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The free energy in the weak-coupling phase of two-dimensional Yang-Mills theory on a sphere for SO(N) and Sp(N) is evaluated in the $1/N$ expansion using the techniques of Gross and Matytsin. Many features of Yang-Mills theory are universal among different gauge groups in the large N limit, but significant differences arise in subleading order in $1/N$. [S0556-2821(96)00114-2]

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I. INTRODUCTION

Two-dimensional (2D) Yang-Mills theories have been used as a laboratory to uncover general nonperturbative features of gauge theories [1–10]. It has been shown that the $1/N$ expansion of these theories may be represented as a formal string theory, for SU(N) and U(N) gauge groups [2,3] as well as for SO(N) and Sp(N) [4]. It is useful to compare these various string theories in order to learn which structures are generic, and what one might expect in a four-dimensional string theory of QCD.

Certain features of 2D Yang-Mills theories are universal, i.e., independent of the gauge group, in the large N limit [8]. For example, the normalized vacuum expectation values (VEV's) of Wilson loops on arbitrary surfaces do not depend on the gauge group to leading order in $1/N$, a fact most naturally understood from the string-theoretic interpretation of these theories [9]. On the other hand, the universality of gauge theory observables breaks down in subleading orders of the $1/N$ expansion. An example of this is the contribution from cross caps which appear on the world sheet for SO(N) and Sp(N), but not for SU(N) or U(N) [4]. It is important to have a clear understanding of the role of the gauge group in the string interpretation.

To further analyze the differences between these theories, in this paper we evaluate the free energy of Yang-Mills theory on the sphere in the small area (weak-coupling) phase, including exponential corrections to the $1/N$ expansion. Our analysis closely parallels that of Gross and Matytsin [10] for U(N), focusing specifically on the gauge groups SO(N) and Sp(N). One of the more interesting results of our analysis is the difference in the double-scaling limit for different gauge groups [see Eq. (34)ff].

II. THE PARTITION FUNCTION

The partition function of two-dimensional Yang-Mills theory on the sphere is

$$Z_0=\sum_R (\dim R)^2 e^{-(\lambda\bar{A}/2N)\,C_2(R)}, \quad (1)$$

where the sum is over all irreducible representations R of the gauge group, $\dim R$ and $C_2(R)$ are the dimension and quadratic Casimir invariant of R, $\bar{A}$ is the area of the sphere, and $\lambda=e^2N$, where e is the gauge coupling. The quadratic Casimir invariant is given by

$$C_2(R)=fN\left[r-U(r)+\frac{T(R)}{N}\right] \quad (2)$$

with

$$f=\begin{cases}1,\\ 1/2,\end{cases}\qquad U(r)=\begin{cases}r/N & \text{for SO}(N),\\ -r/N & \text{for Sp}(N),\end{cases} \quad (3)$$

and

$$T(R)=\sum_{i=1}^{n} n_i(n_i+1-2i)=\sum_{i=1}^{k_1} n_i^2-\sum_{j=1}^{n_1} k_j^2, \quad (4)$$

where $n_i(k_i)$ are the row (column) lengths of the Young diagram associated with R, and n is the rank of the gauge group. [Our convention is rank Sp($2n$) $=n$.] Defining

$$\begin{aligned}&\ell_i=n_i+n-i,\quad m_i=n-i \quad \text{for SO}(2n),\\ &\ell_i=n_i+n-i+\tfrac{1}{2},\quad m_i=n-i+\tfrac{1}{2}\quad \text{for SO}(2n+1),\\ &\ell_i=n_i+n-i+1,\quad m_i=n-i+1\quad \text{for Sp}(2n),\end{aligned} \quad (5)$$

the dimension and quadratic Casimir invariant of R may be expressed as [11]

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$$\dim R=\begin{cases}\displaystyle\prod_{i<j}^{n}\frac{(\ell_i^2-\ell_j^2)}{(m_i^2-m_j^2)} & \text{for SO}(2n),\\ \displaystyle\prod_{i<j}^{n}\frac{(\ell_i^2-\ell_j^2)}{(m_i^2-m_j^2)}\prod_{i=1}^{n}\frac{\ell_i}{m_i} & \text{for SO}(2n+1)\ \text{and Sp}(2n)\end{cases}\tag{6}$$

and

$$C_2(R)=\begin{cases}\displaystyle\sum_{i=1}^{n}\ell_i^2-\frac{1}{24}N(N-1)(N-2) & \text{for SO}(N),\\ \displaystyle\frac{1}{2}\sum_{i=1}^{n}\ell_i^2-\frac{1}{48}N(N+1)(N+2) & \text{for Sp(N)}.\end{cases}\tag{7}$$

These expressions are also valid for spinor representations of Spin(N), which are associated with Young diagrams with $n_i\in\mathbf{Z}+1/2$.

