6-12-2000

Elliptic models and M-theory

Isabel P. Ennes
Brandeis University

Carlos Lozano
Brandeis University

Stephen G. Naculich
Bowdoin College

Howard J. Schnitzer
Brandeis University

Recommended Citation

Ennes, Isabel P.; Lozano, Carlos; Naculich, Stephen G.; and Schnitzer, Howard J., "Elliptic models and M-theory" (2000). *Physics Faculty Publications*. 146.
https://digitalcommons.bowdoin.edu/physics-faculty-publications/146

Elliptic models and M-theory

Isabel P. Ennes [a,1], Carlos Lozano [a,1], Stephen G. Naculich [b,2],
Howard J. Schnitzer [a,c,3,4]

[a] *Martin Fisher School of Physics, Brandeis University, Waltham, MA 02454, USA* [3]
[b] *Department of Physics, Bowdoin College, Brunswick, ME 04011, USA*
[c] *Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA*

Abstract

We give a unified analysis of four-dimensional elliptic models with $\mathcal{N} = 2$ supersymmetry and a simple gauge group, and their relation to M-theory. Explicit calculations of the Seiberg–Witten curves and the resulting one-instanton prepotential are presented. The remarkable regularities that emerge are emphasized. In addition, we calculate the prepotential in the Coulomb phase of the (asymptotically-free) $Sp(2N)$ gauge theory with N_f fundamental hypermultiplets of arbitrary mass. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

The study of $\mathcal{N} = 2$ supersymmetric gauge theories using the Seiberg–Witten (SW) approach [1] to the low energy effective action is now more than five years old. During this period the theory has undergone considerable development, from a variety of approaches. One of the intriguing aspects of SW theory is the connection to integrable models, where M-theory [2] provides one method of constructing the spectral curves of elliptic models. (Another technique is geometric engineering [3–15].) Even though this aspect of elliptic models has been extensively studied [16–20], there remain a number of open questions of

some importance for these theories. In particular, except in certain special cases, the bridge between the spectral curve of the elliptic model and the corresponding curve obtained from an M-theory picture is still absent. This is one of the issues we consider in this paper, with considerable progress, but not a complete resolution of all the issues.

One motivation for understanding the connection between the spectral curve and M-theory picture is to present the instanton expansion of the prepotential $\mathcal{F}$ for the theory in question. As explained in our previous papers in this series [21–26], this will provide tests of M-theory by means of comparison between our results for $\mathcal{F}_{\text{instanton}}$ with the analogous instanton prepotential obtained from the microscopic Lagrangian [5] [27–37].

The breakthrough of Seiberg and Witten [1] was their formulation of the exact solution of 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in terms of a low-energy (Wilsonian) effective action accurate to two derivatives of the fields,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left(\int d^4\theta \, \frac{\partial \mathcal{F}(A)}{\partial A_i} \, \bar{A}_i + \frac{1}{2} \int d^2\theta \, \frac{\partial^2 \mathcal{F}(A)}{\partial A_i \, \partial A_j} \, W_i^\alpha W_{\alpha,j} \right)$$
$$+ \text{ higher derivatives,} \tag{1.1}$$

where A^i are $\mathcal{N} = 1$ chiral superfields ($i = 1$ to $\text{rank}\,\mathcal{G}$), $\mathcal{F}(A)$ is the holomorphic prepotential, and W^i is the gauge field strength. The holomorphic prepotential can be expressed in terms of a perturbative piece and an infinite series of instanton contributions

$$\mathcal{F}(A) = \mathcal{F}_{\text{classical}}(A) + \mathcal{F}_{\text{1-loop}}(A) + \sum_{d=1}^{\infty} L^{2d} \mathcal{F}_{\text{d-inst}}(A), \tag{1.2}$$

where $L^2 = \Lambda^{I(G) - \sum_R I(R)}$ with Λ the quantum scale (Wilson cutoff), $I(G)$ the Dynkin index of the adjoint representation, and $I(R)$ the Dynkin index of a matter hypermultiplet in representation R. The one-loop contribution is given by perturbation theory:

$$\mathcal{F}_{\text{1-loop}}(a) = \frac{i}{4\pi} \sum_{\alpha \in \Delta_+} (a \cdot \alpha)^2 \log \left(\frac{a \cdot \alpha}{\Lambda} \right)^2$$
$$- \frac{i}{8\pi} \sum_j \sum_{w \in R_j} (a \cdot w + m_j)^2 \log \left(\frac{a \cdot w + m_j}{\Lambda} \right)^2, \tag{1.3}$$

where α ranges over the positive roots Δ_+ of $\mathcal{G}$, w runs over the weight vectors for a hypermultiplet with mass m_j in the representation R_j, and a_i parametrizes the Cartan subalgebra of $\mathcal{G}$. For models with zero beta function, the instanton expansion is in powers of q rather than Λ, where $q = e^{2\pi i \tau}$ with τ the coupling constant of the theory.

In order to compute the prepotential (1.2) using the Seiberg–Witten approach (for a recent review, see [40]), one requires:

---

[5] Slater [37] has calculated $\mathcal{F}_{\text{1-inst}}$ for $\mathcal{N} = 2$ $SU(N)$ gauge theory with one symmetric hypermultiplet and N_f fundamental hypermultiplets using the microscopic Lagrangian. His result is in agreement with the predictions of Refs. [22,23] obtained using the M-theory curve of Ref. [38,39]. This provides the first independent check of the predictions obtained using hyperelliptic perturbation theory [21].

(1) a suitable Riemann surface or algebraic curve, appropriate to the gauge group and matter content of the theory, dependent on gauge invariant moduli u_i (equivalently on the order parameters a_i) and the masses of the hypermultiplets;

(2) a preferred meromorphic 1-form λ, the SW differential;

(3) a canonical basis of homology cycles (A_k, B_k) on the surface.

These data allow the computation of period integrals:

$$2\pi i a_k = \oint_{A_k} \lambda, \qquad 2\pi i a_{D,k} = \oint_{B_k} \lambda, \tag{1.4}$$

from which one may compute $\mathcal{F}(a)$ by integrating $a_{D,k} = \frac{\partial \mathcal{F}(a)}{\partial a_k}$.

In this paper we will discuss the SW theory for all simple classical groups $\mathcal{G}$, with matter hypermultiplets in the asymptotically free Coulomb phase, or in the Coulomb phase with zero beta function. The discussion will be comprehensive in the sense that we will consider all generic cases (i.e., of arbitrary rank $\mathcal{G}$) for such models. The SW curves for these models fall into three classes:

(a) hyperelliptic curves [41–60],

(b) cubic (nonhyperelliptic) curves [38,39],[6] and

(c) curves of infinite order.

Our focus in this paper will be primarily on the last class of curves which correspond to decompactified elliptic models. The M-theory pictures for elliptic models for theories with vanishing beta function were given by Uranga [18]; those with simple classical groups are:

(1) $SU(N)$ with two antisymmetric hypermultiplets and four fundamental hypermultiplets,

(2) $SU(N)$ with an antisymmetric and a symmetric hypermultiplet,

(3) $SU(N)$ with an adjoint hypermultiplet,

(4) $SO(N)$ with an adjoint hypermultiplet,

(5) $Sp(2N)$ with an adjoint hypermultiplet, and

(6) $Sp(2N)$ with an antisymmetric hypermultiplet and four fundamental hypermultiplets.

We will explicitly write down the curves (leading-order terms only) for these models, and the resulting one-instanton prepotential. By sending the masses of some of the fundamental hypermultiplets to infinity, we recover the curves for some additional models in the Coulomb phase which also possess curves of infinite order.

A number of methods exist for extracting the instanton expansion from hyperelliptic curves, with the method of asymptotic expansion [62–66] being the most useful for our purposes. In Refs. [21–25], we have extended these ideas to cases (b) and (c), developing methods for obtaining the instanton expansion for nonhyperelliptic SW curves, of finite or infinite order. In this body of work, the order parameter a_k emerges as the natural variable for describing the instanton expansion, rather than the gauge invariant moduli. See Ref. [26] for a review and more details.

---

[6] The curve for $SU(N)$ + one antisymmetric representation was recently derived from an integrable model [61].

In Section 2, we discuss $Sp(2N)$ gauge theory with N_f fundamental hypermultiplets of arbitrary masses, resolving some issues that were left open by previous work [63]. In Section 3, we assemble the results for the one-instanton prepotential for models with different groups and matter content, observing a remarkable empirical regularity among the different cases. In Section 4, we summarize the M-theory pictures for the decompactified elliptic models, from which we obtain the leading-order terms (defined in Section 4) of the coefficient functions of the SW curves, using the geometry of NS 5-branes, D4-branes, and O6$^\pm$ orientifold planes. How to compute subleading terms in elliptic models or their decompactification is one of the open problems of this subject. Using these leading-order curves, we compute the one-instanton prepotential for each theory. In Section 5, we show that the SW curve obtained by Gukov and Kapustin [19] for $SU(N)$ with two antisymmetric hypermultiplets (with equal masses) and four fundamental hypermultiplets, and the curve obtained by Uranga [18] for $SU(N)$ with an antisymmetric and a symmetric hypermultiplet (with equal masses), are equivalent, after a change of variables, to the curves for those theories derived in this paper, giving dramatic confirmation of our methods. Section 6 is devoted to a consideration of $SU(N)$ gauge theory with a massive adjoint hypermultiplet. We explicitly exhibit the relation between the curve derived in this paper, the spectral curve derived from the Calogero–Moser model by D'Hoker and Phong [66], and the curve derived by Donagi and Witten [16,17] in the context of the integrable Hitchin system. We close with conclusions and comments on open problems in Section 7.

2. $Sp(2N) + N_f$ fundamentals

Although this paper primarily concerns elliptic models, we include this section on $Sp(2N)$ gauge theory with N_f fundamental hypermultiplets for completeness, even though the SW curve is hyperelliptic. There are some unresolved issues when all the hypermultiplets in the fundamental representation have nonzero masses [63], and we take the opportunity to clarify these, particularly as the results are needed to complete our tables.

The SW curve for $Sp(2N)$ gauge theory with $N_f < 2N + 2$ hypermultiplets in the fundamental representation is [38,39,63]:

$$y^2 + 2y \left[x^2 \prod_{i=1}^{N} (x^2 - e_i^2) + i^{N_f} L^2 \prod_{j=1}^{N_f} M_j \right] + L^4 \prod_{j=1}^{N_f} (x^2 - M_j^2) = 0, \tag{2.1}$$

where $L^2 = \Lambda^{2N+2-N_f}$. Writing the curve (2.1) as

$$y^2 + 2Ay + B = 0, \tag{2.2}$$

the SW differential is

$$\lambda = \frac{\left(\frac{A'}{A} - \frac{B'}{2B} \right) x}{\sqrt{1 - B/A^2}} \, dx. \tag{2.3}$$

Because of the $x \to -x$ symmetry, and since the genus of the curve (2.1) exceeds the rank of the group (and hence the number of independent moduli), not all period integrals are relevant for the SW problem.

The period integral is [63]:

$$2\pi i a_k = 2 \int_{x_k^-}^{x_k^+} \lambda, \tag{2.4}$$

as the A_k cycles are taken to surround the cut joining the two branch-points $x_k^\pm$. The branch-cuts surrounding $x = e_k$ go from x_k^- to x_k^+, and about $x = -e_k$ from $-x_k^+$ to $-x_k^-$. The B_k cycle for the dual period is chosen to go from $-x_k^-$ to x_k^- on the first sheet, and its counterpart on the second sheet. The dual period is given by [63]:

$$2\pi i a_{D,k} = 2 \int_{-x_k^-}^{x_k^-} \lambda. \tag{2.5}$$

The relevant branch-points are located at

$$x_k^\pm = e_k \pm L^2 \left[S_k \left(x_k^\pm \right) \right]^{1/2} - L^2 R_k \left(x_k^\pm \right) + \mathcal{O}\left(L^4 \right), \tag{2.6}$$

where

$$S_k(x) = \frac{\prod_{j=1}^{N_f} \left(x^2 - M_j^{\,2} \right)}{x^4 (x + a_k)^2 \prod_{i \neq k}^{N} (x^2 - a_i^{\,2})^2}, \tag{2.7}$$

and

$$R_k(x) = \frac{i^{N_f} \prod_{j=1}^{N_f} M_j}{x^2 (x + a_k) \prod_{i \neq k}^{N} (x^2 - a_i^{\,2})}. \tag{2.8}$$

The periods and dual periods are computed by asymptotic expansion, as in Refs. [62,66]. The period integral (2.4) yields

$$a_k = e_k - L^2 R_k(e_k) + L^4 \left(\tfrac{1}{4} \partial_k S_k(e_k) + \tfrac{1}{2} \partial_k \left[R_k(e_k) \right]^2 \right) + \mathcal{O}\left(L^6 \right). \tag{2.9}$$

Equation (2.9) differs from the periods for $SU(N)$ [62] in that for $Sp(2N)$ $S_k(a_k)$ does not contribute to order L^2 (1-instanton). It will contribute to order L^4 (2-instanton), as is already clear from Eq. (2.6) and will be explicitly shown below. To one-instanton accuracy, the dual periods are given by

$$2\pi i a_{D,k} = 2\pi i (a_{D,k})_{\text{classical}} + 2\pi i (a_{D,k})_{\text{1-loop}} - \frac{8L^2}{a_k} \sum_{i=1}^{N} a_i R_i(a_i) + \mathcal{O}\left(L^4 \right). \tag{2.10}$$

In order to integrate Eq. (2.10) to obtain the one-instanton prepotential, define the analytic function:

$$F(x) = \frac{i^{N_f} \prod_{j=1}^{N_f} M_j}{x \prod_{i=1}^{N} (x^2 - a_i^{\,2})}. \tag{2.11}$$

The sum of its residues vanishes, yielding:

$$2 \sum_{i=1}^{N} a_i R_i(a_i) + \frac{i^{N_f} \prod_{j=1}^{N_f} M_j}{\prod_{i=1}^{N} (-a_i{}^2)} = 0. \tag{2.12}$$

With the definition

$$S(x) = \frac{\prod_{j=1}^{N_f} (x^2 - M_j{}^2)}{x^4 \prod_{j=1}^{N} (x^2 - a_j{}^2)^2} = \frac{\bar{S}_0(x)}{x^4}, \tag{2.13}$$

the identity (2.12) becomes

$$2 \sum_{i=1}^{N} a_i R_i(a_i) + \left[\bar{S}_0(0) \right]^{1/2} = 0. \tag{2.14}$$

Since

$$\frac{2[\bar{S}_0(0)]^{1/2}}{a_k} = -\frac{\partial}{\partial a_k} \left[\bar{S}_0(0) \right]^{1/2}, \tag{2.15}$$

Eq. (2.10) may be rewritten:

$$2\pi i a_{D,k} = 2\pi i (a_{D,k})_{\text{classical}} + 2\pi i (a_{D,k})_{\text{1-loop}}$$
$$- 2L^2 \frac{\partial}{\partial a_k} \left[\bar{S}_0(0) \right]^{1/2} + \mathcal{O}(L^4), \tag{2.16}$$

which can be integrated to give

$$2\pi i \mathcal{F}_{\text{1-inst}} = -2 \left[\bar{S}_0(0) \right]^{1/2}, \quad \text{with} \tag{2.17}$$

$$\left[\bar{S}_0(0) \right]^{1/2} = i^{N_f} \frac{\prod_{j=1}^{N_f} M_j}{\prod_{i=1}^{N} (-a_i{}^2)}. \tag{2.18}$$

This result is entered in Table 2. A similar derivation applies to $Sp(2N)$ + adjoint (the corresponding curve is given in Section 4.6), and to $Sp(2N)$ + 1 anti. + N_f fund. (see Section 4.7). The corresponding results are given in Table 2, with the relevant $S(x)$ given in Table 1.

