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A cascading $\mathcal{N} = 1$ $\text{Sp}(2N + 2M) \times \text{Sp}(2N)$ gauge theory

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Abstract

We study the $\mathcal{N} = 1$ $\text{Sp}(2N + 2M) \times \text{Sp}(2N)$ cascading gauge theory on a stack of N physical and M fractional (half) D3-branes at the singularity of an orientifolded conifold. In addition to the D3-branes and an $O7$ -plane, the background contains eight D7-branes, which give rise to matter in the fundamental representation of the gauge group. The moduli space of the gauge theory is analyzed and its structure is related to the brane configurations in the dual type IIB theory and in type IIA/M-theory. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The desire to extend the original AdS/CFT correspondence [1] to examples with less supersymmetry has prompted the study of branes at conical singularities. An important example is that of N D3-branes at the singularity of the conifold [2]. The resulting four-dimensional $\mathcal{N} = 1$ gauge theory has gauge group $\text{SU}(N) \times \text{SU}(N)$ and chiral matter multiplets in the bifundamental representations of the gauge group. The addition of M fractional D3-branes changes the gauge group to $\text{SU}(N + M) \times \text{SU}(N)$ [3] (other models within the same universality class have also recently attracted attention, see, e.g., [4]). This

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non-conformal theory exhibits a duality cascade [5]

$$\mathrm{SU}(N + M) \times \mathrm{SU}(N) \rightarrow \mathrm{SU}(N - M) \times \mathrm{SU}(N) \rightarrow \cdots \rightarrow \mathrm{SU}(M + p) \times \mathrm{SU}(p), \quad (1.1)$$

with $1 \leq p \leq M$, where the simplest case is $p = 1$ for which one finds an $\mathrm{SU}(M + 1) \times \mathrm{SU}(1) \cong \mathrm{SU}(M + 1)$ theory at the end of the cascade.

A richer example is that of D3-branes at the singularity of an orientifolded conifold (where the orientifold arises from an O7-plane together with 8 D7-branes required for consistency) leading to an $\mathcal{N} = 1$ $\mathrm{Sp}(2N) \times \mathrm{Sp}(2N)$ gauge theory with matter in the bifundamental and fundamental representations of the gauge group [6]. The addition of M fractional D3-branes changes the gauge group to $\mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N)$ and leads to a cascade

$$\begin{aligned} \mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N) &\rightarrow \mathrm{Sp}(2N - 2M) \times \mathrm{Sp}(2N) \\ &\rightarrow \cdots \rightarrow \mathrm{Sp}(2M + 2p) \times \mathrm{Sp}(2p). \end{aligned} \quad (1.2)$$

At the end of the cascade one arrives at an $\mathrm{Sp}(2M + 2p) \times \mathrm{Sp}(2p)$ gauge theory, where $2p \leq 2M$, the simplest case being $2p = 2$.

In this paper we study the $\mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N)$ gauge theory on a stack of N physical D3-branes and M fractional (half) D3-branes placed at the singularity of the orientifolded conifold mentioned above. This field theory is dual to type IIB string theory on $\mathrm{AdS}_5 \times T^{11}/\mathbb{Z}_2$ where the \mathbb{Z}_2 is an orientifold operation described in more detail later. In the dual theory the N D3-branes are replaced by an \mathbb{Z}_5 flux on T^{11}/\mathbb{Z}_2 and the M fractional branes are replaced by an F_3 flux on an S^3/\mathbb{Z}_2 inside T^{11}/\mathbb{Z}_2 . This model, which is a natural extension of previously studied models [5,7,8], is interesting because the D7-branes give rise to matter fields in the field theory transforming in the fundamental representation of the gauge group, which leads to an intricate Higgs branch structure of the moduli space of the theory.

This paper is organized as follows. In Section 2 we briefly review the relevant orientifolded conifold theories, while in Section 3 we describe some aspects of the cascade of the $\mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N)$ theory and check that the Klebanov–Strassler supergravity solution [5] is also a solution in the orientifolded theory. Section 4 is devoted to a study of the (classical) moduli space of the $\mathrm{Sp}(2N_1) \times \mathrm{Sp}(2N_2)$ gauge theory with chiral multiplets in both the fundamental and bifundamental representations of the gauge group. The analysis of this section sets the stage for the more detailed analysis in Section 5 of the full quantum moduli space of the $\mathrm{Sp}(2M + 2) \times \mathrm{Sp}(2)$ theory at the end of the duality cascade. We carry out the analysis in Section 5 in two steps, first describing the classical moduli space and then the quantum moduli space. We also discuss the interpretation of the moduli space in terms of the dual string theory. We find that the classical and quantum solutions join smoothly and that a deformation of the conifold arises in the quantum theory as expected. In Section 6 we discuss the interpretation of the moduli space in terms of type IIA and M-theory brane configurations. In Section 7 we summarize our findings.

2. Orientifolded conifold theories

The $\mathcal{N} = 1$ $SU(N) \times SU(N)$ superconformal gauge theory with chiral multiplets in the $2(\mathbf{1}, \overline{\mathbf{1}}) \oplus 2(\overline{\mathbf{1}}, \mathbf{1})$ bifundamental representations arises as the low energy limit of the world-volume theory on N D3-branes at a conifold singularity [2,9]. The conifold [10] can be described as the subspace of \mathbb{C}^4 defined by the equation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$. Via a linear change of basis the conifold can also be written $xy = wz$. The base of the conifold, obtained by intersecting the above space with $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1$, is $T^{11} = [SU(2) \times SU(2)]/U(1)$. A striking example of the AdS/CFT correspondence [1] is the duality between this field theory, and type IIB string theory on $AdS_5 \times T^{11}$ [2].

Orientifolds of the conifold lead to further examples of the AdS/CFT correspondence. The following two models arise as the low-energy theories on the D3-branes in a conifold background with an orientifold \mathbb{Z}_2 symmetry that does not break any supersymmetry, and are dual to type IIB string theory on $AdS_5 \times T^{11}/\mathbb{Z}_2$ [6,8]:

$$\begin{aligned} \text{(i)} \quad & Sp(2N) \times Sp(2N), \quad \text{with } 2(\mathbf{1}, \mathbf{1}) \oplus 4(\mathbf{1}, 1) \oplus 4(1, \mathbf{1}), \\ \text{(ii)} \quad & Sp(2N) \times SO(2N + 2), \quad \text{with } 2(\mathbf{1}, \mathbf{1}). \end{aligned} \tag{2.1}$$

The form of the \mathbb{Z}_2 action on the conifold can be determined from the corresponding IIA brane configurations (see Section 6 for further details).