The partition function (1) depends on the area only through the dimensionless combination $A=\lambda fA$ and is given up to an overall constant by

$$Z_0(A,N)\propto\begin{cases}e^{\beta(A,N)}X^{(+)}(\alpha) & \text{for SO}(2n),\\ e^{\beta(A,N)}Y^{(-)}(\alpha) & \text{for SO}(2n+1),\\ e^{\beta(A,N)}Y^{(+)}(\alpha) & \text{for Sp}(2n),\\ e^{\beta(A,N)}[X^{(+)}(\alpha)+X^{(-)}(\alpha)] & \text{for Spin}(2n),\\ e^{\beta(A,N)}[Y^{(+)}(\alpha)+Y^{(-)}(\alpha)] & \text{for Spin}(2n+1),\end{cases}\tag{8}$$

with

$$\alpha=\frac{A}{2N},$$

$$\beta(A,N)=\begin{cases}\displaystyle\frac{A}{48}(N-1)(N-2) & \text{for SO}(N)\ \text{and Spin}(N),\\ \displaystyle\frac{A}{48}(N+1)(N+2), & \text{for Sp}(N),\end{cases}\tag{9}$$

and

$$X^{(\pm)}(\alpha)=\sum_{\ell_1>\cdots>\ell_n\geqslant 0}\Delta^2(\ell_1^2,\ldots,\ell_n^2)e^{-\alpha\Sigma_{j=1}^n\ell_j^2},$$

$$Y^{(\pm)}(\alpha)=\sum_{\ell_1>\cdots>\ell_n\geqslant 0}\Delta^2(\ell_1^2,\ldots,\ell_n^2)\times\left(\prod_{i=1}^{n}\ell_i^2\right)e^{-\alpha\Sigma_{j=1}^n\ell_j^2},\tag{10}$$

where the ℓ_i are integers in $X^{(+)}$ and $Y^{(+)}$ and half integers in $X^{(-)}$ and $Y^{(-)}$, and

$$\Delta(\ell_1^2,\ldots,\ell_n^2)=\prod_{i<j}^{n}(\ell_i^2-\ell_j^2)=\begin{vmatrix}1 & \cdots & 1\\ \ell_1^2 & \cdots & \ell_n^2\\ \vdots & \ddots & \vdots\\ (\ell_1^2)^{n-1} & \cdots & (\ell_n^2)^{n-1}\end{vmatrix}\tag{11}$$

is the van der Monde determinant in the variables ℓ_i^2. Note that both tensor and spinor representations contribute to the partition function for Spin(N), while only tensor representations contribute for SO(N). As in the U(N) case [10], the expressions (10) are symmetric with respect to the interchange $\ell_j\leftrightarrow\ell_k$, and vanish when $\ell_j=\ell_k$, so the summations can be extended to $-\infty<\ell_j<\infty$ for all ℓ_j, yielding

$$X^{(\pm)}(\alpha)=\frac{1}{2^n n!}\sum_{-\infty<\ell_1,\ldots,\ell_n<\infty}\Delta^2(\ell_1^2,\ldots,\ell_n^2)e^{-\alpha\Sigma_{j=1}^n\ell_j^2},$$

$$Y^{(\pm)}(\alpha)=\frac{1}{2^n n!}\sum_{-\infty<\ell_1,\ldots,\ell_n<\infty}\Delta^2(\ell_1^2,\ldots,\ell_n^2)\times\left(\prod_{i=1}^{n}\ell_i^2\right)e^{-\alpha\Sigma_{j=1}^n\ell_j^2},\tag{12}$$

where, again, the ℓ_i are integers in $X^{(+)}$ and $Y^{(+)}$ and half integers in $X^{(-)}$ and $Y^{(-)}$.