We can make several checks of our expressions (2.17) and (2.18). First of all, for pure $Sp(2N)$ gauge theory, we have:

$$2\pi i \mathcal{F}_{\text{1-inst}} = -2(-1)^N \frac{1}{\prod_{i=1}^{N} a_i{}^2}, \tag{2.19}$$

which agrees with the results of Masuda and Suzuki, and Ito and Sasakura [67,68] up to an overall constant, which can be reabsorbed with a redefinition of Λ.

Next, we can compare our one-instanton prediction for $Sp(4)$ without matter hypermultiplets with the one-instanton result for $SO(5)$ without matter hypermultiplets as given by [63,67,68]. The order parameters a_i of $Sp(4)$ are related to the order parameters d_i of $SO(5)$ by the change of variables

$$a_1 = (d_1 + d_2)/2, \qquad a_2 = (d_1 - d_2)/2. \tag{2.20}$$

Inserting Eq. (2.20) into Eq. (2.19) we find again perfect agreement (up to an overall constant).

We can also compare our result (2.17) for $Sp(2) + N_f$ fundamental hypermultiplets with that for $SU(2) + N_f$ fundamentals as given, for example, in Ref. [62]:

$$2\pi i \mathcal{F}_{\text{1-inst}} = \begin{cases} \frac{\Lambda_0{}^4}{8a^2}, & \text{for } N_f = 0, \\[2mm] \frac{\Lambda_1{}^3}{8a^2} M_1, & \text{for } N_f = 1, \\[2mm] \frac{\Lambda_2{}^2}{8a^2}\big[a^2 + M_1 M_2\big], & \text{for } N_f = 2, \\[2mm] \frac{\Lambda_3}{8a^2}\big[a^2(M_1 + M_2 + M_3) + M_1 M_2 M_3\big], & \text{for } N_f = 3. \end{cases} \qquad (2.21)$$

Again, we find agreement up to a multiplicative constant, and a moduli-independent additive term.

At first glance, it appears that our result disagrees with the result for $Sp(2N)$ with N_f fundamental hypermultiplets given in Ref. [63]. In Ref. [63], however, at least two of the fundamental hypermultiplets had vanishing masses. In that case, Eq. (2.18) yields $[\bar{S}_0(0)]^{1/2} = 0$, and hence $\mathcal{F}_{\text{1-inst}} = 0$ from Eq. (2.17). Thus, for the particular case of $Sp(2N)$ with at least two massless fundamental hypermultiplets, the first non-trivial contribution to the instanton prepotential is $\mathcal{F}_{\text{2-inst}}$.

To make contact with the results in Ref. [63], we calculate the two-instanton contribution to the prepotential following the method of Ref. [69]. The result is

$$2\pi i \mathcal{F}_{\text{2-inst}} = \sum_{k=1}^{N} S_k(a_k) + \frac{1}{4}\left(\frac{\partial^2 \bar{S}_0}{\partial x^2}\right)_{x=0}. \qquad (2.22)$$

Note that $S_k(a_k)$ contributes to two instantons (as it depends on L^4), as we had anticipated. Further, from Eq. (2.13), one may verify that $\frac{\partial^2 \bar{S}_0}{\partial x^2}(0) = 0$ when two or more of the hypermultiplets are massless, so the only contribution to the two-instanton prepotential will be the first term in Eq. (2.22), in complete agreement with [63].

For generic values of the masses of the matter hypermultiplets, we can check our two-instanton result for $Sp(2)$ against the two-instanton result for $SU(2)$, given by the expressions [62]:

$$2\pi i \mathcal{F}_{\text{2-inst}} = \begin{cases} \frac{5\Lambda_0{}^8}{2^{10}a^6}, & \text{for } N_f = 0, \\[2mm] \frac{\Lambda_1{}^6}{2^{10}a^6}\big[5M_1^2 - 3a^2\big], & \text{for } N_f = 1, \\[2mm] \frac{\Lambda_2{}^4}{2^{10}a^6}\big[a^4 - 3a^2(M_1^2 + M_2^2) + 5M_1^2 M_2^2\big], & \text{for } N_f = 2, \\[2mm] \frac{\Lambda_3{}^2}{2^{10}a^6}\big[a^6 + a^4(M_1^2 + M_2^2 + M_3^2) - 3a^2(M_1^2 M_2^2 + M_2^2 M_3^2 + M_1^2 M_3^2) \\[1mm] \qquad + 5M_1^2 M_2^2 M_3^2\big], & \text{for } N_f = 3. \end{cases} \qquad (2.23)$$

Our results (2.22) agree with Eq. (2.23) up to an overall constant. (For $N_f = 3$ there is a moduli-independent additive constant as well.)

Finally, we can compare our two-instanton prediction (2.22) for $Sp(4)$ without matter hypermultiplets with the two-instanton result for $SO(5)$ obtained using the method of Ref. [69]. Using the change of variables (2.20) we again find agreement. [7]

3. Universality

By examining $\mathcal{F}_{1\text{-inst}}$ obtained for all generic cases of classical groups in the Coulomb phase, one finds that the results may be summarized succinctly in terms of a master function $S(x)$ for each case, as we have emphasized previously [25,26]. These functions are collected for each theory in Table 1. (The new results in this table are from Sections 2 and 4 of this paper.)

Given $S(x)$, one defines residue functions $S_k(x)$ and $S_m(x)$ at the quadratic poles of $S(x)$ by

$$S(x) = \frac{S_k(x)}{(x - a_k)^2} = \frac{S_m(x)}{\left(x + \frac{1}{2} m\right)^2}. \tag{3.1}$$

If $S(x)$ has a quartic pole at $x = 0$, one defines:

$$S(x) = \frac{\bar{S}_0(x)}{x^4}. \tag{3.2}$$

In many cases [22,23,62,63,66], the one-instanton prepotential is given by

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k), \tag{3.3}$$

while for models containing one antisymmetric representation of $SU(N)$ [21,23] or the adjoint representation of $SO(N)$, the one-instanton prepotential is

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m\left(-\tfrac{1}{2} m\right), \tag{3.4}$$

and for models containing two antisymmetric representations of $SU(N)$ [25], it is

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}\left(-\tfrac{1}{2} m_1\right) - 2S_{m_2}\left(-\tfrac{1}{2} m_2\right). \tag{3.5}$$

Finally, for $Sp(2N)$ with various matter content, the one-instanton prepotential is

$$2\pi i \mathcal{F}_{1\text{-inst}} = -2\big[\bar{S}_0(0)\big]^{1/2}. \tag{3.6}$$

These results are summarized in Table 2.

---

[7] Apparently, there is a misprint in Eq. (3.9) for the two-instanton prepotential for $SO(N)$ in Ref. [63]. The correct expression appears to be:

$$\mathcal{F}_{2\text{-inst}} \propto \sum_{k \neq l}^{N} \frac{S_k(a_k) S_l(a_l)}{(a_k - a_l)^2} + \sum_{k,l}^{N} \frac{S_k(a_k) S_l(a_l)}{(a_k + a_l)^2} + \frac{1}{4} \sum_{k=1}^{N} S_k(a_k) \frac{\partial^2 S_k(x)}{\partial x^2}\bigg|_{x=a_k}. \tag{2.24}$$

Table 1

Hypermultiplet representations	$S(x)$
$SU(N) + N_f$ fund. (M_j) $(N_f \leqslant 2N)$ (Ref. [62])	$\dfrac{\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
$SU(N) + 1$ sym. (m) $+ N_f$ fund. (M_j) $(N_f \leqslant N-2)$ (Ref. [22,23])	$\dfrac{(-1)^N\left(x+\frac{1}{2}m\right)^2\prod_{i=1}^{N}(x+a_i+m)\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
$SU(N) + 1$ anti. (m) $+ N_f$ fund. (M_j) $(N_f \leqslant N+2)$ (Ref. [21,23])	$\dfrac{(-1)^N\prod_{i=1}^{N}(x+a_i+m)\prod_{j=1}^{N_f}(x+M_j)}{\left(x+\frac{1}{2}m\right)^2\prod_{i=1}^{N}(x-a_i)^2}$
$SU(N) + 2$ anti. (m_1, m_2) $+ N_f$ fund. (M_j) $(N_f \leqslant 4)$ (Ref. [25])	$\dfrac{\prod_{i=1}^{N}(x+a_i+m_1)\prod_{i=1}^{N}(x+a_i+m_2)\prod_{j=1}^{N_f}(x+M_j)}{\left(x+\frac{1}{2}m_1\right)^2\left(x+\frac{1}{2}m_2\right)^2\prod_{i=1}^{N}(x-a_i)^2}$
$SU(N) + 1$ anti. (m_1) $+ 1$ sym. (m_2)	$\dfrac{\left(x+\frac{1}{2}m_2\right)^2\prod_{i=1}^{N}(x+a_i+m_1)\prod_{i=1}^{N}(x+a_i+m_2)}{\left(x+\frac{1}{2}m_1\right)^2\prod_{i=1}^{N}(x-a_i)^2}$
$SU(N) +$ adjoint (Ref. [66])	$\dfrac{\prod_{i=1}^{N}[(x-a_i)^2-m^2]}{\prod_{i=1}^{N}(x-a_i)^2}$
$SO(2N) + N_f$ fund. $(N_f \leqslant 2N-2)$ (Ref. [63])	$\dfrac{x^4\prod_{j=1}^{N_f}(x^2-M_j^2)}{\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$SO(2N) +$ adjoint	$\dfrac{x^4\prod_{i=1}^{N}[(x-m)^2-a_i^2]\prod_{i=1}^{N}[(x+m)^2-a_i^2]}{\left(x+\frac{1}{2}m\right)^2\left(x-\frac{1}{2}m\right)^2\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$SO(2N+1) + N_f$ fund. $(N_f \leqslant 2N-1)$ (Ref. [63])	$\dfrac{x^2\prod_{j=1}^{N_f}(x^2-M_j^2)}{\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$SO(2N+1) +$ adjoint	$\dfrac{x^2(x+m)(x-m)\prod_{i=1}^{N}[(x-m)^2-a_i^2]\prod_{i=1}^{N}[(x+m)^2-a_i^2]}{\left(x+\frac{1}{2}m\right)^2\left(x-\frac{1}{2}m\right)^2\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$Sp(2N) + N_f$ fund. $(N_f \leqslant 2N+2)$	$\dfrac{\prod_{j=1}^{N_f}(x^2-M_j^2)}{x^4\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$Sp(2N) +$ adjoint	$\dfrac{\left(x+\frac{1}{2}m\right)^2\left(x-\frac{1}{2}m\right)^2\prod_{i=1}^{N}[(x-m)^2-a_i^2]\prod_{i=1}^{N}[(x+m)^2-a_i^2]}{x^4\prod_{i=1}^{N}(x^2-a_i^2)^2}$
$Sp(2N) + 1$ anti. $+ N_f$ fund. $(N_f \leqslant 4)$	$\dfrac{\prod_{i=1}^{N}[(x-m)^2-a_i^2]\prod_{i=1}^{N}[(x+m)^2-a_i^2]\prod_{j=1}^{N_f}(x^2-M_j^2)}{x^4\left(x+\frac{1}{2}m\right)^2\left(x-\frac{1}{2}m\right)^2\prod_{i=1}^{N}(x^2-a_i^2)^2}$

Table 2
$\mathcal{F}_{\text{1-inst}}$ for different groups and matter content

Group	Matter content
	$2\pi i \mathcal{F}_{\text{1-inst}} = \sum_{k=1}^{N} S_k(a_k)$
$SU(N)$	N_f fund. $(N_f \leqslant 2N)$
	1 sym. $+ N_f$ fund. $(N_f \leqslant N - 2)$
	Adjoint
$SO(2N)$	N_f fund. $(N_f \leqslant 2N - 2)$
$SO(2N + 1)$	N_f fund. $(N_f \leqslant 2N - 1)$
	$2\pi i \mathcal{F}_{\text{1-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}\left(-\frac{1}{2} m_1\right)$
$SU(N)$	1 anti. $(m_1) + N_f$ fund. $(N_f \leqslant N + 2)$
	1 anti. $(m_1) + 1$ sym. (m_2)
$SO(2N)$	Adjoint (m_1)
$SO(2N + 1)$	Adjoint (m_1)
	$2\pi i \mathcal{F}_{\text{1-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}\left(-\frac{1}{2} m_1\right) - 2S_{m_2}\left(-\frac{1}{2} m_2\right)$
$SU(N)$	2 anti. $(m_1, m_2) + N_f$ fund. $(N_f \leqslant 4)$
	$2\pi i \mathcal{F}_{\text{1-inst}} = -2[\bar{S}_0(0)]^{1/2}$
$Sp(2N)$	N_f fund. $(N_f \leqslant 2N + 2)$
	Adjoint
	1 anti. $+ N_f$ fund. $(N_f \leqslant 4)$

An examination of Table 1 leads to empirical rules for constructing $S(x)$, where $S(x)$ is given as the product of factors, each corresponding to a different $\mathcal{N} = 2$ multiplet in a given representation of a classical group. These rules for the factors that make up $S(x)$ are given in Table 3, which contains some new results, obtained in Sections 2 and 4 of this paper.