For model (i), the action of the orientifold on the conifold becomes [6] $z \leftrightarrow w$, with x, y invariant, or, equivalently, $(z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, z_3, -z_4)$. The fixed point set of this action is the $w = z$ subspace of the conifold, whose three-dimensional intersection with $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1$ was called X_3 in Ref. [6]. The model thus contains an O7-plane, and also for consistency 8 D7-branes. The world volume of the O7-plane and D7-branes is $AdS_5 \times X_3$.

For model (ii), the orientifold action can be shown to be $x \rightarrow -x, y \rightarrow -y$ and $z \leftrightarrow w$, or, equivalently, $(z_1, z_2, z_3, z_4) \rightarrow (-z_1, -z_2, z_3, z_4)$ using the approach of [6]. This result was recently obtained in [8] using a slightly different approach. This action has no fixed points inside T^{11} , so model (ii) has no orientifold-planes or D7-branes.

For other discussions of various orientifolds of the conifold, see, e.g., [11–14].

3. Cascading theories

Generalizations of the orientifolded models considered in the previous section may be obtained by including fractional D3-branes at the conifold singularity, breaking the superconformal invariance. The addition of the fractional branes increases the rank of first factor in the product gauge groups. The resulting field theories have gauge groups $Sp(2N + 2M) \times Sp(2N)$ and $Sp(2N + 2M) \times SO(2N + 2)$, respectively.

The lack of conformal invariance causes the gauge couplings to run. The first factor of the $Sp(2N + 2M) \times Sp(2N)$ theory has effectively $2N_f = 4N + 4$ fields in the fundamental representation: four from the fundamentals, and $4N$ from the bifundamentals. The beta function is therefore negative, and the coupling becomes strong in the infrared. Seiberg duality [15] can be used to transform this to another weakly-coupled gauge theory. Seiberg duality relates a strongly-coupled $Sp(2N_c)$ theory with $2N_f$ chiral superfields in the

fundamental representation to a weakly-coupled $\text{Sp}(2N_f - 2N_c - 4)$ theory with the same number of fundamental superfields [16]. In our case, Seiberg duality implies

$$\text{Sp}(2N + 2M) \times \text{Sp}(2N) \rightarrow \text{Sp}(2N - 2M) \times \text{Sp}(2N). \quad (3.1)$$

In the new theory, the gauge coupling of the second group factor now becomes strong in the infrared, leading to a second duality transformation. This process continues, leading to a duality cascade

$$\begin{aligned} \text{Sp}(2N + 2M) \times \text{Sp}(2N) &\rightarrow \text{Sp}(2N - 2M) \times \text{Sp}(2N) \\ &\rightarrow \text{Sp}(2N - 2M) \times \text{Sp}(2N - 4M) \rightarrow \dots \end{aligned} \quad (3.2)$$

just as in the case of the $\text{SU}(N + M) \times \text{SU}(N)$ theory [5]. At the end of the cascade one arrives at a $\text{Sp}(2M + 2p) \times \text{Sp}(2p)$ theory, where $2p \leq 2M$. A similar cascading phenomenon was shown for the $\text{Sp}(2N + 2M) \times \text{SO}(2N + 2)$ case in Refs. [7,8].

The dual supergravity solution describing the cascade of the $\text{SU}(N + M) \times \text{SU}(N)$ model was found in [5] (following earlier work in [3]); see also [17]. At the end of the cascade the conifold is replaced by its deformed version. The solution in [5] is also a solution of the orientifolded theory dual to the $\text{Sp} \times \text{Sp}$ gauge theory (for the theory dual to the $\text{Sp} \times \text{SO}$ gauge theory, this was shown in Ref. [8]). This follows because F_3 and H_3 change sign under the interchange of z and w , whereas the metric and F_5 are invariant. Combining this with the action of $\Omega(-1)^{F_L}$, this shows that all fields are invariant under the orientifold projection.

Based on the properties of the supergravity solution one expects to find the deformed conifold at the end of the flow. To understand the geometry at the end of the flow, we can probe the background with a single D3-brane as in Ref. [5], i.e., we will assume that $p = 1$ (note that the probe brane has a mirror). To probe the theory, therefore, we must analyze the moduli space of the $\text{Sp}(2M + 2) \times \text{Sp}(2)$ gauge theory. This analysis will be carried out in Section 5. First, however, we consider the more general case of the $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ gauge theory moduli space.

4. The $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ gauge theory moduli space

In this section, we analyze the (classical) moduli space of the $\mathcal{N} = 1$ $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ gauge theory with four chiral matter multiplets in the fundamental representation of each factor of the gauge group, and two in the bifundamental representation. To obtain the $\mathcal{N} = 1$ superpotential for this theory, we start with the $\mathcal{N} = 2$ version of the theory, turn on (opposite sign) masses for the adjoint chiral superfields, which breaks the supersymmetry to $\mathcal{N} = 1$, and integrate out the massive fields. This procedure is analogous to the way one obtains the $\mathcal{N} = 1$ $\text{SU}(N_1) \times \text{SU}(N_2)$ theory from its $\mathcal{N} = 2$ cousin [2].

The $\mathcal{N} = 2$ $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ theory can be obtained by orientifolding the $\mathcal{N} = 2$ $\text{SU}(2N_1) \times \text{SU}(2N_2)$ theory. However, for both calculational and notational purposes it is convenient to view the superpotential for the $\mathcal{N} = 2$ $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ theory as arising from that of another $\mathcal{N} = 2$ $\text{SU}(2N_1) \times \text{SU}(2N_2)$ gauge theory, with matter hypermultiplets in both the bifundamental *and* the fundamental representations, by imposing a projection on all the fields.

