, because then $z_0=1$ minimizes both the classical and quantum contributions to the energy; the classical kink is an extremum of the effective action restricted to the subspace of functions (4.2). One might suspect from this that the classical kink extremizes the effective action over the space of all functions. Campbell and Liao [4] proved this to be the case by using inverse scattering methods (which were tractable only when $y=1$).² Thus, the quantum kink exactly coincides with the classical kink (for all values of v) when $y=1$.³ As we

²The stationary phase approximation of Ref. [4] is equivalent to our large- N approximation.

³Interestingly, the theory is supersymmetric precisely when $y=1$ [4,21].

have seen, they differ when $y \neq 1$.

The energy of the kink when $y = 1$ is

$$E_{\text{eff}} = \left[\frac{4}{3}v^2 + \frac{1}{\pi} \right] \frac{N}{x_{\text{cl}}}, \quad y = 1, \quad (4.10)$$

so a kink-antikink pair will have less energy than N widely separated fermions when $v < [\frac{3}{8}(1 - 2/\pi)]^{1/2} \approx 0.37$. It turns out that for $y = 1$, N fermions are unstable to kink formation only if $v < (1/4\pi)^{1/2} \approx 0.28$; the most energetically favorable configuration of N fermions for $y = 1$ and $v > 1/(4\pi)^{1/2}$ is a bag [4]. (See also Ref. [9].)

Finally, for small Yukawa coupling, $y < 1$, the fermion Compton wavelength is larger than the scalar field Compton wavelength, so quantum corrections tend to increase the size of the kink. When $v^2 \ll y < 1$, the quantum contribution dominates the energy, and the kink has size $z_0 \approx 1/y$ and energy $E_{\text{eff}} \approx (y/\pi)(N/x_{\text{cl}})$.

V. CONCLUSIONS

We have examined the effects of quantum corrections on solitons in a $(1+1)$ -dimensional ϕ^4 theory with a large Yukawa coupling y to fermions. To treat the Yukawa coupling nonperturbatively, we have solved the theory in the large- N limit, where N is the number of flavors. The solitons in this theory are kinks which carry fermion number ranging from $-\frac{1}{2}N$ to $\frac{1}{2}N$. In the classical approximation, the energy of the kink is independent of y , and its size is proportional to the scalar field Compton wavelength. We have found that fermion loop corrections increase the energy of the kink and (when $y > 1$) reduce its size. As a result of the fermion vacuum contribution, the kink energy is bounded below by $(y/\pi)(N/x_{\text{cl}}) = Ngv/\pi$, and its size can be as small as the fermion Compton wavelength.

When y is large, a state of N fermions is expected on classical grounds to be unstable to the formation of a kink and antikink, each carrying fermion number $\frac{1}{2}N$. Quantum corrections eliminate this instability for $v \gtrsim 1$ by increasing the kink-antikink energy. The instability persists for $v \lesssim 1$, but the difference in energy between the N fermions and the kink-antikink pair is only about 36% because the kink energy is proportional to the Yukawa coupling in the large- y limit.

In the large- N limit, scalar loops are suppressed. The energy of scalar field fluctuations is of order $1/x_{\text{cl}}$, small compared to the classical kink energy $\sim Nv^2/x_{\text{cl}}$. What happens when N is not large? Will a single fermion decay into a kink-antikink pair when $N=1$? Scalar field fluctuations are still relatively unimportant as long as v is large; $1/v^2$ is the usual semiclassical expansion parameter. We found fermions to be stable in this regime. Sca-

lar corrections become more important for small v , but on the other hand $1/x_{\text{cl}}$ is still small relative to the quantum kink energy $y/\pi x_{\text{cl}}$ when y is large. It is difficult to say whether a single fermion is unstable when $v \lesssim 1$.

Some aspects of the model discussed in this paper are peculiar to two dimensions. Presumably only in two dimensions can a fermion split into a pair of solitons, each carrying fermion number $\frac{1}{2}$. We expect other features of the quantum kink to be more universal, however. First, its energy acquires a linear dependence on the Yukawa coupling in the strong coupling limit through the fermion vacuum energy. Second, for large Yukawa coupling, fermion loop corrections tend to reduce the size of the soliton in the direction of the fermion Compton wavelength. Both of these features apply not only to the $(1+1)$ -dimensional solitons described in this paper, but also to $(3+1)$ -dimensional (large- N) nontopological solitons [10].