By examining Table 3, one observes that certain pairs of (mass-deformed) elliptic models have identical $S(x)$, and therefore identical $\mathcal{F}_{\text{1-inst}}$, for suitable choices of mass parameters and moduli. These equivalences are presented in Table 4. In each of the cases, one may verify that $\mathcal{F}_{\text{1-loop}}$ is also identical for both sides.[8] Finally, one can verify that the curves are identical on both sides of the first line of the table, and that the leading-order terms (see Section 4) of the curves are identical on both sides of the remaining lines of the table. Since it is very plausible that $S(x)$ determines the complete instanton expansion for a theory, we claim that Table 4 likely represents pairs of theories with identical prepotentials.

---

[8] This has been previously observed for the third entry of the table in Ref. [70].

Table 3
Factors of $S(x)$

Group	Representation	Factor of $S(x)$
$SU(N)$	Gauge multiplet	$\dfrac{1}{\prod_{i=1}^{N}(x-a_i)^2}$
	N_f fundamental (M_j)	$\displaystyle\prod_{j=1}^{N_f}(x+M_j)$
	Symmetric (m)	$(-1)^N\left(x+\tfrac{1}{2}m\right)^2\displaystyle\prod_{i=1}^{N}(x+a_i+m)$
	Antisymmetric (m)	$(-1)^N\left(x+\tfrac{1}{2}m\right)^{-2}\displaystyle\prod_{i=1}^{N}(x+a_i+m)$
	Adjoint (m)	$\displaystyle\prod_{i=1}^{N}[(x-a_i)^2-m^2]$
$SO(2N)$	Gauge multiplet	$\dfrac{x^4}{\prod_{i=1}^{N}(x^2-a_i^2)^2}$
	N_f fundamental (M_j)	$\displaystyle\prod_{j=1}^{N_f}(x^2-M_j^2)$
	Adjoint (m)	$\dfrac{\prod_{i=1}^{N}[(x+m)^2-a_i^2]\prod_{i=1}^{N}[(x-m)^2-a_i^2]}{\left(x+\tfrac{1}{2}m\right)^2\left(x-\tfrac{1}{2}m\right)^2}$
$SO(2N+1)$	Gauge multiplet	$\dfrac{x^2}{\prod_{i=1}^{N}(x^2-a_i^2)^2}$
	N_f fundamental (M_j)	$\prod_{j=1}^{N_f}(x^2-M_j^2)$
	Adjoint (m)	$\dfrac{(x+m)(x-m)\prod_{i=1}^{N}[(x+m)^2-a_i^2]\prod_{i=1}^{N}[(x-m)^2-a_i^2]}{\left(x+\tfrac{1}{2}m\right)^2\left(x-\tfrac{1}{2}m\right)^2}$
$Sp(2N)$	Gauge multiplet	$\dfrac{1}{x^4\prod_{i=1}^{N}(x^2-a_i^2)^2}$
	N_f fundamental (M_j)	$\displaystyle\prod_{j=1}^{N_f}(x^2-M_j^2)$
	Adjoint (m)	$\left(x+\tfrac{1}{2}m\right)^2\left(x-\tfrac{1}{2}m\right)^2\displaystyle\prod_{i=1}^{N}[(x+m)^2-a_i^2]\prod_{i=1}^{N}[(x-m)^2-a_i^2]$
	Antisymmetric (m)	$\dfrac{\prod_{i=1}^{N}[(x+m)^2-a_i^2]\prod_{i=1}^{N}[(x-m)^2-a_i^2]}{\left(x+\tfrac{1}{2}m\right)^2\left(x-\tfrac{1}{2}m\right)^2}$

Table 4
Table of equivalences (from $S(x)$)

$SU(N) + 1$ sym. $+ 1$ anti.	$SU(N) + 2$ anti. $+ 4$ fund.
(m, m)	$(m, m) + \left(\frac{1}{2}m, \frac{1}{2}m, \frac{1}{2}m, \frac{1}{2}m \right)$

<div align="center">Valid for all moduli
Also verified using elliptic curves (Refs. [18,19]) (see Section 5)</div>

$SO(2N) + $ adjoint	$Sp(2N) + 1$ anti. $+ 4$ fund.
(m)	$(m) + (0, 0, 0, 0)$

<div align="center">Valid for all moduli</div>

$Sp(2N) + $ adjoint	$Sp(2N) + 1$ anti $+ 4$ fund.
(m)	$(m) + \left(\frac{1}{2}m, \frac{1}{2}m, \frac{1}{2}m, \frac{1}{2}m \right)$

<div align="center">Valid for all moduli</div>

$SU(2N) + 2$ anti. $+ 4$ fund.	$SO(2N) + $ adjoint
$(m, -m) + (0, 0, 0, 0)$	(m)
Moduli: $a_1, a_2, \ldots, a_N, -a_1, -a_2, \ldots, -a_N$	Moduli: $a_1, a_2, \ldots, a_N$

$SU(2N) + 2$ anti. $+ 4$ fund.	$Sp(2N) + 1$ anti. $+ 4$ fund.
$(m, -m) + (M_1, M_2, -M_1, -M_2)$	$(m) + (M_1, M_2, 0, 0)$

<div align="center">Same relations for moduli as in the previous case</div>

4. Curves for decompactified elliptic models

In Sections 4 and 5 of Ref. [25], we reverse-engineered a curve for $\mathcal{N} = 2$ $SU(N)$ gauge theory with two hypermultiplets in the antisymmetric representation and $N_f < 4$ hypermultiplets in the fundamental representation using the regularities of the function $S(x)$ observed in Section 3 of this paper. This curve can be regarded as the decompactification of an elliptic model with zero beta function, having two hypermultiplets in the antisymmetric representation and four fundamental hypermultiplets, with the mass of one or more of the fundamental representations sent to infinity. The resulting SW curve remains of infinite order in this limit.

A number of $\mathcal{N} = 2$ theories with simple classical gauge groups can be understood as decompactifications of elliptic models. (By decompactification, we mean that the curve is formulated on the covering space of the circular base space of the elliptic model.) The M-theory description for these theories has been considered by Uranga [18], but only for those with zero beta function. The "basic cell" for these models (except for $SU(N)$ with an adjoint hypermultiplet) contains two O6 planes (with the same or opposite charges depending on the group and matter content), together with one or two NS 5-branes, and a number of D4-branes and (possibly) D6-branes. (We use the language of type IIA theory, which is then considered to be lifted to M-theory.) In the most cases, no explicit curve, suitable for studying the instanton expansion, has been previously presented (except for $SU(N)$ with a massive adjoint hypermultiplet, for which

the prepotential was calculated in Ref. [66]). In this section, we present curves and the resulting prepotentials for those theories obtained as decompactifications of the elliptic models discussed in [18].

The curve for a decompactified elliptic model has the form

$$\sum_{n=-\infty}^{\infty} L^{rn^2} J_n(x) P_n(x) t^n = 0, \tag{4.1}$$

where $t = \exp[-(x_6 + ix_{10})/R]$, $x = x_4 + ix_5$, and $r = 1$ $(r = 2)$ if there are two (one) NS 5-branes per unit cell. The coefficient functions $P_n(x)$ and $J_n(x)$ themselves have (in principle) expansions in L:

$$P_n(x) = P_n(x)\big|_{\text{leading}} + \mathcal{O}(L^2), \qquad J_n(x) = J_n(x)\big|_{\text{leading}} + \mathcal{O}(L^2). \tag{4.2}$$

The leading-order terms in $P_n(x)$ are determined by the positions of the D4-branes (i.e., moduli of the gauge theory) and the positions of the orientifold planes (i.e., masses of hypermultiplets in two-index representations of the gauge group), while the leading-order terms of $J_n(x)$ are associated with the positions of the D6-branes (i.e., masses of fundamental representations), if present. We have not been able to uniquely determine the subleading terms in Eq. (4.2).

The M-theory pictures corresponding to elliptic models with zero beta function are periodic in both the x_6 and x_{10} directions. If we let z parameterize the torus with the identifications $z \equiv z + 2\omega_1 \equiv z + 2\omega_2$, then $t = e^{\beta z}$ with $\beta = -i\pi/\omega_1$. The shift $z \to z + 2\omega_1$ (or $t \to e^{-2\pi i} t$) corresponds to a translation by $2\pi R$ in the x_{10} direction. The shift $z \to z + 2\omega_2$ (or $t \to q^{-1} t$), where $q = e^{2\pi i \tau}$ and $\tau = \omega_2/\omega_1$, corresponds to a translation in the x_6 direction (accompanied by a translation in x_{10} if $\text{Re}\,\tau \neq 0$). The curves (4.1) derived from these pictures would likewise be expected to be doubly periodic (up to a shift in x). Periodicity in the x_{10} direction is automatic, but periodicity in the x_6 direction requires that L be replaced by $q^{1/4}$, so (4.1) becomes

$$\sum_{n=-\infty}^{\infty} q^{rn^2/4} e^{\beta nz} J_n(x) P_n(x) = 0, \tag{4.3}$$

for theories with zero beta function. It further requires that $P_n(x)$ and $J_n(x)$ possess certain symmetry properties. If there are two NS 5-branes per unit cell (as is the case for the $SU(N)$ theories with symmetric or antisymmetric hypermultiplets that we consider), then

$$P_{n+2\ell}(x) = P_n(x - \ell\Delta), \qquad J_{n+2\ell}(x) = J_n(x - \ell\Delta), \tag{4.4}$$

implies that the curves (4.1) and (4.3), with $r = 1$, are invariant under $t \to tL^{-4}$ (or $z \to z + 2\omega_2$) and $x \to x + \Delta$ (where Δ is the "global mass", the relative mass of the two hypermultiplets in two-index representations of $SU(N)$). If there is only one NS 5-brane per unit cell (as in all the $SO(N)$ and $Sp(2N)$ theories that we consider, or the $SU(N)$ theory with a massive adjoint hypermultiplet), then

$$P_{n+\ell}(x) = P_n(x - \ell m), \qquad J_{n+\ell}(x) = J_n(x - \ell m), \tag{4.5}$$

guarantees that the curves (4.1) and (4.3), with $r = 2$, are invariant under $t \to tL^{-4}$ (or $z \to z + 2\omega_2$) and $x \to x + m$ (where m is the global mass, the mass of the adjoint or antisymmetric hypermultiplet).

The factors in the leading terms of $P_n(x)$ and $J_n(x)$ correspond not only to the D4-branes which depend on the moduli, but also to the semiinfinite "nondynamical" D4-branes associated with the O6 planes and D6-branes [18,71]. The placement of these nondynamical D4-branes is not unique, because they can extend either to the left or the right of the O6 plane or D6-brane. Different choices correspond precisely to different parameterizations of the curve $t \to t/G(x)$, where $G(x)$ is a rational function of x and positions of O6 planes and D6-branes. The symmetries (4.4) or (4.5) will not be present unless the nondynamical D4-branes are distributed symmetrically to the left and the right. In this paper, therefore, we will always choose a parametrization of the curves that respects these symmetries, so that the invariance of the curve under translations in x_6 (together with a shift in x) will be manifest.

To determine the prepotential to 1-instanton accuracy, it is sufficient [24,25] to consider only the quartic truncation of the curve (4.1):

$$L^{4r} J_2(x) P_2(x) t^2 + L^r J_1(x) P_1(x) t + J_0(x) P_0(x)$$
$$+ L^r J_{-1}(x) P_{-1}(x) \frac{1}{t} + L^{4r} J_{-2}(x) P_{-2}(x) \frac{1}{t^2} = 0. \qquad (4.6)$$

The prepotential for the decompactified elliptic models may then be obtained by calculating the period integrals a_k and $a_{D,k}$ from the curve (4.1), and then integrating $a_{D,k} = \partial \mathcal{F}/\partial a_k$. Applying residue methods [62] and hyperelliptic perturbation theory [21–25] to the quartic truncation (4.6), one obtains: [9]

$$\mathcal{F}(A) = \mathcal{F}_{\text{classical}}(A) + \mathcal{F}_{\text{1-loop}}(A) + L^{2r} \mathcal{F}_{\text{1-inst}}(A), \qquad (4.7)$$

where $\mathcal{F}_{\text{1-inst}}(A)$ is given by either of Eqs. (3.3)–(3.6). In each case, the function $S(x)$ is given by

$$S(x) = \left. \frac{J_1(x) J_{-1}(x) P_1(x) P_{-1}(x)}{J_0^2(x) P_0^2(x)} \right|_{\text{leading}}, \qquad (4.8)$$

where only the leading-order terms of $P_n(x)$ and $J_n(x)$ are used in defining $S(x)$. If there are no D6-branes in the model, the factors of $J_n(x)$ are absent in (4.8). The $O(L^{4r})$ terms in (4.6) are essential in obtaining the one-instanton prepotential, which involves a delicate cancellation [21] between these terms and the subleading terms in $P_n(x)$. This cancellation places constraints on the form of these subleading terms, but does not necessarily uniquely determine them.

In the rest of this section, we present results for each of several models, giving in each case:

(a) a figure containing the M-theory picture of the model;
(b) the leading-order terms of the coefficient functions $P_n(x)$ and $J_n(x)$, i.e., those terms with the lowest power in L or q for a given n;

---

[9] For the $Sp(N)$ theories, however, the one instanton prepotential is proportional to L^2, not L^{2r}.