We therefore consider the $SU(2N_1) \times SU(2N_2)$ theory with two $\mathcal{N} = 2$ vector multiplets in the adjoint representations of $SU(2N_1)$ and $SU(2N_2)$, respectively, two $\mathcal{N} = 2$ hypermultiplets in the bifundamental representations, and also an additional four $\mathcal{N} = 2$ hypermultiplets in the fundamental representation of each gauge group. Our notation is such that a lower/upper index $a = 1, \dots, 2N_1$ denotes a component in the fundamental/antifundamental representation of $SU(2N_1)$, and a lower/upper index $\bar{a} = 1, \dots, 2N_2$ denotes a component in the fundamental/antifundamental representation of $SU(2N_2)$. In $\mathcal{N} = 1$ language the two $\mathcal{N} = 2$ vector multiplets consist of two vector multiplets corresponding to the two gauge groups, and two chiral multiplets $\phi_{1a}{}^b$ and $\phi_{2\bar{a}}{}^{\bar{b}}$. The two $\mathcal{N} = 2$ bifundamental hypermultiplets consist of two $\mathcal{N} = 1$ chiral multiplets $A_{i\bar{a}}{}^{\bar{b}}$ ($i = 1, 2$) in the $(\bar{\quad}, \bar{\quad})$ of $SU(2N_1) \times SU(2N_2)$ and two $\mathcal{N} = 1$ chiral multiplets $B_{i\bar{a}}{}^b$ ($i = 1, 2$) in the $(\bar{\quad}, \quad)$ representation. In addition, the four $\mathcal{N} = 2$ multiplets in the fundamental representation consist of four $\mathcal{N} = 1$ chiral multiplets Q_{1a}^I ($I = 1, \dots, 4$) in the $(\quad, 1)$ and four $\mathcal{N} = 1$ chiral multiplets \tilde{Q}_{1I}^a in the $(\bar{\quad}, 1)$, as well as $Q_{2\bar{a}}^I$ and $\tilde{Q}_{2I}^{\bar{a}}$ in the $(1, \quad)$ and $(1, \bar{\quad})$, respectively.

The $\mathcal{N} = 2$ superpotential for this theory is

$$\mathcal{W}_{=2} = \sqrt{2} \{ \text{Tr}[\phi_1(A_1 B_1 + A_2 B_2) + \phi_2(B_1 A_1 + B_2 A_2)] + \tilde{Q}_{1I} \phi_1 Q_1^I - \tilde{Q}_{2I} \phi_2 Q_2^I \}. \tag{4.1}$$

We may reduce the gauge group to $Sp(2N_1) \times Sp(2N_2)$ by imposing the projections

$$\phi_{1a}{}^b = J_{ac} J^{bd} \phi_{1d}{}^c, \quad \phi_{2\bar{a}}{}^{\bar{b}} = J_{\bar{a}\bar{c}} J^{\bar{b}\bar{d}} \phi_{2\bar{d}}{}^{\bar{c}}, \tag{4.2}$$

on the adjoint hypermultiplets (and the vector multiplets). Here J^{ab} and $J^{\bar{a}\bar{b}}$ are the symplectic units of $Sp(2N_1)$ and $Sp(2N_2)$, respectively, which are used to raise and lower indices. Projections on the other hypermultiplet fields

$$\begin{aligned} B_{1\bar{a}}{}^b &= J_{\bar{a}\bar{c}} J^{bd} A_{2d}{}^{\bar{c}}, & B_{2\bar{a}}{}^b &= J_{\bar{a}\bar{c}} J^{bd} A_{1d}{}^{\bar{c}}, \\ \tilde{Q}_{1I}^a &= g_{IJ} J^{ab} Q_{1b}^J, & \tilde{Q}_{2I}^{\bar{a}} &= g_{IJ} J^{\bar{a}\bar{b}} Q_{2\bar{b}}^J, \end{aligned} \tag{4.3}$$

result in the $\mathcal{N} = 2$ $Sp(2N_1) \times Sp(2N_2)$ gauge theory with two $\mathcal{N} = 1$ chiral multiplets in the bifundamental (\quad, \quad) and four $\mathcal{N} = 1$ chiral multiplets in each of the fundamental representations $(\quad, 1)$ and $(1, \quad)$ (as well as chiral multiplets in the adjoint representation of the gauge group). In subsequent calculations we use the explicit basis choices $g_{IJ} = \sigma_x \otimes \epsilon_{2 \times 2}$, and $J^{ab} = i\sigma_y \otimes \epsilon_{N_1 \times N_1}$ (and similarly for $J^{\bar{a}\bar{b}}$). In more readable matrix notation, the projections (4.2) and (4.3) become

$$\begin{aligned} \phi_1 &= J_1 \phi_1^T J_1, & \phi_2 &= J_2 \phi_2^T J_2, \\ B_1 &= J_2 A_2^T J_1, & B_2 &= J_2 A_1^T J_1, \\ \tilde{Q}_{1I}^T &= g_{IJ} J_1 Q_1^J, & \tilde{Q}_{2I}^T &= g_{IJ} J_2 Q_2^J, \end{aligned} \tag{4.4}$$

where $J^{ab} = J_1$ and $J_{ab} = J_1^{-1} = -J_1$ (and similarly for $J^{\bar{a}\bar{b}} = J_2$). We could use the constraints (4.4) to eliminate half the fields in (4.1) but it will be clearer to continue to write the superpotential as (4.1), with the constraints understood.

Now we include a bare mass μ for the adjoint hypermultiplets in the superpotential

$$\mathcal{W}_{\text{mass}} = \mu \text{Tr} \left(\phi_1^2 - \phi_2^2 \right), \tag{4.5}$$

breaking the $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. Taking μ to be large, we may integrate out the adjoint fields from the superpotential, giving the quartic superpotential for the $\mathcal{N} = 1$ $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ gauge theory:

$$\begin{aligned} \mathcal{W}_{=1} = \frac{1}{\mu} \Big[& \text{Tr}(A_1 B_1 A_2 B_2 - B_1 A_1 B_2 A_2) \\ & + \frac{1}{2} \tilde{Q}_{1I} Q_1^I \tilde{Q}_{1J} Q_1^J - \frac{1}{2} \tilde{Q}_{2I} Q_2^I \tilde{Q}_{2J} Q_2^J \\ & + \tilde{Q}_{1I} (A_1 B_1 + A_2 B_2) Q_1^I - \tilde{Q}_{2I} (B_1 A_1 + B_2 A_2) Q_2^I \Big]. \end{aligned} \tag{4.6}$$

When integrating out ϕ_1 and ϕ_2 , we must implement the constraint (4.2), but this will be automatic as long as the matter hypermultiplets obey the constraints (4.3).