To further evaluate Eq. (12), we introduce several sets of polynomials in x^2, $q_j^{(\pm)}(x|\alpha)=x^{2j}+\cdots$, and $r_j^{(\pm)}(x|\alpha)=x^{2j}+\cdots$. They are defined to be mutually orthogonal with respect to the discrete measures

$$\sum_x e^{-\alpha x^2}q_i^{(\pm)}(x|\alpha)q_j^{(\pm)}(x|\alpha)=\delta_{ij}f_j^{(\pm)}(\alpha),$$

$$\sum_x e^{-\alpha x^2}x^2 r_i^{(\pm)}(x|\alpha)r_j^{(\pm)}(x|\alpha)=\delta_{ij}g_j^{(\pm)}(\alpha),\tag{13}$$

where the sums on x are over integers for $q^{(+)}$ and $r^{(+)}$ and half integers for $q^{(-)}$ and $r^{(-)}$. Defining

$$q_j^{(\pm)}(x|\alpha)=p_{2j}^{(\pm)}(x|\alpha),\quad f_j^{(\pm)}(\alpha)=h_{2j}^{(\pm)}(\alpha),$$
$$xr_j^{(\pm)}(x|\alpha)=p_{2j+1}^{(\pm)}(x|\alpha),\quad g_j^{(\pm)}(\alpha)=h_{2j+1}^{(\pm)}(\alpha),\quad (14)$$

the orthogonality relations (13) reduce to

$$\sum_x e^{-\alpha x^2}p_i^{(\pm)}(x|\alpha)p_j^{(\pm)}(x|\alpha)=\delta_{ij}h_j^{(\pm)}(\alpha),\quad x\in\begin{cases}\mathbf{Z},\\ \mathbf{Z}+\frac{1}{2}.\end{cases}\quad (15)$$

The $p_j^{(\pm)}(x|\alpha)=x^j+\cdots$ are the polynomials introduced by Gross and Matytsin [10] in their study of U(N). They showed that the $h_j^{(\pm)}(\alpha)$ are given by

$$h_j^{(\pm)}(\alpha)=h_0^{(\pm)}(\alpha)\prod_{i=1}^{j}R_i^{(\pm)}(\alpha),\quad h_0^{(\pm)}(\alpha)=\sum_x e^{-\alpha x^2},$$
$$x\in\begin{cases}\mathbf{Z},\\ \mathbf{Z}+\frac{1}{2},\end{cases}\quad (16)$$

where the $R_j^{(\pm)}(\alpha)$ are defined through the recursion relations

$$xp_j^{(\pm)}(x|\alpha)=p_{j+1}^{(\pm)}(x|\alpha)+R_j^{(\pm)}(\alpha)p_{j-1}^{(\pm)}(x|\alpha),$$
$$R_0^{(\pm)}(\alpha)=0,\quad (17)$$

and satisfy the differential relations [10]

$$\frac{d}{d\alpha}\ln R_j^{(\pm)}(\alpha)=R_{j-1}^{(\pm)}(\alpha)-R_{j+1}^{(\pm)}(\alpha),$$
$$\frac{d}{d\alpha}h_0^{(\pm)}(\alpha)=-R_1^{(\pm)}(\alpha).\quad (18)$$

Rewriting the van der Monde determinants in Eq. (12) in terms of the polynomials $q_n^{(\pm)}(x|\alpha)$ and $r_n^{(\pm)}(x|\alpha)$ and using the orthogonality relations, we find that the partition function is given up to a constant by

$$Z_0(A,N)\propto\begin{cases}e^{\beta}\prod_{j=0}^{n-1}h_{2j}^{(+)}(\alpha) & \text{for SO}(2n),\\ e^{\beta}\prod_{j=0}^{n-1}h_{2j+1}^{(\pm)}(\alpha) & \text{for }\begin{cases}\text{Sp}(2n)\\ \text{SO}(2n+1),\end{cases}\\ e^{\beta}\left[\prod_{j=0}^{n-1}h_{2j}^{(+)}(\alpha)+\prod_{j=0}^{n-1}h_{2j}^{(-)}(\alpha)\right] & \text{for Spin}(2n),\\ e^{\beta}\left[\prod_{j=0}^{n-1}h_{2j+1}^{(+)}(\alpha)+\prod_{j=0}^{n-1}h_{2j+1}^{(-)}(\alpha)\right] & \text{for Spin}(2n+1).\end{cases}\quad (19)$$