   (c) the infinite curve for each model (with leading-order coefficients only);

   (d) the function $S(x)$ calculated from each curve;

   (e) the one-instanton contribution to the prepotential;

   (f) various checks on the proposed curve and prepotential.

4.1. $SU(N) + 2$ antisymmetric $+ N_f$ fundamentals

Consider $SU(N)$ gauge theory with two matter hypermultiplets (masses m_1 and m_2) in the antisymmetric representation, and N_f matter hypermultiplets (masses M_j) in the fundamental (defining) representation. The M-theory picture for this case, represented in Fig. 1, contains an infinite chain of NS 5-branes, with an O6$^-$ plane coincident with each one. Between each pair of consecutive 5-branes there are N D4-branes and N_f D6-branes.

The curve for this theory is given by Eq. (4.1), with $r = 1$. A particular parametrization for the functions $P_n(x)$ and $J_n(x)$ was given in Ref. [25]. In this paper, as discussed above, we choose a different parametrization, one in which the $P_n(x)$ and $J_n(x)$ have the symmetries

$$P_{n+2\ell}(x) = P_n(x - \ell\Delta), \qquad J_{n+2\ell}(x) = J_n(x - \ell\Delta), \tag{4.1.1}$$

where $m = \frac{1}{2}(m_1 + m_2)$ and $\Delta = m_1 - m_2$. In this parametrization, the curve becomes

$$\sum_{n \text{ even}} L^{n^2} t^n J_0\left(x - \tfrac{1}{2}n\Delta\right) P_0\left(x - \tfrac{1}{2}n\Delta\right)$$
$$+ \sum_{n \text{ odd}} L^{n^2} t^n J_1\left(x - \tfrac{1}{2}(n-1)\Delta\right) P_1\left(x - \tfrac{1}{2}(n-1)\Delta\right) = 0, \tag{4.1.2}$$
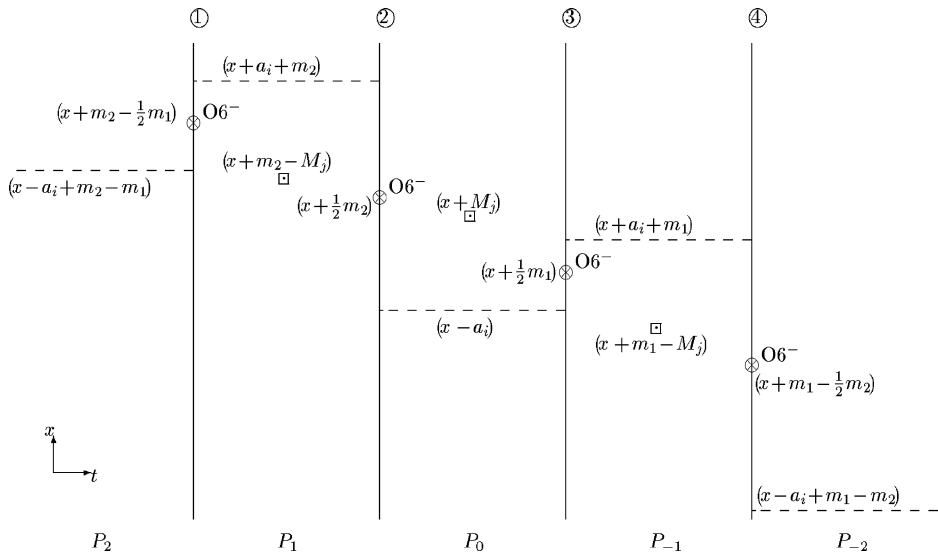
with

Fig. 1.

$$J_0(x) = \prod_{j=1}^{N_f} \prod_{p=1}^{\infty} \Big(\big[x + \tfrac{1}{2}m + (-1)^p\big(M_j - \tfrac{1}{2}m\big) - \tfrac{1}{2}\,p\Delta\big]$$

$$\times \big[x + \tfrac{1}{2}m + (-1)^p\big(M_j - \tfrac{1}{2}m\big) + \tfrac{1}{2}\,p\Delta\big]\Big)^{p/2}, \tag{4.1.3}$$

and with the leading term (in L) of $P_0(x)$ given by

$$P_0(x) = \prod_{p=1}^{\infty} \Big(\big[x + \tfrac{1}{2}m + \tfrac{1}{2}(p - \tfrac{1}{2})\Delta\big]\big[x + \tfrac{1}{2}m - \tfrac{1}{2}(p - \tfrac{1}{2})\Delta\big]\Big)^{-2p} \prod_{i=1}^{N}(x - a_i). \tag{4.1.4}$$

Using the involution symmetry in the $O6^-$ plane at $x = -\tfrac{1}{2}m_2$, one obtains:

$$P_1(x) = P_0\big(-x - m + \tfrac{1}{2}\Delta\big), \qquad J_1(x) = J_0\big(-x - m + \tfrac{1}{2}\Delta\big). \tag{4.1.5}$$

For $N_f = 4$, the beta function vanishes, so the curve for this model is (4.3), with $r = 1$. When the masses of the hypermultiplets satisfy $m_1 = m_2 = 2M_j$ ($j = 1, \ldots, 4$), the curve becomes:

$$0 = \sum_{n \text{ even}} q^{n^2/4} e^{\beta nz} \prod_{i=1}^{N}(x - a_i) + \sum_{n \text{ odd}} q^{n^2/4} e^{\beta nz} (-1)^N \prod_{i=1}^{N}(x + a_i + m)$$

$$= \theta_3\Big(\frac{z}{\omega_1}\Big|2\tau\Big) \prod_{i=1}^{N}(x - a_i) + \theta_2\Big(\frac{z}{\omega_1}\Big|2\tau\Big)(-1)^N \prod_{i=1}^{N}(x + a_i + m), \tag{4.1.6}$$

with no subleading terms, where θ_2 and θ_3 are Jacobi theta functions (5.1.12). The curve (4.1.6) is manifestly invariant under $z \to z + 2\omega_2$.

Using Eq. (4.8), we find:

$$S(x) = \frac{\prod_{i=1}^{N}(x + a_i + m_1) \prod_{i=1}^{N}(x + a_i + m_2) \prod_{j=1}^{N_f}(x + M_j)}{\big(x + \tfrac{1}{2}m_1\big)^2 \big(x + \tfrac{1}{2}m_2\big)^2 \prod_{i=1}^{N}(x - a_i)^2}, \tag{4.1.7}$$

in agreement with the empirical rules given in Table 3. The form of $S(x)$ is independent of the parametrization used for $P_n(x)$ and $J_n(x)$. The one-instanton contribution to the prepotential is [25]

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}\big(-\tfrac{1}{2}m_1\big) - 2S_{m_2}\big(-\tfrac{1}{2}m_2\big), \quad \text{where} \tag{4.1.8}$$

$$S_k(a_k) = \frac{\prod_{i=1}^{N}(a_k + a_i + m_1) \prod_{i=1}^{N}(a_k + a_i + m_2) \prod_{j=1}^{N_f}(a_k + M_j)}{\big(a_k + \tfrac{1}{2}m_1\big)^2 \big(a_k + \tfrac{1}{2}m_2\big)^2 \prod_{i \neq k}^{N}(a_k - a_i)^2},$$

$$S_{m_1}\big(-\tfrac{1}{2}m_1\big) = \frac{\prod_{i=1}^{N}\big(a_i + m_2 - \tfrac{1}{2}m_1\big) \prod_{j=1}^{N_f}\big(M_j - \tfrac{1}{2}m_1\big)}{\big(\tfrac{1}{2}m_2 - \tfrac{1}{2}m_1\big)^2 \prod_{i=1}^{N}\big(a_i + \tfrac{1}{2}m_1\big)},$$

$$S_{m_2}\big(-\tfrac{1}{2}m_2\big) = \frac{\prod_{i=1}^{N}\big(a_i + m_1 - \tfrac{1}{2}m_2\big) \prod_{j=1}^{N_f}\big(M_j - \tfrac{1}{2}m_2\big)}{\big(\tfrac{1}{2}m_1 - \tfrac{1}{2}m_2\big)^2 \prod_{i=1}^{N}\big(a_i + \tfrac{1}{2}m_2\big)}. \tag{4.1.9}$$

Various checks of this result were made in Ref. [25]. (In Ref. [25], the masses of the antisymmetric hypermultiplets were defined to be $2m_1$ and $2m_2$ rather than m_1 and m_2, and the definition of $S(x)$ differed by a factor of 4.)

4.2. $SU(N) + 1$ antisymmetric $+ 1$ symmetric

Consider $SU(N)$ gauge theory with one matter hypermultiplet (mass m_1) in the antisymmetric representation, and one (mass m_2) in the symmetric representation. The corresponding M-theory picture, represented in Fig. 2, contains O6$^+$ planes (related to the symmetric hypermultiplet) coincident with the even NS 5-branes, and O6$^-$ planes (related to the antisymmetric hypermultiplet) coincident with the odd NS 5-branes.

Choosing the parametrization of the coefficient functions of the curve to have the properties (4.4), the curve (4.3) for this theory takes the form:

$$\sum_{n \text{ even}} q^{n^2/4} e^{\beta n z} P_0\left(x - \tfrac{1}{2} n \Delta\right) + \sum_{n \text{ odd}} q^{n^2/4} e^{\beta n z} P_1\left(x - \tfrac{1}{2}(n-1)\Delta\right) = 0, \qquad (4.2.1)$$

where the leading term of $P_0(x)$ is

$$P_0(x) = \prod_{p=1}^{\infty} \left[\frac{x + \tfrac{1}{2}m + \tfrac{1}{2}(-1)^p \left(p - \tfrac{1}{2}\right)\Delta}{x + \tfrac{1}{2}m - \tfrac{1}{2}(-1)^p \left(p - \tfrac{1}{2}\right)\Delta} \right]^{2p} \prod_{i=1}^{N} (x - a_i), \qquad (4.2.2)$$

and, using the involution property,

$$P_1(x) = P_0\left(-x - m + \tfrac{1}{2}\Delta\right), \qquad (4.2.3)$$

with $m = \tfrac{1}{2}(m_1 + m_2)$ and $\Delta = m_1 - m_2$. The curve (4.2.1) is manifestly invariant under $z \to z + 2\omega_2$ together with $x \to x + \Delta$.
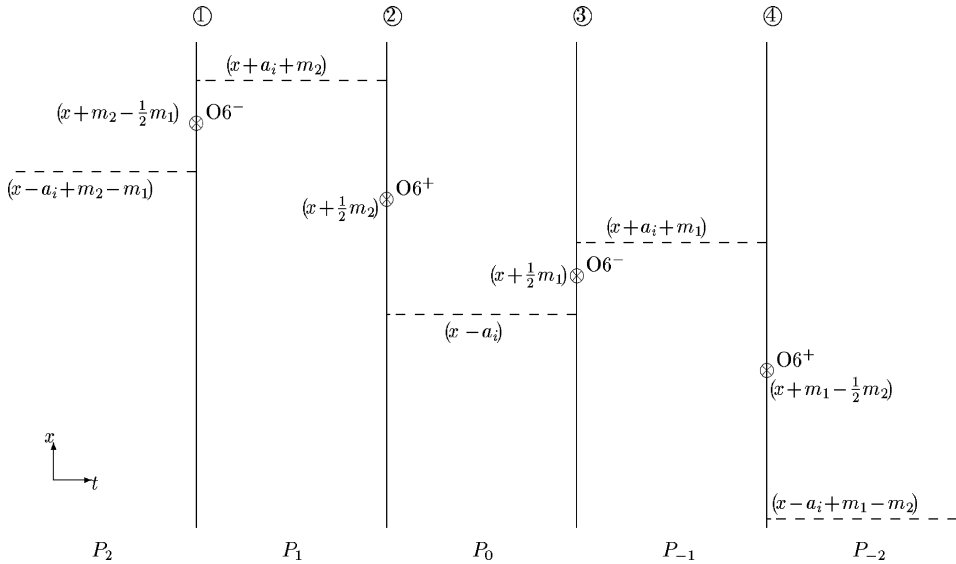

Fig. 2.

It may be verified that in the $m_2 \to \infty$ ($m_1 \to \infty$) limit, the curve (4.2.1) reduces to the curve (leading-order coefficients only) for $SU(N)$ with one antisymmetric (symmetric) hypermultiplet [38,39].

In the case $m_1 = m_2$, i.e., zero global mass Δ, the subleading terms of $P_n(x)$ vanish (the effects of the orientifolds of opposite charge cancel, as they are located at the same position in the x plane), and the curve (4.2.1) reduces to Eq. (4.1.6), which, surprisingly, also describes $SU(N)$ with two antisymmetric hypermultiplets (masses m_1 and m_2) and four fundamental hypermultiplets (masses M_j), with $m_1 = m_2 = 2M_j$. (See Section 5 for more details.)

Using Eq. (4.8), we obtain:

$$S(x) = \frac{\left(x + \frac{1}{2}m_2\right)^2 \prod_{i=1}^{N}(x + a_i + m_1) \prod_{i=1}^{N}(x + a_i + m_2)}{\left(x + \frac{1}{2}m_1\right)^2 \prod_{i=1}^{N}(x - a_i)^2}. \tag{4.2.4}$$

The one-instanton contribution to the prepotential is given by

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}\left(-\tfrac{1}{2}m_1\right), \quad \text{with} \tag{4.2.5}$$

$$S_k(a_k) = \frac{\left(a_k + \frac{1}{2}m_2\right)^2 \prod_{i=1}^{N}(a_k + a_i + m_1) \prod_{i=1}^{N}(a_k + a_i + m_2)}{\left(a_k + \frac{1}{2}m_1\right)^2 \prod_{i \neq k}^{N}(a_k - a_i)^2},$$

$$S_{m_1}\left(-\tfrac{1}{2}m_1\right) = \frac{(m_1 - m_2)^2 \prod_{i=1}^{N}\left(a_i + m_2 - \frac{1}{2}m_1\right)}{4\prod_{i=1}^{N}\left(a_i + \frac{1}{2}m_1\right)}. \tag{4.2.6}$$

We can test the expression (4.2.5) in two particular cases. For $SU(2)$, we can compare our results against those for $SU(2) +$ adjoint, as given in Eqs. (4.3.5) and (4.3.6). Setting $m_2 = m$ in (4.2.6), with m the mass of the adjoint hypermultiplet, and $a_1 + a_2 = 0$, we find that both expressions agree up to a moduli independent additive constant. For $SU(3)$, we can compare the prepotential (4.2.5) against that for $SU(3)$ with one symmetric representation and one fundamental representation [23], finding agreement (after setting $a_1 + a_2 + a_3 = 0$ and $m_1 = -m_f$) up to a moduli independent additive constant.