Since we will later be interested in the regime where the first gauge group is strongly coupled, we define a set of fields that are singlets under $\text{Sp}(2N_1)$:

$$\begin{aligned} (N_{ij})_{\bar{a}}^{\bar{b}} &= B_{j\bar{a}}^c A_{ic}^{\bar{b}}, & M_I^J &= \tilde{Q}_{1I}^a Q_{1a}^J, \\ u_{iI}^{\bar{a}} &= \tilde{Q}_{1I}^b A_{ib}^{\bar{a}}, & v_{i\bar{a}}^I &= B_{i\bar{a}}^b Q_{1b}^I \quad (i, j = 1, 2) \end{aligned} \tag{4.7}$$

in terms of which the superpotential (4.6) becomes

$$\begin{aligned} \mathcal{W}_{=1} = \frac{1}{\mu} \Big[& \text{Tr}(N_{12} N_{21} - N_{11} N_{22}) + \frac{1}{2} M_I^J M_J^I - \frac{1}{2} \tilde{Q}_{2I} Q_2^I \tilde{Q}_{2J} Q_2^J \\ & + u_{iI} v_i^I - \tilde{Q}_{2I} (N_{11} + N_{22}) Q_2^I \Big], \end{aligned} \tag{4.8}$$

where the trace is over $\text{Sp}(2N_2)$ indices. For later convenience we also define the fields

$$\sigma_{\bar{a}}^{\bar{b}} = Q_{2\bar{a}}^I \tilde{Q}_{2I}^{\bar{b}} \tag{4.9}$$

even though the $Q_{2\bar{a}}^I$ are themselves singlets under $\text{Sp}(2N_1)$.

The constraints (4.3) imply that the $\text{Sp}(2N_1)$ gauge-invariant fields obey

$$N_{11} = J_2 N_{22}^T J_2, \quad N_{12} = J_2 N_{12}^T J_2, \quad N_{21} = J_2 N_{21}^T J_2, \tag{4.10}$$

$$u_{iI}^T = \epsilon_{ij} g_{IJ} J_2 v_j^I, \tag{4.11}$$

$$M_I^J = g_{IK} g^{JL} M_L^K, \tag{4.12}$$

$$\sigma = J_2 \sigma^T J_2, \tag{4.13}$$

where $\epsilon_{12} = 1$. The 4×4 matrix M_I^J parametrized by

$$M = \begin{pmatrix} W & 0 & Y & P \\ 0 & W & Q & X \\ X & P & Z & 0 \\ Q & Y & 0 & Z \end{pmatrix} \tag{4.14}$$

automatically satisfies the constraint (4.12).

The classical F-term equations are obtained by varying the superpotential (4.6) with respect to the independent variables A_i and Q_i (recall that B_i and \tilde{Q}_i are not independent variables, cf. (4.3)). However, it is easy to see that one obtains the same equations by treating A_i , B_i , Q_i , and \tilde{Q}_i as independent when performing the variation. Varying with respect to A_1 and A_2 gives

$$\begin{aligned} N_{21}B_2 \quad N_{22}B_1 + v_1^I \tilde{Q}_{1I} \quad \sigma B_1 &= 0, \\ N_{12}B_1 \quad N_{11}B_2 + v_2^I \tilde{Q}_{1I} \quad \sigma B_2 &= 0. \end{aligned} \tag{4.15}$$

Multiplying these equations on the right by A_i , we obtain

$$\begin{aligned} N_{21}N_{12} \quad N_{22}N_{11} + v_1^I u_{1I} \quad \sigma N_{11} &= 0, \\ N_{21}N_{22} \quad N_{22}N_{21} + v_1^I u_{2I} \quad \sigma N_{21} &= 0, \\ N_{12}N_{11} \quad N_{11}N_{12} + v_2^I u_{1I} \quad \sigma N_{12} &= 0, \\ N_{12}N_{21} \quad N_{11}N_{22} + v_2^I u_{2I} \quad \sigma N_{22} &= 0. \end{aligned} \tag{4.16}$$

The F-term equations obtained by varying (4.6) with respect to B_1 and B_2 ,

$$\begin{aligned} A_2N_{12} \quad A_1N_{22} + Q_1^I u_{1I} \quad A_1\sigma &= 0, \\ A_1N_{21} \quad A_2N_{11} + Q_1^I u_{2I} \quad A_2\sigma &= 0, \end{aligned} \tag{4.17}$$

are equivalent to (4.15), using the constraints (4.3). Varying the superpotential with respect to Q_1 and \tilde{Q}_1 yields

$$\begin{aligned} u_{1I}B_1 + u_{2I}B_2 + M_I^J \tilde{Q}_{1J} &= 0, \\ A_1v_1^I + A_2v_2^I + Q_1^J M_J^I &= 0, \end{aligned} \tag{4.18}$$

where the second equation follows from the first using (4.3). Finally,

$$\begin{aligned} (N_{11} + N_{22} + \sigma)Q_2^I &= 0, \\ \tilde{Q}_{2I}(N_{11} + N_{22} + \sigma) &= 0, \end{aligned} \tag{4.19}$$

follow by varying with respect to Q_2 and \tilde{Q}_2 (the second equation follows from the first using (4.3)).

When both $A_{i\bar{a}} Q_{2\bar{a}}^I \tilde{Q}_{2I}^{\bar{b}}$ and $Q_{1a}^I \tilde{Q}_{1I}^{\bar{b}} A_{ib}^{\bar{b}}$ vanish, the F-term equations (4.16), and the corresponding equations that follow from (4.17), imply that the set of $2N_2 \times 2N_2$ matrices N_{ij} mutually commute, and hence they can be diagonalized. The eigenvalues can therefore be interpreted as the positions of the D3-branes. By virtue of (4.16),

$$N_{21}N_{12} \quad N_{22}N_{11} = 0, \tag{4.20}$$

so these D3-branes live on a conifold.

For the unorientifolded $SU(N + M) \times SU(N)$ model and for the $Sp(2N + 2M) \times SO(2N + 2)$ orientifolded theory, Eq. (4.20) describes the entire classical moduli space. However, for the $Sp(2N_1) \times Sp(2N_2)$ theory the moduli space has additional structure. One way to ensure that $A_{i\bar{a}} Q_{2\bar{a}}^I \tilde{Q}_{2I}^{\bar{b}}$ and $Q_{1a}^I \tilde{Q}_{1I}^{\bar{b}} A_{ib}^{\bar{b}}$ both vanish is to set Q_1 and Q_2 to zero, but there are also other solutions. As an example, let us assume that $v_{iI}^b = \tilde{Q}_{1I}^{\bar{b}} A_{ib}^{\bar{b}}$