General arguments can be adduced to suggest that these are generic features of quantum solitons in any (large- N) strongly coupled Yukawa theory in $3+1$ dimensions. The fermion loop contribution to the effective action is $-iN \ln \det(i\mathcal{D})$. After renormalization, its contributions to the effective potential (of order g^4) and to the two-derivative term (of order g^2) overwhelm the tree-level contributions when g is large.⁴ For large Yukawa coupling, therefore, quantum solitons are determined by the fermion vacuum energy $(iN/T) \ln \det(i\mathcal{D})$. If this has a minimum for given boundary conditions, the resulting configuration must have size $R \sim 1/gv$ (the only scale present) and energy $\sim (gv)^4 R^{3-d}$, where d is the dimension of the soliton. For pointlike solitons ($d=0$), the energy is proportional to the Yukawa coupling. Assuming that the large- N restriction is only a technical one to facilitate calculations at strong coupling, we conjecture that these properties hold for quantum solitons in any strongly coupled Yukawa theory.

Note added in proof. D. Wasson [Nucl. Phys. **A535**, 456 (1991)] has also considered Dirac sea effects on the kink.

ACKNOWLEDGMENTS

I would like to thank R. Perry for a useful conversation, and for drawing my attention to Ref. [17]. I am also grateful to J. Bagger, E. Poppitz, and S. Mrenna for helpful discussions. This work has been supported by the NSF under Grant No. PHY-90-96198.

⁴Skyrmions, in which the fermion loop contribution to the two-derivative term vanishes after renormalization, apparently present an exception [5,6,14,16].

[1] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985), Chap. 6; R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[2] S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, *ibid.* **11**, 3026 (1975).

[3] P. Vinciarelli, Lett. Nuovo Cimento **4**, 905 (1972); T. D. Lee and G. Wick, Phys. Rev. D **9**, 2291 (1974); A. Chodos,

- R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, *ibid.* **9**, 3471 (1974); M. Creutz, *ibid.* **10**, 1749 (1974); M. Creutz and K. Soh, *ibid.* **12**, 443 (1975); W. Bardeen, M. Chanowitz, S. Drell, M. Weinstein, and T. Yan, *ibid.* **11**, 1094 (1975); R. Friedberg and T. D. Lee, *ibid.* **15**, 1694 (1977); **16**, 1096 (1977); **18**, 2623 (1978).
- [4] D. Campbell and Y.-T. Liao, Phys. Rev. D **14**, 2093 (1976).
- [5] R. MacKenzie, F. Wilczek, and A. Zee, Phys. Rev. Lett. **53**, 2203 (1984); R. MacKenzie, Ph.D. thesis, ITP Report No. NSF-ITP-84-135 (unpublished); Mod. Phys. Lett. A **7**, 293 (1992).
- [6] E. D'Hoker and E. Farhi, Nucl. Phys. **B241**, 109 (1984); **B248**, 77 (1984).
- [7] R. L. Davis, Phys. Rev. D **38**, 3722 (1988).
- [8] D. Kaplan, Phys. Lett. B **235**, 163 (1990).
- [9] R. MacKenzie and W. Palmer, Phys. Rev. D **42**, 701 (1990).
- [10] J. Bagger and S. Naculich, Phys. Rev. Lett. **67**, 2252 (1991); Phys. Rev. D **45**, 1395 (1992).
- [11] T. Banks and A. Dabholkar, Phys. Rev. D **46**, 4016 (1992).
- [12] S. Peris, Phys. Lett. B **251**, 603 (1990).
- [13] A somewhat different large- N limit is used in M. Einhorn and G. Goldberg, Phys. Rev. Lett. **57**, 2115 (1986); K. Aoki, Phys. Rev. D **44**, 1547 (1991); K. Aoki and S. Peris, Report No. UCLA/92/TEP/23, OHSTPY-HEPT-92-008 (unpublished).
- [14] I. Aitchison and C. Fraser, Phys. Lett. **146B**, 63 (1984); Phys. Rev. D **31**, 2605 (1985); C. Fraser, Z. Phys. C **28**, 101 (1985); L.-H. Chan, Phys. Rev. Lett. **54**, 1222 (1985); **56**, 404(E) (1985); **55**, 21 (1985); O. Cheyette, *ibid.* **55**, 2394 (1985); J. Zuk, Z. Phys. C **29**, 303 (1985).
- [15] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4114 (1974); **10**, 4130 (1974); **12**, 2443 (1975).
- [16] G. Ripka and S. Kahana, Phys. Lett. **155B**, 327 (1985); S. Kahana, R. Perry, and G. Ripka, *ibid.* **163B**, 37 (1985); R. Perry, Nucl. Phys. **A467**, 717 (1987).
- [17] D. Wasson and S. Koonin, Phys. Rev. D **43**, 3400 (1991).
- [18] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
- [19] S.-J. Chang and T.-M. Yan, Phys. Rev. D **12**, 3225 (1975).
- [20] U. Ascher, J. Christiansen, and R. Russell, ACM Trans. Math. Sftw. **7**, 223 (1981).
- [21] P. DiVecchia and S. Ferrara, Nucl. Phys. **B130**, 93 (1977); J. Hruby, *ibid.* **B131**, 275 (1977).