4.3. $SU(N) +$ adjoint

Consider $SU(N)$ gauge theory with a matter hypermultiplet (mass m) in the adjoint representation. This is an elliptic model which can be described in terms of the M-theory picture in Fig. 3a. In this figure, there are N D4-branes suspended between a single NS 5-brane with a periodicity in t, but with a shift in x of m (the global mass) for each circuit of t. The covering space of the S^1 (the t-variable) is shown in Fig. 3b.

The coefficient functions (which have no subleading terms) are:

$$P_0(x) = \prod_{i=1}^{N}(x - a_i), \qquad P_n(x) = P_0(x - nm), \tag{4.3.1}$$
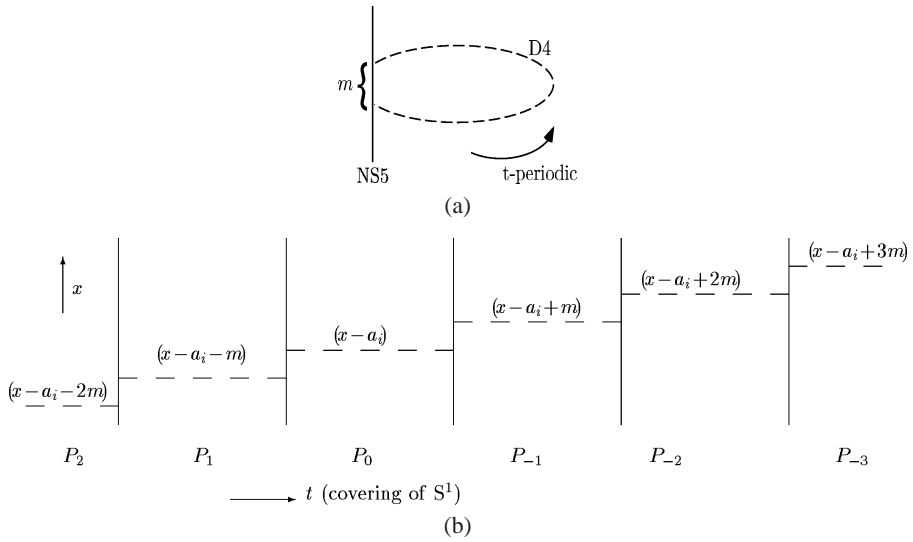
Fig. 3.

so the curve (4.3), with $r = 2$, becomes

$$\sum_n q^{n^2/2} e^{\beta nz} \prod_{i=1}^{N} (x - a_i - nm) = 0. \tag{4.3.2}$$

This is manifestly invariant under $z \to z + 2\omega_2$ together with $x \to x + m$. Shifting $z \to z + \omega_1 + \omega_2$, Eq. (4.3.2) becomes identical to the result of D'Hoker and Phong [66],

$$\sum_n (-1)^n q^{n(n-1)/2} e^{\beta nz} \prod_{i=1}^{N} (x - a_i - nm) = 0. \tag{4.3.3}$$

In Section 6, we show its relation to the curve of Donagi and Witten [16] for this theory.

Using Eq. (4.8), we obtain:

$$S(x) = \frac{\prod_{i=1}^{N} [(x - a_i)^2 - m^2]}{\prod_{i=1}^{N} (x - a_i)^2}. \tag{4.3.4}$$

The one-instanton contribution to the prepotential is given by [66]

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k), \quad \text{where} \tag{4.3.5}$$

$$S_k(a_k) = \frac{\prod_{i=1}^{N} [(a_k - a_i)^2 - m^2]}{\prod_{i \neq k}^{N} (a_k - a_i)^2}. \tag{4.3.6}$$

No subtraction is required in Eq. (4.3.5), as there are no spurious singularities to remove from $\sum_k S_k(a_k)$. This fact is related to the absence of subleading terms.

4.4. SO(2N) + adjoint

Consider $SO(2N)$ gauge theory with a matter hypermultiplet (mass m) in the adjoint representation. The corresponding M-theory picture (Fig. 4) contains O6$^-$ planes on top of each NS 5-brane, O6$^+$ planes between each pair of NS 5-branes, and D4-branes. There are additional four-branes in mirror positions with respect to the O6$^+$ orientifolds that are *not* represented in the figure.

In this theory, the unit cell contains only one NS 5-brane, so the parametrization of the coefficient functions of the curve are chosen to respect

$$P_{n+\ell}(x) = P_n(x - \ell m). \tag{4.4.1}$$

With this, the curve (4.3) becomes

$$\sum_{n=-\infty}^{\infty} q^{n^2/2} e^{\beta n z} P_0(x - nm) = 0, \tag{4.4.2}$$

where the leading term of $P_0(x)$ is given by

$$P_0(x) = \prod_{p=1}^{\infty} \left[\frac{(x + pm)(x - pm)}{(x + (p - \frac{1}{2})m)(x - (p - \frac{1}{2})m)} \right]^{2p} \prod_{i=1}^{N} [x^2 - a_i^2]. \tag{4.4.3}$$

The curve (4.4.2) is manifestly invariant under $z \to z + 2\omega_2$ and $x \to x + m$.

There are several decoupling limits [72] that can be considered to check Eq. (4.4.2).

(i) Let
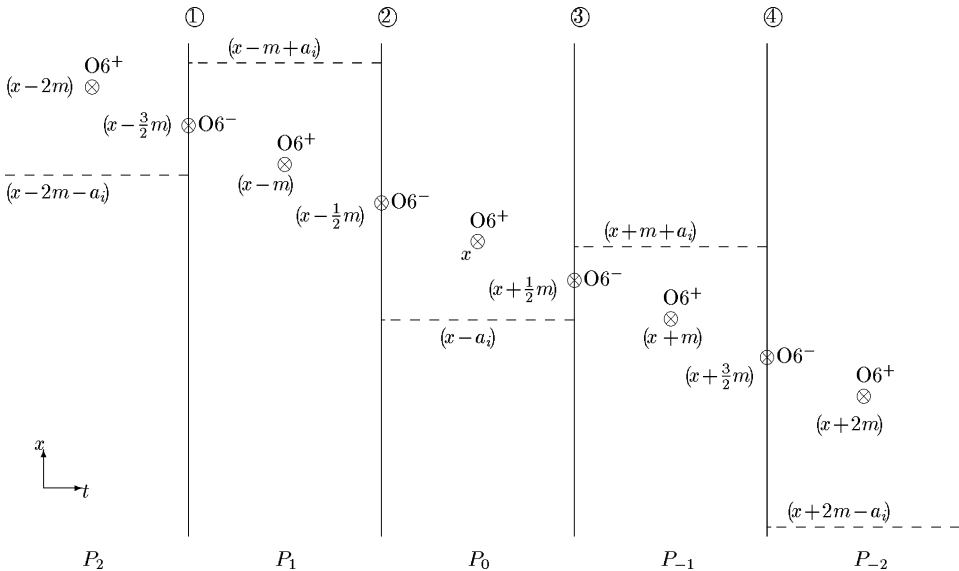
$$x_0 = \frac{1}{N} \sum_{i=1}^{N} a_i, \tag{4.4.4}$$

Fig. 4.

be the average position of the D4-branes in a single cell which lie above the $O6^+$ inside that cell. Then change variables

$$x \to x + x_0, \qquad a_i \to a_i + x_0, \tag{4.4.5}$$

and let $x_0 \to \infty$. After taking these limits in Eq. (4.4.2), the final curve coincides with the one for $\mathcal{N} = 4$ $SU(N)$ + adjoint.

(ii) There is another decoupling limit we can consider. If we take

$$q \to 0 \ (\tau \to i\infty), \qquad m \to \infty, \qquad \Lambda^{b_0} = qm^{b_0} \text{ fixed};$$
$$b_0 = 4N_1 - 2N_2 - 4, \qquad N_1 + N_2 = N, \tag{4.4.6}$$

we obtain the curve corresponding to $\mathcal{N} = 2$ $SO(2N_1) + N_2$ fundamentals. Using Eq. (4.8), we obtain:

$$S(x) = \frac{x^4 \prod_{i=1}^{N}[(x - m)^2 - a_i^2] \prod_{i=1}^{N}[(x + m)^2 - a_i^2]}{\left(x + \frac{1}{2}m\right)^2 \left(x - \frac{1}{2}m\right)^2 \prod_{i=1}^{N}(x^2 - a_i^2)^2}. \tag{4.4.7}$$

The one-instanton contribution to the prepotential is given by

$$2\pi i \mathcal{F}_{\text{1-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m\left(-\frac{1}{2}m\right), \quad \text{with} \tag{4.4.8}$$

$$S_k(a_k) = \frac{a_k^2 \prod_{i=1}^{N}[(a_k - m)^2 - a_i^2] \prod_{i=1}^{N}[(a_k + m)^2 - a_i^2]}{4\left(a_k + \frac{1}{2}m\right)^2 \left(a_k - \frac{1}{2}m\right)^2 \prod_{i \neq k}^{N}(a_k^2 - a_i^2)^2},$$

$$S_m\left(-\frac{1}{2}m\right) = \frac{m^2 \prod_{i=1}^{N}\left(\frac{9}{4}m^2 - a_i^2\right)}{16 \prod_{i=1}^{N}\left(\frac{1}{4}m^2 - a_i^2\right)}. \tag{4.4.9}$$

We can compare our result for $SO(2N)$ with a massive adjoint hypermultiplet with that of Minahan et al. [73], who obtained a mass expansion for $\mathcal{F}_{\text{instanton}}$, using a conjectured recursion relation for $SO(2N)$ plus adjoint. Expanding Eqs. (4.4.8) and (4.4.9) in powers of m, we find the first moduli-dependent contribution to be

$$2\pi i \mathcal{F}_{\text{1-inst}} = 4m^4 \sum_{k=1}^{N} \sum_{j \neq k} \frac{a_k^2}{(a_k^2 - a_j^2)^2} + \mathcal{O}(m^6). \tag{4.4.10}$$

This coincides with Eq. (4.7) of Ref. [73], identifying their ϕ_k with our a_k.

We can make a further check of our proposed expression (4.4.8) by comparing the result for $SO(6)$ with that for $SU(4)$. The one-instanton prepotential for $SU(4)$ with a massive adjoint hypermultiplet is given by Eq. (4.3.5) with moduli a_k', $k = 1, \ldots, 4$, restricted by $\sum_{k=1}^{4} a_k' = 0$. This constraint allows us to eliminate a_4'. Using the change of variables

$$a_1 = a_1' + a_2', \qquad a_2 = a_2' + a_3', \qquad a_3 = a_1' + a_3', \tag{4.4.11}$$

where a_i are the moduli of $SO(6)$, we find that the one-instanton prepotentials (4.3.5) and (4.4.8) agree, up to a moduli-independent additive constant.

To construct our curve, we have used the M-theory picture suggested by Uranga [18] in terms of $O6^{\pm}$ planes. Yokono [72] has constructed curves for $SO(2N)$ using orientifold

four-planes instead of O6$^\pm$ planes. His curves, while satisfying the correct decoupling lim-
its (i) and (ii) above, differ from ours. Although the curves in Ref. [72] have a smooth
limit when $m \to 0$, the brane configuration from which it is constructed is not consistent
in this limit. When $m = 0$, the four-branes must change their charge when crossing an
NS 5-brane [74], but this is not the case for the M-theory picture in Ref. [72]. (The same
comments apply to the curves proposed for $Sp(2N) + adjoint$ and $SO(2N + 1) + adjoint$
in Ref. [72].) Moreover, the one-loop prepotential derived from Yokono's curves disagrees
with the perturbation theory result, and the one-instanton prepotential for $SO(6)$ with mas-
sive adjoint calculated from his curve conflicts with that for $SU(4)$ with massive adjoint.

4.5. SO(2N + 1) + adjoint

Consider $SO(2N + 1)$ gauge theory with a matter hypermultiplet (mass m) in the adjoint
representation. The M-theory picture in Fig. 5 is similar to the one for $SO(2N)$ with
adjoint hypermultiplet with the following difference: there is an additional four-brane
whose position is fixed at the O6$^+$ plane in each cell [18]. As in last sections, there are
additional four-branes at mirror positions that are not included in the figure.

As before, the parametrization is chosen to obey (4.4.1) so the curve becomes

$$\sum_n q^{n^2/2} e^{\beta nz} P_0(x - nm) = 0, \qquad (4.5.1)$$

with the leading term of $P_0(x)$ given by

$$P_0(x) = \prod_{p=1}^{\infty} \left[\frac{(x + pm)(x - pm)}{(x + (p - \frac{1}{2})m)(x - (p - \frac{1}{2})m)} \right]^{2p} x \prod_{i=1}^{N} [x^2 - a_i^2]. \qquad (4.5.2)$$

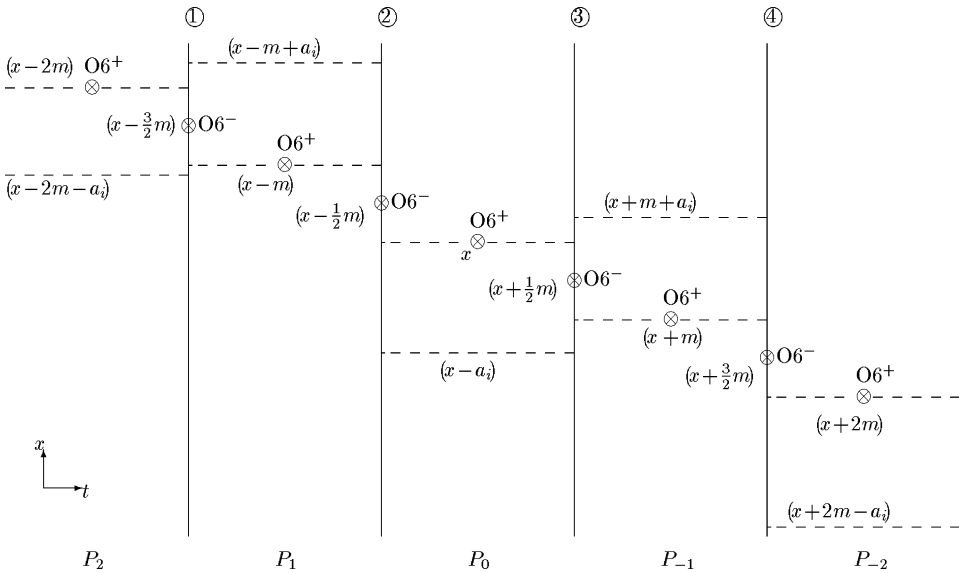Fig. 5.