The free energy for the orthogonal and symplectic groups is, therefore,

$$F(A,N)=\ln Z_0=\beta(A,N)+F_N(A)+\text{const}\quad (20)$$

with

$$F_N(A)=\begin{cases}n\ln h_0^{(+)}(\alpha)+\sum_{j=1}^{n-1}(n-j)[\ln R_{2j-1}^{(+)}(\alpha)+\ln R_{2j}^{(+)}(\alpha)] & \text{for SO}(2n),\\ n\ln[h_0^{(\pm)}(\alpha)R_1^{(\pm)}(\alpha)]+\sum_{j=1}^{n-1}(n-j)[\ln R_{2j}^{(\pm)}(\alpha)+\ln R_{2j+1}^{(\pm)}(\alpha)] & \text{for }\begin{cases}\text{Sp}(2n)\\ \text{SO}(2n+1),\end{cases}\end{cases}\quad (21)$$

to be compared with the result for U(N) [10]:

$$F_N(A)=N\ln h_0^{(\pm)}(\alpha)+\sum_{j=1}^{N-1}(N-j)\ln R_j^{(\pm)}(\alpha)\quad\text{for }\begin{cases}\text{U}(N\ \text{odd})\\ \text{U}(N\ \text{even}).\end{cases}\quad (22)$$

Using Eq. (18), we obtain from Eqs. (21) and (22) the specific heat capacities

$$\frac{d^2F(A)}{dA^2}=\begin{cases}\frac{1}{4N^2}[R_N^{(\pm)}R_{N-1}^{(\pm)}] & \text{for } \begin{cases}\mathrm{SO}(N\ \text{even})\\ \mathrm{SO}(N\ \text{odd}),\end{cases}\\ \frac{1}{4N^2}[R_N^{(+)}R_{N+1}^{(+)}] & \text{for } \mathrm{Sp}(N),\\ \frac{1}{4N^2}[R_N^{(\pm)}(R_{N+1}^{(\pm)}+R_{N-1}^{(\pm)})] & \text{for } \begin{cases}\mathrm{U}(N\ \text{odd})\\ \mathrm{U}(N\ \text{even}).\end{cases}\end{cases} \tag{23}$$

To obtain more explicit expressions for the free energies, one may expand $R_j^{(\pm)}(\alpha)$, keeping the leading exponential correction:

$$R_j^{(\pm)}(\alpha)=\frac{j}{2\alpha}\mp\frac{2\pi^2}{\alpha^2}e^{-\pi^2/\alpha}G_j(\alpha)+\cdots. \tag{24}$$

Gross and Matytsin [10] use the recursion relations (18) to show that

$$G_j(\alpha)=\oint\frac{dt}{2\pi i}\left(1+\frac{1}{t}\right)^n e^{-2\pi^2 t/\alpha} \tag{25}$$

with the contour circling $t=0$ and passing to the right of $t=-1$. This can then be used to evaluate the free energy (22) for U(N) below the phase transition[1]

$$F_N=-\frac{N^2}{2}\ln A\pm 2e^{-\frac{2\pi^2N}{A}}G_N(\alpha)+\cdots \quad \text{for } \begin{cases}\mathrm{U}(N\ \text{odd})\\ \mathrm{U}(N\ \text{even}).\end{cases} \tag{26}$$

In the large N limit, the $G_j(\alpha)$ have the form [10]

$$G_j(\alpha)\approx(-1)^{j+1}\sqrt{\frac{j}{32\pi n_c^2}}\left(1-\frac{j}{n_c}\right)^{-1/4}\times\exp\left\{-\frac{2\pi^2N}{A}[\gamma(j/n_c)-1]\right\},$$

$$\gamma(x)=\sqrt{1-x}-\frac{x}{2}\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right), \tag{27}$$

$$n_c=\frac{\pi^2}{2\alpha}.$$

Using Eqs. (24) and (25), we calculate the free energy for the orthogonal and symplectic groups (21) below the phase transition

$$F_N=\begin{cases}\left(-\frac{N^2}{4}+\frac{N}{4}\right)\ln A+e^{-(2\pi^2N/A)}[\pm G_{2n}(\alpha)-I_{2n}(\alpha)]+\cdots & \text{for } \begin{cases}\mathrm{SO}(N=2n)\\ \mathrm{SO}(N=2n+1),\end{cases}\\ \left(-\frac{N^2}{4}-\frac{N}{4}\right)\ln A+e^{-(2\pi^2N/A)}[G_{2n}(\alpha)+I_{2n}(\alpha)]+\cdots & \text{for } \mathrm{Sp}(N=2n),\end{cases} \tag{28}$$

where

$$I_{2n}(\alpha)=-\frac{2\pi^2}{\alpha}\sum_{j=1}^{n}\frac{G_{2j-1}(\alpha)}{2j-1}=\oint\frac{dt}{2\pi i}\left(1+\frac{1}{t}\right)^{2n}\frac{e^{-2\pi^2t/\alpha}}{2t+1}. \tag{29}$$