and choose a basis such that $Q_{2\bar{a}}^I$ is only non-zero for the first four entries ($\bar{a} = 1, 2, 3, 4$, say). Let us also assume that the N_{ij} 's are block diagonal with one (4×4) -dimensional block and one $((2N_2 - 4) \times (2N_2 - 4))$ -dimensional block (it is not clear whether all solutions have this block-diagonal form). In this case the F-term equations split into two parts. For the $((2N_2 - 4) \times (2N_2 - 4))$ -dimensional block it follows as above that the N_{ij} 's commute; hence the eigenvalues in this sector satisfy the conifold equation (4.20). For the (4×4) -dimensional block it follows from (4.19) that if $\sigma = 0$ then $N_{11} + N_{22} = 0$ has to hold (assuming that the Q_2^I 's span the 4×4 space). As we will see in more detail in the next section, $N_{11} + N_{22} = 0$ corresponds in the dual type IIB geometry to the point where the O7-plane and the 8 D7-branes are localized. The implications of this solution is that when the Q_2^I 's are non-zero, four of the D3-branes are stuck to the D7-branes. When σ is not zero the generic solution to Eq. (4.19) is given by $\sigma = N_{11} - N_{22}$. Inserting this relation into (4.16) leads to the equations

$$\begin{aligned} N_{21}N_{12} + N_{11}^2 &= 0, & N_{12}N_{21} + N_{22}^2 &= 0, \\ N_{21}N_{22} + N_{11}N_{21} &= 0, & N_{12}N_{11} + N_{22}N_{12} &= 0. \end{aligned} \tag{4.21}$$

It can be shown that there exist (4×4) -dimensional matrices satisfying these equations which are not mutually commuting. The interpretation of this non-commutative solution on the string theory side is unclear. Since the non-zero Q_2^I 's only affect a (4×4) -dimensional subspace, they are essentially a $1/N$ effect. Perhaps the general framework discussed in [18] can be used to shed some light on this sector of the moduli space.

So far we have only analyzed the classical moduli space. In general there are quantum corrections to the classical moduli space and some solutions may not have counterparts in the full quantum moduli space. The quantum modification of the superpotential for the $Sp(2N_1) \times Sp(2N_2)$ theory is not known. However, for the theory at the end of the cascade, it is possible, with certain assumptions, to determine the quantum superpotential. In the next section we will study the full quantum moduli space for the theory at the end of the cascade.

5. The $Sp(2N_1) \times Sp(2N_2)$ moduli space

At the end of the cascade, we have an $Sp(2N_1) \times Sp(2N_2)$ gauge theory. For this case, the 2×2 matrices N_{ij} satisfying (4.10) can be explicitly parametrized as

$$\begin{aligned} N_{11} &= \begin{pmatrix} w & p \\ q & z \end{pmatrix}, & N_{12} &= \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \\ N_{21} &= \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, & N_{22} &= \begin{pmatrix} z & p \\ q & w \end{pmatrix}. \end{aligned} \tag{5.1}$$

These satisfy

$$N_{11}N_{22} - N_{12}N_{21} = \begin{pmatrix} xy & wz + pq & 0 \\ 0 & xy & wz + pq \end{pmatrix}, \tag{5.2}$$

and mutually commute

$$[N_{ij}, N_{kl}] = 0. \tag{5.3}$$

From this result it follows that for the theory at the end of the cascade there are no non-commutative solutions of the type discussed at the end of Section 4.

We now analyze the moduli space of this theory, first considering the classical moduli space, then turning to the quantum modifications due to the dynamically-generated superpotential.

5.1. Classical moduli space

We do not consider the most general case, but rather analyze regions of the moduli space where, roughly speaking, the scalar vev of one or the other (or both) of the fundamental fields Q_1 and Q_2 vanishes.

Case I. $v_i^I = 0$ and $Q_2^I = 0$

First we consider solutions of the F-term equations for which both $v_i^I = B_i Q_1^I = 0$ and $Q_2^I = 0$ (thus $\sigma = 0$). The constraints (4.11) and (4.3) then imply $u_{iI} = 0$ and $\tilde{Q}_{2I} = 0$. The F-term equations (4.16) reduce to

$$N_{11}N_{22} \quad N_{12}N_{21} = 0. \tag{5.4}$$

We may use an $Sp(2)$ gauge transformation to diagonalize Eqs. (5.1), corresponding to setting $p = q = 0$. The eigenvalues of N_{ij} then correspond to the position of the D3-brane probe and its orientifold mirror. Eq. (5.4) implies

$$xy \quad wz = 0, \tag{5.5}$$

so the probe brane (and its mirror) move on a conifold. Moreover, the orientifold action on the conifold described in Section 2, $z \leftrightarrow w, x \rightarrow x, y \rightarrow y$, exchanges the positions of the probe and its mirror, so our choice of parametrization (5.1) is consistent with the variables used for the geometry in Section 2.

The simplest way to satisfy $B_i Q_1^I = 0$ is to set $Q_1^I = 0$, in which case M_I^J vanishes. M_I^J may, however, be non-zero if not all the Q_1^I vanish. Multiplying the first F-term equation in (4.18) on the right by Q_1^K , we obtain

$$M_I^J M_J^K = 0 \quad \Rightarrow \quad \det M = 0, \tag{5.6}$$

which implies

$$XY \quad WZ = 0, \quad W = Z, \quad P = Q = 0, \tag{5.7}$$

in terms of the parametrization (4.14).

Case II. $v_i^I = 0$

Next we consider the case where $v_i^I = B_i Q_1^I = 0$, but some of the Q_2^I are non-vanishing. The constraints (4.10) and (4.13) imply $N_{11} + N_{22} + \sigma = J_2(N_{11} + N_{22} + \sigma)^T J_2$. Consequently $N_{11} + N_{22} + \sigma$ is proportional to a linear combination of the Pauli matrices, and therefore is invertible if it does not vanish. If it is invertible, then Eq. (4.19) implies $Q_2^I = 0$, contrary to assumption. Therefore, it vanishes:

$$\sigma = N_{11} \quad N_{22}. \tag{5.8}$$

Setting $v_i = 0$ in Eq. (4.16), and using Eqs. (5.3) and (5.8), we see that

$$\begin{aligned} N_{11}N_{22} - N_{12}N_{21} &= 0, \\ N_{11} + N_{22} &= 0, \\ \sigma &= 0, \end{aligned} \tag{5.9}$$

which implies

$$xy - wz = 0, \quad w = z, \quad p = q = 0, \tag{5.10}$$

so the probe brane moves on the $w = z$ subspace of the conifold (5.5). The restriction to this subspace occurs only because some of the Q_2^I have non-zero vevs. By comparing with the results in Section 2 we see that the D3-brane probe (5.10) is stuck to the D7-branes which are located at the orientifold fixed point, $z = w$, so the minimal length of D3–D7 strings vanishes. This is consistent with the fact that the induced masses of the Q_2^I fields, which are given by the eigenvalues of σ , are zero in this case, since σ vanishes identically.