As in Section 4.4, there are several decoupling limits that can be considered to check Eq. (4.5.1).

(i) This limit works exactly as in Section 4.4, to yield the curve for $\mathcal{N} = 4$ $SU(N) +$ adjoint.

(ii) If we take

$$q \to 0 \ (\tau \to i\infty), \qquad m \to \infty, \qquad \Lambda^{b_0} = qm^{b_0} \ \text{fixed};$$
$$b_0 = 4N_1 - 2N_2 - 2, \qquad N_1 + N_2 = N, \tag{4.5.3}$$

the resulting curve agrees with the one for $SO(2N_1 + 1) + N_2$ fundamentals. (We disagree with the curve proposed in Ref. [72], for reasons discussed in Section 4.4.) From (4.8), we find:

$$S(x) = \frac{x^2(x + m)(x - m) \prod_{i=1}^{N}[(x - m)^2 - a_i^2] \prod_{i=1}^{N}[(x + m)^2 - a_i^2]}{\left(x + \frac{1}{2}m\right)^2 \left(x - \frac{1}{2}m\right)^2 \prod_{i=1}^{N}(x^2 - a_i^2)^2}. \tag{4.5.4}$$

The one-instanton contribution to the prepotential is given by

$$2\pi i \mathcal{F}_{1\text{-inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m\left(-\tfrac{1}{2}m\right), \quad \text{with} \tag{4.5.5}$$

$$S_k(a_k) = \frac{(a_k + m)(a_k - m) \prod_{i=1}^{N}[(a_k - m)^2 - a_i^2] \prod_{i=1}^{N}[(a_k + m)^2 - a_i^2]}{4\left(a_k + \frac{1}{2}m\right)^2 \left(a_k - \frac{1}{2}m\right)^2 \prod_{i \neq k}^{N}(a_k^2 - a_i^2)^2},$$

$$S_m\left(-\tfrac{1}{2}m\right) = \frac{-3m^2 \prod_{i=1}^{N}\left(\frac{9}{4}m^2 - a_i^2\right)}{16 \prod_{i=1}^{N}\left(\frac{1}{4}m^2 - a_i^2\right)}. \tag{4.5.6}$$

In the case of $SO(5)$, we will be able to test Eq. (4.5.5) against the result we will obtain in the next subsection for $Sp(4)$ with a massive adjoint hypermultiplet.

4.6. $Sp(2N) +$ adjoint

Consider $Sp(2N)$ gauge theory with a matter hypermultiplet (mass m) in the adjoint representation. The corresponding M-theory picture (Fig. 6) contains $O6^+$ planes on top of each NS 5-brane, $O6^-$ planes between each pair of NS 5-branes, and D4-branes. As in the previous section, there are additional four-branes at mirror symmetric positions with respect to the $O6^-$ orientifolds that are not exhibited in Fig. 6 for clarity.

Choosing the coefficient functions to obey (4.4.1), the curve for this theory becomes:

$$\sum_n q^{n^2/2} e^{\beta n z} P_0(x - nm) = 0, \tag{4.6.1}$$

where the leading term of $P_0(x)$ is

$$P_0(x) = \prod_{p=1}^{\infty} \left[\frac{\left(x + (p - \frac{1}{2})m\right)\left(x - (p - \frac{1}{2})m\right)}{(x + pm)(x - pm)}\right]^{2p} \prod_{i=1}^{N}[x^2 - a_i^2]. \tag{4.6.2}$$

Notice that the nondynamical factors of (4.6.2) are the inverse of those in (4.4.3). The curve (4.6.1) is manifestly invariant under $z \to z + 2\omega_2$ and $x \to x + m$.
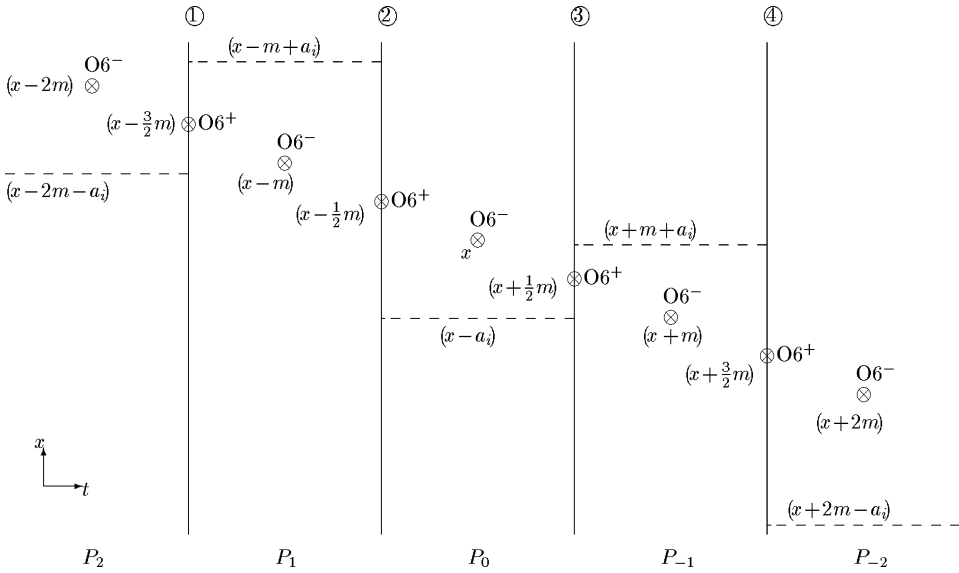
Fig. 6.

As in Section 4.4, we can consider some decoupling limits to check Eq. (4.6.1).

 (i) This limit is again the same as in Section 4.4.

(ii) If we consider

$$q \to 0 \ (\tau \to i\infty), \qquad m \to \infty, \qquad \Lambda^{b_0} = q m^{b_0} \text{ fixed};$$
$$b_0 = 4N_1 - 2N_2 + 4, \qquad N_1 + N_2 = N, \tag{4.6.3}$$

the curve we obtain agrees with the one corresponding to $\mathcal{N} = 2 \ Sp(2N_1) + N_2$ fundamentals, as it should. (We disagree with the curve proposed in Ref. [72], for the reasons given in Section 4.4.)

From (4.8), we obtain:

$$S(x) = \frac{\left(x + \frac{1}{2}m\right)^2 \left(x - \frac{1}{2}m\right)^2 \prod_{i=1}^{N}[(x - m)^2 - a_i^2] \prod_{i=1}^{N}[(x + m)^2 - a_i^2]}{x^4 \prod_{i=1}^{N}(x^2 - a_i^2)^2}. \tag{4.6.4}$$

Using the methods of Section 2 of this paper, one obtains the following one-instanton contribution to the prepotential:

$$2\pi i \mathcal{F}_{1\text{-inst}} = -2\big[\bar{S}_0(0)\big]^{1/2}, \quad \text{where} \tag{4.6.5}$$

$$\bar{S}_0(0) = \frac{(-m^2)^2 \prod_{i=1}^{N}(m^2 - a_i^2)^2}{16 \prod_{i=1}^{N}(-a_i^2)^2}. \tag{4.6.6}$$

We can check Eqs. (4.6.5) and (4.6.6) by specializing to $Sp(2)$ and comparing with $SU(2)$ plus adjoint hypermultiplet with mass m. The corresponding prepotentials agree up to a rescaling and a moduli-independent additive constant. We can also test Eqs. (4.6.5) and (4.6.6) for $Sp(4)$ with a massive adjoint hypermultiplet against Eqs. (4.5.5) and (4.5.6) for $SO(5)$ with a massive adjoint hypermultiplet. Using the change of variables (2.20)

relating the moduli of $SO(5)$ and $Sp(4)$, one can show that the two results agree up to a rescaling, and a moduli-independent additive constant. This is actually a consistency check of our methods rather than a truly independent test.

4.7. $Sp(2N) + 1$ antisymmetric $+ N_f$ fundamentals

Consider $Sp(2N)$ gauge theory with a matter hypermultiplet (mass m) in the antisymmetric representation, and $N_f \leqslant 4$ matter hypermultiplets (masses M_j) in the fundamental representation. The M-theory picture in Fig. 7 contains O6$^-$ planes on top of each NS 5-brane, O6$^-$ planes between each pair of NS 5-branes, together with D4-branes and D6-branes. As in previous examples, there are additional D4-branes and D6-branes at mirror symmetric positions with respect to the O6$^-$ orientifolds that are not depicted in Fig. 7.

There is one NS 5-brane per unit cell ($r = 2$), so we choose a parametrization for the coefficient functions obeying (4.5), yielding the curve

$$\sum_{n=-\infty}^{\infty} L^{2n^2} t^n J_0(x - nm) P_0(x - nm) = 0, \tag{4.7.1}$$

where $L^2 = \Lambda^{4-N_f}$, with the D6-branes responsible for the function

$$J_0(x) = \prod_{j=1}^{N_f} \prod_{p=1}^{\infty} \big[(x - pm - M_j)(x - pm + M_j)(x + pm - M_j) $$
$$\times (x + pm + M_j)\big]^{p/2}, \tag{4.7.2}$$
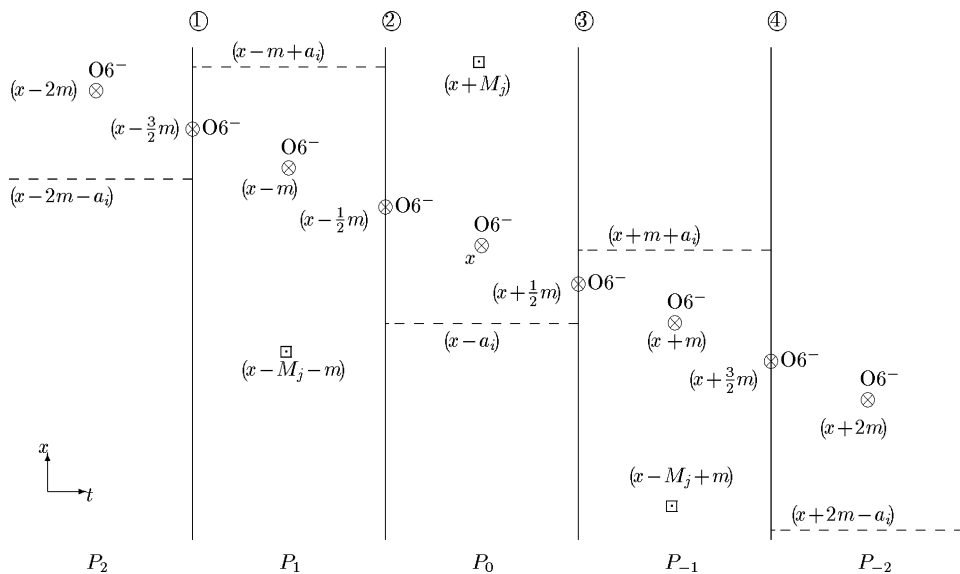
and the leading term (in L) of $P_0(x)$ given by

Fig. 7.

$$P_0(x) = \prod_{p=1}^{\infty} \left[\left(x + (p - \tfrac{1}{2})m \right) \left(x - (p - \tfrac{1}{2})m \right) (x + pm)(x - pm) \right]^{-2p}$$

$$\times \prod_{i=1}^{N} [x^2 - a_i^2]. \tag{4.7.3}$$

One may verify that in the $m \to \infty$ limit, the curve reduces to that for $Sp(2N)$ with N_f fundamental hypermultiplets.

Using Eq. (4.8), one obtains:

$$S(x) = \frac{\prod_{i=1}^{N}[(x-m)^2 - a_i^2] \prod_{i=1}^{N}[(x+m)^2 - a_i^2] \prod_{j=1}^{N_f}(x^2 - M_j^2)}{x^4 \left(x + \tfrac{1}{2}m \right)^2 \left(x - \tfrac{1}{2}m \right)^2 \prod_{i=1}^{N}(x^2 - a_i^2)^2}. \tag{4.7.4}$$

Using the methods of Section 2, one obtains the one-instanton contribution to the prepotential given by

$$2\pi i \mathcal{F}_{1\text{-inst}} = -2 \left[\bar{S}_0(0) \right]^{1/2}, \quad \text{where} \tag{4.7.5}$$

$$\bar{S}_0(0) = \frac{16 \prod_{i=1}^{N}(m^2 - a_i^2)^2 \prod_{j=1}^{N_f}(-M_j^2)}{(-m^2)^2 \prod_{i=1}^{N}(-a_i^2)^2}. \tag{4.7.6}$$

Several checks may be made of this result. For $Sp(2)$, Eqs. (4.7.5) and (4.7.6) yield

$$2\pi i \mathcal{F}_{1\text{-inst}} = 8i^{N_f} \frac{\prod_{j=1}^{N_f} M_j}{m^2} - 8i^{N_f} \frac{\prod_{j=1}^{N_f} M_j}{a_1^2}, \tag{4.7.7}$$

what agrees with the one-instanton prepotential for $Sp(2)$ with N_f fundamental hypermultiplets (up to a moduli-independent additive constant and an overall rescaling), as expected, since the antisymmetric representation of $Sp(2)$ is trivial. The result for $Sp(4)$ with 1 antisymmetric hypermultiplet and no fundamental hypermultiplets agrees with the one-instanton prepotential for $SO(5)$ with one fundamental hypermultiplet [63] (see Table 1) up to a rescaling and a moduli independent additive constant, after making the change of variables (2.20).