In the large N limit, this yields

$$F_N=\begin{cases}\left(-\frac{N^2}{4}+\frac{N}{4}\right)\ln A\pm\left(1-\frac{1}{\sqrt{1-A/\pi^2}}\right)e^{-(2\pi^2N/A)}G_N(\alpha)+\cdots & \text{for } \begin{cases}\mathrm{SO}(N\ \text{even})\\ \mathrm{SO}(N\ \text{odd}),\end{cases}\\ \left(-\frac{N^2}{4}-\frac{N}{4}\right)\ln A+\left(1+\frac{1}{\sqrt{1-A/\pi^2}}\right)e^{-(2\pi^2N/A)}G_N(\alpha)+\cdots & \text{for } \mathrm{Sp}(N),\end{cases} \tag{30}$$

[1] We correct a sign error in Ref. [10] for even N.

but these expressions break down if the area A nears the critical area π^2. For the Spin(N) groups, the $O(e^{-2\pi^2 N/A})$ correction vanishes due to cancellation between the tensor and spinor representations, so that the leading correction is $O(e^{-4\pi^2 N/A})$ in that case.

Approaching the phase transition from below in the double-scaling limit, defined by

$$A\rightarrow\pi^2 \quad \text{and} \quad N\rightarrow\infty \quad \text{with} \quad N^2(\pi^2-A)^3\equiv g_{\rm str}^{-2}=\text{const}, \tag{31}$$

Gross and Matytsin [10] show that $R_j^{(\pm)}(\alpha)$ behaves as

$$R_j^{(\pm)}=\frac{n_c^2}{\pi^2}\mp(-)^j n_c^{5/3} f_1(x)+O(n_c^{4/3}), \quad x=n_c^{2/3}\left(1-\frac{j}{n_c}\right), \quad n_c\rightarrow\infty, \tag{32}$$

where $f_1(x)$ obeys the Painlevé II equation

$$f_1''-4xf_1-\tfrac{1}{2}\pi^2 f_1^3=0. \tag{33}$$

Using this, we may show that in the double-scaling limit the specific heat capacity (23) satisfies

$$\frac{d^2F_N}{dA^2}=\frac{n_c^4}{4\pi^4N^2}\left[1-\frac{2x}{n_c^{2/3}}-\frac{\pi^4}{2n_c^{2/3}}f_1^2(x)\pm\frac{\pi^2}{n_c^{2/3}}f_1'(x)+\cdots\right]_{x=x_N} \quad \text{for} \begin{cases}\text{SO}(N)\\ \text{Sp}(N),\end{cases} \tag{34}$$

which has an additional term proportional to $f_1'(x)$ compared with [10]

$$\frac{d^2F_N}{dA^2}=\frac{n_c^4}{2\pi^4N^2}\left[1-\frac{2x}{n_c^{2/3}}-\frac{\pi^4}{2n_c^{2/3}}f_1^2(x)+\cdots\right]_{x=x_N} \quad \text{for } \text{U}(N). \tag{35}$$

Equation (34) gives the one instanton contribution to the specific heat for SO(N) and Sp(N) in the double-scaling limit. The computation of the specific heat for Spin(N) is more complicated due to the contributions to the partition function equation (19) from both tensor and spinor representations.

III. CONCLUSIONS

Many features of two-dimensional Yang-Mills theory are universal in the large N limit [8,9], but differ in subleading order in $1/N$. In this paper, we have explicitly evaluated the free energy on the sphere in the weak-coupling phase, and shown how it compares among the different gauge groups. The double-scaling limit does not appear to be universal. Any proposed world-sheet action for two-dimensional Yang-Mills string theory must accommodate both the universal behavior as well as the differences among the gauge groups.

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