The fields Q_1^I may also have non-zero vevs, as long as they satisfy $B_i Q_1^I = 0$ and $M_I^J M_J^K = 0$. The latter condition implies that M_I^J satisfies Eq. (5.7).

Case III. $Q_2^I = 0$

Finally, we consider the case in which $Q_2^I = 0$ (therefore $\tilde{Q}_{2I} = 0$), but some of the Q_1^I are non-zero. Setting $\sigma = 0$ in Eqs. (4.16), and using (5.3), we see that

$$v_{j\bar{a}}^I u_{iI}^{\bar{b}} \propto \delta_{ij} \delta_{\bar{a}}^{\bar{b}}. \tag{5.11}$$

We assume that the constant of proportionality does not vanish, otherwise this reduces to case I. Viewing $v_{j\bar{a}}^I$ as vectors whose components are labelled by I we choose a basis in which

$$v_{1\bar{1}}^I = \begin{pmatrix} v_{1\bar{1}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_{2\bar{2}}^I = \begin{pmatrix} 0 \\ v_{2\bar{2}} \\ 0 \\ 0 \end{pmatrix}, \quad v_{2\bar{1}}^I = \begin{pmatrix} 0 \\ 0 \\ v_{2\bar{1}} \\ 0 \end{pmatrix}, \quad v_{1\bar{2}}^I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_{1\bar{2}} \end{pmatrix}. \tag{5.12}$$

The constraints (4.11) then imply

$$u_{1I}^{\bar{1}} = \begin{pmatrix} v_{2\bar{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_{2I}^{\bar{2}} = \begin{pmatrix} 0 \\ v_{1\bar{1}} \\ 0 \\ 0 \end{pmatrix}, \quad u_{2I}^{\bar{1}} = \begin{pmatrix} 0 \\ 0 \\ v_{1\bar{2}} \\ 0 \end{pmatrix}, \quad u_{1I}^{\bar{2}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_{2\bar{1}} \end{pmatrix}, \tag{5.13}$$

and the relations (5.11) imply

$$v_{2\bar{1}} v_{1\bar{2}} = v_{1\bar{1}} v_{2\bar{2}}. \tag{5.14}$$

The F-term equations (4.16) then give

$$(N_{11}N_{22} - N_{12}N_{21})_{\bar{a}}^{\bar{b}} = v_{1\bar{1}} v_{2\bar{2}} \delta_{\bar{a}}^{\bar{b}}. \tag{5.15}$$

Multiplying the first equation of (4.18) on the right by $A_{ja} \bar{b}$ and on the left by $v_{i\bar{a}}^I$ we get

$$v_{1\bar{1}} v_{2\bar{2}} (N_{ji})_{\bar{a}} \bar{b} + v_{i\bar{a}}^I M_I^J u_{j\bar{J}} \bar{b} = 0. \tag{5.16}$$

Using (5.1), (5.12), and (5.13), this can be used to show that M has the form (4.14) with

$$\begin{aligned} X &= \frac{v_{1\bar{1}}}{v_{2\bar{1}}} x, & Y &= \frac{v_{2\bar{1}}}{v_{1\bar{1}}} y, & W &= w, & Z &= z, \\ P &= \frac{v_{1\bar{2}}}{v_{1\bar{1}}} p, & Q &= \frac{v_{1\bar{1}}}{v_{1\bar{2}}} q. \end{aligned} \tag{5.17}$$

Next, we multiply Eqs. (4.15) and (4.17) by Q_1 and \tilde{Q}_1 , Eq. (4.18) by A_i and B_i , and compare the results to show

$$N_{11} + N_{22} = 0. \tag{5.18}$$

Eq. (5.18) arises only when the Q_1^I vevs are not all zero. Eqs. (5.15) and (5.18) imply

$$xy \quad wz = v_{1\bar{1}} v_{2\bar{2}}, \quad w = z, \quad p = q = 0. \tag{5.19}$$

The matrix M_I^J is completely determined in terms of the v_i^I and N_{ij} as

$$M = \begin{pmatrix} z & 0 & v_{2\bar{1}} y / v_{1\bar{1}} & 0 \\ 0 & z & 0 & v_{1\bar{1}} x / v_{2\bar{1}} \\ v_{1\bar{1}} x / v_{2\bar{1}} & 0 & z & 0 \\ 0 & v_{2\bar{1}} y / v_{1\bar{1}} & 0 & z \end{pmatrix} \tag{5.20}$$

and obeys $\det M = (v_{1\bar{1}} v_{2\bar{2}})^2$.

The geometrical interpretation of Eq. (5.19) is not entirely clear. The induced masses of the Q_1^I fields, which are given by the eigenvalues of M , are non-vanishing when $v_{1\bar{1}} v_{2\bar{2}} \neq 0$. This would appear to imply that the length of the D3–D7 strings in this case is non-vanishing.

It would be interesting to find the generalization of the solution in [5] describing this sector of the moduli space.

5.2. Quantum moduli space

At the end of the flow, the first gauge group of the $\text{Sp}(2N_1) \times \text{Sp}(2)$ theory becomes strongly coupled, and a quantum superpotential is dynamically generated. We effectively have an $\mathcal{N} = 1$ $\text{Sp}(2N_1)$ gauge theory with $2N_f = 8$ hypermultiplets q_a^M , which we parametrize as

$$q_a^M = A_{1a} \bar{1} \quad A_{1a} \bar{2} \quad A_{2a} \bar{1} \quad A_{2a} \bar{2} \quad Q_{1a}^1 \quad Q_{1a}^2 \quad Q_{1a}^3 \quad Q_{1a}^4. \tag{5.21}$$