5. Comparison to elliptic curves with zero global mass

Complementary to our strategy of deriving curves from the M-theory pictures of Uranga [2–18], there exist methods developed for elliptic models by Donagi and Witten [16,17], Uranga [18], Gukov and Kapustin [19], and others. Although a curve results from their considerations, the extraction of the instanton expansion has not been carried out for these curves. In short, the issue is how to extract $P_n(x)$ and $J_n(x)$, and from these, $S(x)$, from the curves of the Donagi–Witten type. In this section, we accomplish this for two nontrivial models with zero beta function and zero global mass.

The curves for theories in Section 4 with only one NS 5-brane per unit cell become trivial (i.e., factorize into a function of x and a function of t [18]) when the global mass m

vanishes. This yields $S(x) = $ constant, and a vanishing one-loop and instanton prepotential. On the other hand, the curves for theories with two NS 5-branes per unit cell, viz., $SU(N)$ with two antisymmetric hypermultiplets and four fundamental hypermultiplets, and $SU(N)$ with one antisymmetric and one symmetric hypermultiplet, both have the nontrivial limit (4.1.6) when the global mass Δ vanishes. We show that the curves obtained by Gukov and Kapustin, and Uranga, respectively, for these two models agree, after a suitable change of variables, with Eq. (4.1.6).

5.1. $SU(N) + 2$ antisymmetric + 4 fundamentals

Consider the $SU(N)$ gauge theory, with two antisymmetric hypermultiplets (masses m_1 and m_2) and four fundamental hypermultiplets (masses M_j), with their masses related by $m_1 = m_2 = 2M_j$. This is an elliptic model with zero global mass. Gukov–Kapustin [19] give the curve for $SU(2n)$

$$v^{2n} + f_1(x, y)v^{2n-1} + \cdots + f_{2n}(x, y) = 0, \tag{5.1.1}$$

with the coefficient functions

$$f_{2j}(x, y) = A_j, \qquad f_{2j-1}(x, y) = \frac{y B_j}{(x - e_3)} = \frac{(x - e_1)(x - e_2)}{y} B_j, \tag{5.1.2}$$

where $v = x_4 + i x_5$ is what we called x in earlier sections, A_j and B_j are constants, and x and y parameterize the torus base space via

$$y^2 = (x - e_1)(x - e_2)(x - e_3). \tag{5.1.3}$$

We will assume that precisely the same functions (5.1.2) appear in the curve for $SU(N)$ with N odd, so the curve for any N takes the form:

$$v^N + f_1(x, y)v^{N-1} + \cdots + f_{N-1}(x, y)v + f_N(x, y) = 0. \tag{5.1.4}$$

Let

$$H_0(v) = \prod_{j=1}^{N} \left(v - a_j - \tfrac{1}{2}m\right) = \sum_{j=0}^{N} u_j v^{N-j},$$

$$H_1(v) = H_0(-v) = (-1)^N \prod_{j=1}^{N} \left(v + a_j + \tfrac{1}{2}m\right) = (-1)^N \sum_{j=0}^{N} (-1)^j u_j v^{N-j}, \tag{5.1.5}$$

which defines the u_j as gauge invariant combinations of m and the order parameters a_j (with $u_0 = 1$). Let also

$$H_{\text{even}} = \sum_{i \text{ even}} u_i v^{N-i}, \qquad H_{\text{odd}} = \sum_{i \text{ odd}} u_i v^{N-i},$$

$$H_0 = H_{\text{even}} + H_{\text{odd}}, \qquad (-1)^N H_1 = H_{\text{even}} - H_{\text{odd}}. \tag{5.1.6}$$

Then (5.1.4) can be written as

$$H_{\text{even}}(v) + \frac{(x - e_1)(x - e_2)}{y} H_{\text{odd}}(v) = 0, \tag{5.1.7}$$

where we identify $u_{2j} = A_j$, and $u_{2j-1} = B_j$.

When $\sum_i e_i = 0$ in Eq. (5.1.3), the variables x and y are related to the Weierstrass elliptic functions by

$$x = \wp(z), \qquad 2y = \wp'(z), \tag{5.1.8}$$

where z parametrizes the base torus with the identifications $z \equiv z + 2\omega_1 \equiv z + 2\omega_2$, and therefore

$$\frac{y}{(x-e_1)(x-e_2)} = -ic\frac{\theta_1(v|\tau)\theta_4(v|\tau)}{\theta_2(v|\tau)\theta_3(v|\tau)} = -ic\frac{\theta_1\left(v\left|\frac{\tau}{2}\right.\right)}{\theta_2\left(v\left|\frac{\tau}{2}\right.\right)} = cf(v|\tau), \tag{5.1.9}$$

where $v = z/2\omega_1$ and

$$c = -\frac{2i\omega_1}{\pi\theta_4(0|\tau)^2}, \tag{5.1.10}$$

$$f(v|\tau) = -i\frac{\theta_1\left(v\left|\frac{\tau}{2}\right.\right)}{\theta_2\left(v\left|\frac{\tau}{2}\right.\right)}, \tag{5.1.11}$$

and (see, for example, [75])

$$\theta_1(v|\tau) = i\sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n-1/2)^2} e^{2\pi iv(n-1/2)},$$

$$\theta_2(v|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n-1/2)^2} e^{2\pi iv(n-1/2)},$$

$$\theta_3(v|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi ivn}, \tag{5.1.12}$$

$$\theta_4(v|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau n^2} e^{2\pi ivn}.$$

In view of Eqs. (5.1.9)–(5.1.11) we can rewrite the curve (5.1.7) as follows:

$$H_{\text{even}}(v) + \left[cf(v|\tau)\right]^{-1} H_{\text{odd}}(v) = 0. \tag{5.1.13}$$

One may redefine the gauge invariant moduli as follows:

$$u'_j = \begin{cases} u_j, & j \text{ even}, \\ u_j/c, & j \text{ odd}, \end{cases} \tag{5.1.14}$$

which is valid since (5.1.10) is independent of v and v. Next, we shift $v \to v + \frac{\tau}{4} + \frac{1}{2}$ if N is even, and $v \to v + \frac{\tau}{4}$ if N is odd. Since

$$f\left(v+\frac{\tau}{4}\Big|\tau\right) = \frac{\theta_4\left(v\left|\frac{\tau}{2}\right.\right)}{\theta_3\left(v\left|\frac{\tau}{2}\right.\right)}, \qquad f\left(v+\frac{\tau}{4}+\frac{1}{2}\Big|\tau\right) = \frac{\theta_3\left(v\left|\frac{\tau}{2}\right.\right)}{\theta_4\left(v\left|\frac{\tau}{2}\right.\right)}, \tag{5.1.15}$$

this converts the curve (5.1.13) into

$$\begin{cases} \theta_3\left(v\left|\frac{\tau}{2}\right.\right)H_{\text{even}}(v) + \theta_4\left(v\left|\frac{\tau}{2}\right.\right)H_{\text{odd}}(v) = 0, & N \text{ even}, \\ \theta_4\left(v\left|\frac{\tau}{2}\right.\right)H_{\text{even}}(v) + \theta_3\left(v\left|\frac{\tau}{2}\right.\right)H_{\text{odd}}(v) = 0, & N \text{ odd}. \end{cases} \tag{5.1.16}$$

Using the identities

$$\theta_3\left(v\Big|\frac{\tau}{2}\right) = \theta_3(2v|2\tau) + \theta_2(2v|2\tau),$$
$$\theta_4\left(v\Big|\frac{\tau}{2}\right) = \theta_3(2v|2\tau) - \theta_2(2v|2\tau),$$

(5.1.17)

Eq. (5.1.16) becomes, for both even or odd N,

$$H_0(v)\theta_3(2v|2\tau) + H_1(v)\theta_2(2v|2\tau) = 0.$$

(5.1.18)

This result exactly agrees with our result (4.1.6), when we set $v = x + \frac{1}{2}m$. This supports the validity of the methods used in obtaining the curves in Section 4. Further, we know that for this case of zero global mass, the functions $P_n(x)$ do *not* have any subleading terms, from two points of view: (a) the exact agreement of the leading terms with Eq. (5.1.18), and (b) the absence of "subtractions" for the one-instanton prepotential.

5.2. $SU(N) + 1$ antisymmetric $+ 1$ symmetric

For zero global mass, Uranga [18] gives the curve (5.1.4) with

$$f_{2j}(x, y) = C_j, \qquad f_{2j-1}(x, y) = \frac{yD_j}{(x - e_1)(x - e_2)} = \frac{(x - e_3)}{y}D_j.$$

(5.2.1)

The curve for this case, with $m_1 = m_2$, is therefore

$$H_{\text{even}}(v) + \frac{y}{(x - e_1)(x - e_2)}H_{\text{odd}}(v) = 0,$$

(5.2.2)

where we identify $u_{2j} = C_j$ and $u_{2j-1} = D_j$. Following (5.1.8)–(5.1.13), we have:

$$H_{\text{even}}(v) + cf(v|\tau)H_{\text{odd}}(v) = 0,$$

(5.2.3)

with c and $f(v|\tau)$ defined by Eqs. (5.1.10) and (5.1.11), respectively. We redefine the gauge invariant moduli as

$$u'_j = \begin{cases} u_j, & j \text{ even}, \\ cu_j, & j \text{ odd}. \end{cases}$$

(5.2.4)

Further, we shift $v \to v + \frac{\tau}{4}$ if N is even, and $v \to v + \frac{\tau}{4} + \frac{1}{2}$ if N is odd. (Note that the shifts for even and odd N are the reverse of those in the previous section.) This transforms the curve (5.2.3) into Eq. (5.1.18), agreeing with the result (4.1.6) obtained in Section 4, with no subleading terms for the $P_n(x)$.

Thus, for $SU(N) + 2$ anti. $+ 4$ fund. and $SU(N) + 1$ sym. $+ 1$ anti. with zero global mass in both cases (and $m = 2M_j$ for the former), the curves are identical, i.e., the two theories have identical prepotentials. This was already noted in Table 4, at the one-instanton level. One may wonder from the M-theory point of view why this has occurred. From Fig. 1 for $SU(N) + 2$ anti. $+ 4$ fund., we see that, if all the masses are equal, the positions of the D6-branes have the same value of v (x in the figure) as that of the two O6$^-$ orientifold planes. One may then bring the four D6-branes (plus mirrors) coincident with one of the O6$^-$ planes, converting this effectively to an O6$^+$ plane. The resulting configuration is that

of $SU(N) + 1$ sym. $+ 1$ anti. for zero global mass. Hence, the identity of the low-energy theories with zero global mass could have been anticipated. On the other hand, if there are global masses, this construction is not possible and the curves no longer coincide.

6. Comparison with the curve of Donagi and Witten

The M-theory picture corresponding to the $SU(N)$ gauge theory with massive adjoint hypermultiplet [2] is described in Section 4.3 (see Fig. 3). Using this, we obtained the curve (4.3.3):

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} e^{\beta n z} H(v - nm) = 0, \quad \text{where} \tag{6.1}$$

$$H(v) = \prod_{i=1}^{N} (v - a_i), \tag{6.2}$$

with $v = x_4 + i x_5$ (previously referred to as x), and $q = e^{2\pi i \tau}$. As before, z parametrizes the base torus with the identifications $z \equiv z + 2\omega_1 \equiv z + 2\omega_2$, and throughout this section we fix $\omega_1 = -\pi i$ (hence $\beta = 1$) for convenience. This is exactly the curve derived in Refs. [65,76,77], for the Calogero–Moser model.

On the other hand, Witten [2] shows that the curve for this model is precisely that derived by Donagi and Witten [16,17] in the context of the integrable Hitchin system:

$$F(v, x, y) = \sum_{j=0}^{N} A_j P_{N-j}(v), \tag{6.3}$$

where A_j are gauge invariant polynomials in a_i and m, and where x and y are related by (5.1.3). They show that

$$P_n(v) = \sum_{i=0}^{n} \binom{n}{i} f_i v^{n-i}. \tag{6.4}$$

It can be shown that

$$F(v, x, y) = \sum_{j=0}^{N} \frac{m^j}{j!} f_j H^{(j)}(v), \quad \text{where} \tag{6.5}$$

$$H^{(j)}(v) = \frac{d^j H(v)}{dv^j}. \tag{6.6}$$

Explicit calculation gives, using $\sum_i e_i = 0$ in (5.1.3):

$$\begin{array}{llll} f_0 = 1, & f_1 = 0, & f_2 = -x, \\ f_3 = 2y, & f_4 = -3x^2, & f_5 = 4xy, & \text{etc.} \end{array} \tag{6.7}$$

We will establish the connection between the curves (6.1) and (6.5), which has not been done explicitly previously. The curve (6.1) can be recast as

$$\sum_{j=0}^{N} \frac{(-m)^j}{j!} h_j(z) H^{(j)}\big(v - \tfrac{1}{2}m\big) = 0, \quad \text{where} \tag{6.8}$$

$$h_j(z) = \frac{1}{\theta_1(\frac{z}{-2\pi i}|\tau)} \frac{\partial^j}{\partial z^j} \theta_1\Big(\frac{z}{-2\pi i}\Big|\tau\Big), \tag{6.9}$$

with θ_1 defined in Eq. (5.1.12). Making the change of variables

$$v \longrightarrow v + m h_1(z) + \tfrac{1}{2}m, \tag{6.10}$$

Eq. (6.8) becomes

$$\sum_{j=0}^{N} \frac{m^j}{j!} \tilde{f}_j(z) H^{(j)}(v) = 0, \quad \text{with} \tag{6.11}$$

$$\tilde{f}_j(z) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i h_i(z) h_1(z)^{j-i}. \tag{6.12}$$

Explicitly:

$$
\begin{aligned}
\tilde{f}_0 &= h_0 = 1, \\
\tilde{f}_1 &= 0, \\
\tilde{f}_2 &= h_2 - h_1^2 = h_1' = -\wp(z) - \frac{\pi^2}{12\omega_1^2} E_2 = -\wp(z) + \frac{1}{12} E_2, \\
\tilde{f}_3 &= -h_3 + 3h_2 h_1 - 2h_1^3 = -h_1'' = 2y, \\
\tilde{f}_4 &= h_4 - 4h_3 h_1 + 6h_2 h_1^2 - 3h_1^4 = -3x^2 - \tfrac{1}{2} E_2 x + \tfrac{1}{48} E_2^2 + \tfrac{1}{2} g_2, \\
\tilde{f}_5 &= -h_5 + 5h_4 h_1 - 10 h_3 h_1^2 + 10 h_2 h_1^3 - 4h_1^5 = 4xy + \tfrac{5}{3} E_2\, y, \quad \text{etc.,}
\end{aligned} \tag{6.13}
$$

where E_2 is the Eisenstein series of weight two, and g_2 is defined via the Weierstrass form

$$y^2 = x^3 - \tfrac{1}{4} g_2 x - \tfrac{1}{4} g_3, \tag{6.14}$$

of the elliptic curve (5.1.3). The relations (5.1.8) have been used in computing (6.13).

Comparing (6.13) with (6.7), we find that f_j and $\tilde{f}_j$ differ by the τ-dependent change of basis:

$$
\begin{aligned}
\tilde{f}_0 &= f_0, \\
\tilde{f}_1 &= f_1, \\
\tilde{f}_2 &= f_2 + \tfrac{1}{12} E_2(\tau) f_0, \\
\tilde{f}_3 &= f_3, \\
\tilde{f}_4 &= f_4 + \tfrac{1}{2} E_2(\tau) f_2 + \big[\tfrac{1}{48} E_2(\tau)^2 + \tfrac{1}{2} g_2\big] f_0, \\
\tilde{f}_5 &= f_5 + \tfrac{5}{6} E_2(\tau) f_3, \quad \text{etc.}
\end{aligned} \tag{6.15}
$$

This is similar but not identical to the comparison of the spectral curve with the Donagi–Witten curve made by Itoyama and Morozov [78], to which we refer the reader for further discussion of this issue.

In this section and in Section 5 we have dealt with elliptic models that have no subleading terms for the coefficient functions $P_n(x)$. It remains an open question how to carry out the analogous studies for elliptic models with a global mass and nonvanishing subleading terms.

7. Concluding remarks

In this paper, we have provided a rather comprehensive presentation of the relationship of elliptic models to M-theory, and related topics. There are, however, a number of topics which deserve further attention, as they represent issues not completely understood.

First, the information in Tables 1 and 3 would appear to have an underlying group-theoretic explanation. That is, given the group and matter content, one should be able to predict the factors in Table 3, without appealing to a SW curve. We know of no such explanation.

Second, for a number of elliptic models, only the leading terms of the coefficient functions $P_n(x)$ are known. (The leading term is that with the lowest power of Λ or q for a given power of t.) This occurs for all models in Section 4 with nonzero global mass except $SU(N) +$ adjoint. In these models, the subleading terms are not known. Some non-elliptic models, namely $SU(N) + 1$ anti. $+ N_f$ fund. and $Sp(2N) + N_f$ fund. (treated in Section 2), also contain subleading terms, which are explicitly known in these cases. Comparison with Table 2 correlates models with subleading terms with entries in the table in which $\mathcal{F}_{1\text{-inst}}$ involves terms other than $S_k(a_k)$ (e.g., $[\bar{S}_0(0)]^{1/2}$, or subtractions of factors such as $S_m(-\frac{1}{2}m)$). These additional terms are closely connected to the residue functions $R_k(x)$, e.g., Eq. (2.8), which originate in the subleading terms, and which generate the additional terms via identities such as Eq. (2.14) (leading in the case of $Sp(2N) + N_f$ fund. to Eq. (2.17)). Thus, the existence of subleading terms and the need for the subtractions listed in Table 2 are different aspects of the same issue. The subtractions listed in Table 2 for models described in Sections 4.1, 4.2, and 4.4 through 4.7, do not, however, give enough information to determine the subleading terms. A greater understanding of these subleading terms would be desirable.

Finally, there are a number of elliptic models described in Section 4 which have not been identified with known integrable models. It would improve our understanding of the subject if these connections could be made.

In conclusion, this paper represents considerable progress toward a unified description of elliptic models associated with M-theory. Nevertheless, as outlined above, there remain a number of interesting issues to consider.

Acknowledgements

References

[1] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19; Erratum, Nucl. Phys. B 430 (1994) 485; hep-th/9407087; Nucl. Phys. B 431 (1994) 484; hep-th/9408099.

[2] E. Witten, Nucl. Phys. B 500 (1997) 3; hep-th/9703166.

[3] S. Kachru, C. Vafa, Nucl. Phys. B 450 (1995) 69; hep-th/9505105.

[4] S. Kachru, A. Klemm, W. Lerche, P. Mayr, C. Vafa, Nucl. Phys. B 459 (1996) 537; hep-th/9508155.

[5] M. Bershadsky, V. Sadov, C. Vafa, Nucl. Phys. B 463 (1996) 420; hep-th/9511222.

[6] A. Klemm, P. Mayr, Nucl. Phys. B 469 (1996) 37; hep-th/9601014.

[7] S. Katz, D. Morrison, M. Plesser, Nucl. Phys. B 477 (1996) 105; hep-th/9601108.

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, Nucl. Phys. B 477 (1996) 746; hep-th/9604034.

[9] P. Berglund, S. Katz, A. Klemm, P. Mayr, Nucl. Phys. B 483 (1997) 209; hep-th/9605154.

[10] S. Katz, A. Klemm, C. Vafa, Nucl. Phys. B 497 (1997) 173; hep-th/9609239.

[11] S. Katz, P. Mayr, C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53; hep-th/9706110.

[12] M. Aganagic, M. Gremm, Nucl. Phys. B 524 (1998) 207; hep-th/9712011.

[13] For reviews see: W. Lerche, Nucl. Phys. Proc. Suppl. B 55 (1997) 83; Fortsch. Phys. 45 (1997) 293; hep-th/9611190.

[14] A. Klemm, hep-th/9705131.

[15] S. Katz, P. Mayr, C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53; hep-th/9706110.

[16] R. Donagi, E. Witten, Nucl. Phys. B 460 (1996) 299; hep-th/9510101.

[17] R. Donagi, alg-geom/9705010.

[18] A. Uranga, Nucl. Phys. B 526 (1998) 241; hep-th/9803054.

[19] S. Gukov, A. Kapustin, Nucl. Phys. B 545 (1999) 283; hep-th/9808175.

[20] A. Gorsky, S. Gukov, A. Mironov, Nucl. Phys. B 517 (1998) 409; hep-th/9707120; Nucl. Phys. B 518 (1998) 689; hep-th/9710239.

[21] S. Naculich, H. Rhedin, H. Schnitzer, Nucl. Phys. B 533 (1998) 275; hep-th/9804105.

[22] I. Ennes, S. Naculich, H. Rhedin, H. Schnitzer, Int. J. Mod. Phys. A 14 (1999) 301; hep-th/9804151.

[23] I. Ennes, S. Naculich, H. Rhedin, H. Schnitzer, Nucl. Phys. B 536 (1998) 245; hep-th/9806144.

[24] I. Ennes, S. Naculich, H. Rhedin, H. Schnitzer, Phys. Lett. B 452 (1999) 260; hep-th/9901124.

[25] I. Ennes, S. Naculich, H. Rhedin, H. Schnitzer, Nucl. Phys. B 558 (1999) 41; hep-th/9904078.

[26] I. Ennes, S. Naculich, H. Rhedin, H. Schnitzer, hep-th/9911022.

[27] D. Finnell, P. Pouliot, Nucl. Phys. B 453 (1995) 225; hep-th/9503115.

[28] F. Fucito, G. Travaglini, Phys. Rev. D 155 (1997) 1099; hep-th/9605215.

[29] N. Dorey, V. Khoze, M. Mattis, Phys. Rev. D 54 (1996) 2921; hep-th/9603136; Phys. Lett. B 390 (1997) 205; hep-th/9606199; Phys. Lett. B 388 (1996) 324; hep-th/9607066; Phys. Rev. D 54 (1996) 7832; hep-th/9607202.

[30] K. Ito, N. Sasakura, Phys. Lett. B 382 (1996) 95; hep-th/9602073; Mod. Phys. Lett. A 12 (1997) 205; hep-th/9609104.

[31] H. Aoyama, T. Harano, M. Sato, S. Wada, Phys. Lett. B 388 (1996) 331; hep-th/9607076.

[32] T. Harano, M. Sato, Nucl. Phys. B 484 (1997) 167; hep-th/9608060.

[33] Y. Yoshida, hep-th/9610211.

[34] M. Slater, Phys. Lett. B 403 (1997) 57; hep-th/9701170.

[35] V. Khoze, M. Mattis, M. Slater, Nucl. Phys. B 536 (1998) 69; hep-th/9804009.

[36] I. Sacho, W. Weir, hep-th/9910015.

[37] M. Slater, hep-th/9912011.

[38] K. Landsteiner, E. Lopez, Nucl. Phys. B 516 (1998) 273; hep-th/9708118.

[39] K. Landsteiner, E. Lopez, D. Lowe, J. High Energy Phys. 07 (1998) 011; hep-th/9805158.

[40] E. D'Hoker, D.H. Phong, hep-th/9912271.

[41] A. Hanany, E. Witten, Nucl. Phys. B 492 (1997) 152; hep-th/9611230.

[42] N. Evans, C. Johnson, A. Shapere, Nucl. Phys. B 505 (1997) 251; hep-th/9703210.

[43] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, Nucl. Phys. B 502 (1997) 125; hep-th/9704044; Nucl. Phys. B 504 (1997) 175; hep-th/9705232.

[44] K. Landsteiner, E. Lopez, D. Lowe, Nucl. Phys. B 507 (1997) 197; hep-th/9705199.

[45] S. Terashima, S.-K. Yang, Phys. Lett. B 430 (1998) 102; hep-th/9803014.

[46] A. Klemm, W. Lerche, S. Theisen, S. Yankielowicz, Phys. Lett. B 344 (1995) 169; hep-th/9411048.

[47] P. Argyres, A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931; hep-th/9411057.

[48] M. Douglas, S. Shenker, Nucl. Phys. B 447 (1995) 271; hep-th/9503163.

[49] U. Danielsson, B. Sundborg, Phys. Lett. B 358 (1995) 273; hep-th/9504102; Phys. Lett. B 370 (1996) 83; hep-th/9511180.

[50] P. Argyres, M. Douglas, Nucl. Phys. B 448 (1995) 93; hep-th/9505062.

[51] A. Hanany, Y. Oz, Nucl. Phys. B 452 (1995) 73; hep-th/9505075.

[52] P. Argyres, M. Plesser, A. Shapere, Phys. Rev. Lett. B 75 (1995) 1699; hep-th/9505100.

[53] A. Klemm, W. Lerche, S. Theisen, Int. J. Mod. Phys. A 11 (1996) 1929; hep-th/9505150.

[54] I. Pesando, Mod. Phys. Lett. A 10 (1995) 1871; hep-th/9506139.

[55] A. Brandhuber, K. Landsteiner, Phys. Lett. B 358 (1995) 73; hep-th/9507008.

[56] J. Minahan, D. Nemeschansky, Nucl. Phys. B 464 (1996) 3; hep-th/9507032.

[57] P. Argyres, A. Shapere, Nucl. Phys. B 461 (1996) 437; hep-th/9509175.

[58] A. Hanany, Nucl. Phys. B 466 (1996) 85; hep-th/9509176.

[59] M. Alishahiha, F. Ardalan, F. Mansouri, Phys. Lett. B 381 (1996) 446; hep-th/9512005.

[60] M. Abolhasani, M. Alishahiha, A. Ghezelbash, Nucl. Phys. B 480 (1996) 279; hep-th/9606043.

[61] I. Krichever, D.H. Phong, hep-th/9912180.

[62] E. D'Hoker, I.M. Krichever, D.H. Phong, Nucl. Phys. B 489 (1997) 179; hep-th/9609041.

[63] E. D'Hoker, I.M. Krichever, D.H. Phong, Nucl. Phys. B 489 (1997) 211; hep-th/9609145.

[64] E. D'Hoker, I.M. Krichever, D.H. Phong, Nucl. Phys. B 494 (1997) 89; hep-th/9610156.

[65] E. D'Hoker, D.H. Phong, Phys. Lett. B 397 (1997) 94; hep-th/9701055; Nucl. Phys. B 530 (1998) 537; hep-th/9804124; Nucl. Phys. B 530 (1998) 611; hep-th/9804125; Nucl. Phys. B 534 (1998) 697; hep-th/9804126; hep-th/9808156; hep-th/9903002; hep-th/9903068; hep-th/9906027.

[66] E. D'Hoker, D.H. Phong, Nucl. Phys. B 513 (1998) 405; hep-th/9709053.

[67] T. Masuda, H. Suzuki, Int. J. Mod. Phys. A 13 (1998) 1495; hep-th/9609065.

[68] K. Ito, N. Sasakura, Nucl. Phys. B 484 (1997) 141; hep-th/9608054.

[69] G. Chan, E. D'Hoker, hep-th/9906193.

[70] M. Douglas, D. Lowe, J. Schwarz, Phys. Lett. B 394 (1997) 297; hep-th/9612062.

[71] A. Hanany, E. Witten, Nucl. Phys. B 492 (1997) 152; hep-th/9611230.

[72] T. Yokono, Nucl. Phys. B 532 (1998) 210; hep-th/9803123.

[73] J.A. Minahan, D. Nemeschansky, N.P. Warner, Nucl. Phys. B 528 (1998) 109; hep-th/9710146.

[74] N. Evans, C.V. Johnson, A.D. Shapere, Nucl. Phys. B 505 (1997) 251; hep-th/9703210.

[75] J. Polchinski, String Theory, Vol. I, Cambridge University Press, Cambridge, 1998, p. 215.

[76] E. Martinec, N. Warner, Nucl. Phys. B 459 (1996) 97; hep-th/9509161.

[77] E. Martinec, Phys. Lett. B 367 (1996) 91; hep-th/9510204.

[78] H. Itoyama, A. Morozov, Nucl. Phys. B 477 (1996) 855; hep-th/9511126.