The gauge indices of the $\text{Sp}(2N_2)$ factor act as flavor indices. When the $\mathcal{N} = 2$ superpotential for such a theory has the form $\sqrt{2} q_a^M \phi^{ab} q_b^M$, where $M = 1, \dots, 2N_f$, the Affleck–Dine–Seiberg superpotential [19] is given by [16,20] (when $N_1 + 1 > N_f$)

$$\mathcal{W}_{\text{ADS}} = (N_1 + 1 - N_f) \left(\frac{=1}{\text{Pf } V} \right)^{\frac{1}{N_1+1 - N_f}}, \tag{5.22}$$

where V is the antisymmetric $2N_f \times 2N_f$ meson matrix $V^{MN} = q_a^M J^{ab} q_b^N$. The $\mathcal{N} = 2$ superpotential (4.1) for the $\text{Sp}(2N_1) \times \text{Sp}(2)$ theory is not flavor diagonal in the basis (5.21), but can be written as $\sqrt{2} q_a^M g_{MN} \phi^{ab} q_b^N$ where

$$g_{MN} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.23}$$

where the lower 4×4 block is just the matrix g_{IJ} introduced in Eqs. (4.3). However, g_{MN} can be diagonalized by a change of basis without altering the Pfaffian. Hence, the superpotential of the $\mathcal{N} = 1$ $\text{Sp}(2N_1) \times \text{Sp}(2)$ theory can be written as

$$\mathcal{W} = \mathcal{W}_{=1} + (N_1 - 3) \left(\frac{3N_1 - 1}{\text{Pf} V} \right)^{\frac{1}{N_1 - 3}}, \tag{5.24}$$

with $\mathcal{W}_{=1}$ given by Eq. (4.8) and with V given by

$$V^{MN} = q_a^M J^{ab} q_b^N = \begin{pmatrix} 0 & x & q & w & 0 & v_{2\bar{2}} & 0 & 0 \\ x & 0 & z & p & 0 & 0 & v_{2\bar{1}} & 0 \\ q & z & 0 & y & 0 & 0 & 0 & v_{1\bar{2}} \\ w & p & y & 0 & v_{1\bar{1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{1\bar{1}} & 0 & W & Q & X \\ v_{2\bar{2}} & 0 & 0 & 0 & W & 0 & Y & P \\ 0 & v_{2\bar{1}} & 0 & 0 & Q & Y & 0 & Z \\ 0 & 0 & v_{1\bar{2}} & 0 & X & P & Z & 0 \end{pmatrix}, \tag{5.25}$$

where we have chosen the basis (5.12) and (5.13) for $v_{i\bar{a}}^I$ and $u_{i\bar{I}}^{\bar{a}}$, and used the parametrizations (5.1) and (4.14). Using the same parametrization, the superpotential (5.24) becomes

$$\begin{aligned} \mathcal{W} = & \frac{1}{\mu} \left[2(xy + wz - pq) + 2(v_{1\bar{1}}v_{2\bar{2}} - v_{1\bar{2}}v_{2\bar{1}}) + W^2 + Z^2 - 2XY + 2PQ \right. \\ & \left. + (z - w) \sigma_{\bar{1}}^{\bar{1}} \sigma_{\bar{2}}^{\bar{2}} - 2p\sigma_{\bar{2}}^{\bar{1}} - 2q\sigma_{\bar{1}}^{\bar{2}} - \frac{1}{2} \tilde{Q}_{2I} Q_2^J \tilde{Q}_{2J} Q_2^I \right] \\ & + (N_1 - 3) \left(\frac{3N_1 - 1}{\text{Pf} V} \right)^{\frac{1}{N_1 - 3}}, \end{aligned} \tag{5.26}$$

with

$$\begin{aligned} \text{Pf}(V) = \sqrt{\det V} = & (xy - wz + pq)(XY - WZ + PQ) - v_{1\bar{1}}v_{1\bar{2}}v_{2\bar{1}}v_{2\bar{2}} \\ & + xYv_{1\bar{1}}v_{1\bar{2}} - yXv_{2\bar{1}}v_{2\bar{2}} + wWv_{1\bar{2}}v_{2\bar{1}} + zZv_{1\bar{1}}v_{2\bar{2}} \\ & + qPv_{1\bar{1}}v_{2\bar{1}} - pQv_{1\bar{2}}v_{2\bar{2}}. \end{aligned} \tag{5.27}$$

The F-term equations are derived from the superpotential (5.26) by varying with respect to the gauge invariant fields N_{ij} , M_I^J , v_i^I , and Q_2^I . These equations differ from the classical F-term Eqs. (4.15), (4.17), and (4.18), even in the limit $\epsilon \rightarrow 0$, because the latter were obtained by varying (4.6) with respect to A_i , B_i and Q_1^I . The F-term equations (4.19), obtained by varying with respect to Q_2^I , are the same in the classical and quantum cases, because the AdS superpotential does not depend on Q_2 . Even though their derivations are different we will find that the quantum and classical solutions join smoothly.

Case I. $v_i^I = 0$ and $Q_2^I = 0$

As a simplification we can set $v_i^I = 0$ and $\sigma = 0$ directly in (5.26), (5.27) since these expressions are quadratic in v_i^I and Q_2^I and hence will not contribute to the variation. Comparing the F-term equations derived by varying (5.26) with respect to x , y , w , z , p , and q , and with respect to X , Y , W , Z , P , and Q , we obtain

$$\begin{aligned} xy - wz + pq &= XY - WZ + PQ, \\ W &= Z, \\ P = Q &= 0, \end{aligned} \tag{5.28}$$

implying $\text{Pf}(V) = (xy - wz + pq)^2$. The F-term equations become

$$\frac{2}{\mu} \left. \frac{3N_1 - 1}{N_1 - 3} \right|_{=1} (\text{Pf } V) \left. \frac{2}{N_1 - 3} \right|_{=1} (xy - wz - pq) = 0, \tag{5.29}$$

which implies

$$(N_{11}N_{22} - N_{12}N_{21}) \bar{a}^{\bar{b}} = \epsilon \delta_{\bar{a}}^{\bar{b}}, \quad \text{where } \epsilon = \left(\frac{\mu}{2} \right)^{\frac{N_1 - 3}{N_1 - 1}} \left. \frac{3N_1 - 1}{N_1 - 1} \right|_{=1}. \tag{5.30}$$

Setting $p = q = 0$ using an $\text{Sp}(2)$ gauge transformation, we find that the probe branes move on a deformed conifold

$$xy - wz = \epsilon. \tag{5.31}$$

From (5.28), the matrix M_I^J has the form

$$M_I^J = \begin{pmatrix} Z & 0 & Y & 0 \\ 0 & Z & 0 & X \\ X & 0 & Z & 0 \\ 0 & Y & 0 & Z \end{pmatrix}, \tag{5.32}$$

where the matrix elements of M are arbitrary, but by (5.28) and (5.31) must satisfy

$$\det M = (XY - Z^2)^2 = \epsilon^2. \tag{5.33}$$

Unlike in the classical case, $M = 0$ is not a solution. (If M_I^J were to vanish, then \mathcal{W}_{AdS} would blow up.) The lower 4×4 block of the antisymmetric matrix V has the form

$$V^{IJ} = Q_{1a}^I J^{ab} Q_{1b}^J = g^{IK} M_K^J = \begin{pmatrix} 0 & Z & 0 & X \\ Z & 0 & Y & 0 \\ 0 & Y & 0 & Z \\ X & 0 & Z & 0 \end{pmatrix}. \tag{5.34}$$

A flavor transformation allows us to block-diagonalize this matrix, so that the above relations reduce to

$$V^{IJ} = \begin{pmatrix} 0 & \tilde{Z} & 0 & 0 \\ \tilde{Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{Z} \\ 0 & 0 & \tilde{Z} & 0 \end{pmatrix}, \quad \text{with} \quad \tilde{Z} = \epsilon^{1/2} = \begin{pmatrix} \frac{\mu}{2} & \\ & \frac{3N_1}{2N_1-2} \\ & & \frac{3N_1}{2N_1-2} \\ & & & 1 \end{pmatrix} \quad (5.35)$$

which is exactly the meson matrix in Eq. (3.12) of de Boer et al. [20] for the $\mathcal{N} = 1$ $\text{Sp}(2N_1)$ theory with $2N_f = 4$ fundamental fields (see also Ref. [21]). In Section 6, we will explain this in terms of the M-theory configuration corresponding to this branch of moduli space. Note that (5.35) is simply a rewriting of (5.33) since the determinant is invariant under the flavor rotation.

In the limit $\epsilon \rightarrow 0$, the solution (5.31) and (5.28) reduces to the classical solution (5.5) and (5.7).

Case II. $v_i^I = 0$

The F-term equation obtained by varying the full superpotential (5.26) with respect to Q_2 is equivalent to the classical F-term equation (4.19). By the previous arguments given for the classical case II above, this yields

$$\sigma = N_{11} N_{22}. \quad (5.36)$$

The F-term equations derived by varying (5.26), (5.27) with respect to N_{ij} and M_I^J , after setting $v_i^I = 0$, yield (5.28), (5.29) as in case I. In addition, they imply

$$\sigma_a^{\bar{b}} \propto \delta_a^{\bar{b}}. \quad (5.37)$$

This, together with the constraint (4.13), implies that σ vanishes. Hence we have

$$\begin{aligned} N_{11}N_{22} - N_{12}N_{21} &= \epsilon, \\ N_{11} + N_{22} &= 0, \\ \sigma &= 0. \end{aligned} \quad (5.38)$$

The second equation in (5.38) implies $w = z$ and $p = q = 0$, so the probe brane moves on the $w = z$ subspace of the deformed conifold (5.31). As in case I, the field M is of the form (5.32) satisfying (5.33).

When $\epsilon \rightarrow 0$, the quantum case II solution (5.38) reduces to the classical case II solution (5.9).

Case III. $Q_2^I = 0$

Setting $\sigma = 0$ in Eq. (5.26), and varying with respect to N_{ij} , M_I^J , and $v_{i\bar{a}}$ we find

$$\begin{aligned} xy - wz &= v_{1\bar{1}}v_{2\bar{2}} + \epsilon, \\ w &= z, \\ p &= q = 0, \\ v_{1\bar{2}}v_{2\bar{1}} &= v_{1\bar{1}}v_{2\bar{2}}. \end{aligned} \quad (5.39)$$

Thus

$$\begin{aligned} N_{11}N_{22} - N_{12}N_{21} &= (v_{1\bar{1}}v_{2\bar{2}} + \epsilon) , \\ N_{11} + N_{22} &= 0. \end{aligned} \tag{5.40}$$

The F-term equations also show that the matrix elements of M_I^J are related to those of N_{ij} by

$$\begin{aligned} X &= \left(\frac{v_{1\bar{1}}}{v_{2\bar{1}}}\right)x, & Y &= \left(\frac{v_{2\bar{1}}}{v_{1\bar{1}}}\right)y, \\ W &= w, & Z &= z, & P &= Q = 0, \end{aligned} \tag{5.41}$$

which yields (5.20) but with $x, y, z,$ and w satisfying (5.39). The above solution reduces to the classical case III solution when $\epsilon \rightarrow 0$.

5.3. Summary

The various branches of the $Sp(2N_1) \times Sp(2)$ moduli space relevant to the end of the cascade, and their type IIB brane interpretations have appeared throughout this section. Here we collect these results.

The 2×2 matrices N_{ij} (5.1) mutually commute and their eigenvalues can be interpreted as the position of the D3-brane probe (and its mirror).

We first summarize the structure of the classical moduli space. In case I we found $\det N_{ij} = 0$ (5.4), or equivalently $xy - wz = 0$ in the parametrization (5.1), so the D3-brane probe moves on the orientifolded conifold. For case II we again found $xy - zw = 0$ and in addition $N_{11} + N_{22} = 0$ (5.9), or $w - z = 0$. The latter condition implies that the D3-brane is stuck to O7-plane/D7-brane stack. In case III we also found $w - z = 0$ together with $xy - wz = v_{1\bar{1}}v_{2\bar{2}}$ (5.19). The geometrical interpretation of these equations is less clear, but some suggestions were presented in the text.

For the quantum moduli space we found a similar structure with the quantum and classical solutions joining smoothly. In case I we found $\det N_{ij} = \epsilon$ (5.30), or $xy - wz = \epsilon$, so the D3-brane probe moves on the deformed orientifolded conifold. For case II we again found $xy - zw = \epsilon$ and in addition $N_{11} + N_{22} = 0$ (5.38), or $w - z = 0$, so the D3-brane is stuck to O7-plane/D7-brane stack. In case III we also found $w - z = 0$ together with $xy - wz = v_{1\bar{1}}v_{2\bar{2}} + \epsilon$ (5.39). As in the classical case, the geometrical interpretation of these equations is unclear.

Some insight into the various branches of the quantum moduli space can be gleaned from the M-theory lift of the type IIA brane configuration corresponding to the $Sp(2N_1) \times Sp(2N_2)$ gauge theory, to which we turn next.

6. Type IIA and M-theory interpretations

In the previous sections we have seen that the moduli space of the $Sp(2N_1) \times Sp(2N_2)$ gauge theory and its modification by the AdS superpotential has a richer structure compared to that of its unorientifolded cousin, the $SU(N_1) \times SU(N_2)$ gauge theory.

It is fruitful to study the structure of the moduli space of the $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ theory from the viewpoint of the associated type IIA string theory configuration and its lift to M-theory, where some of the results obtained in the previous sections can be understood. We will start by briefly reviewing the type IIA setup to make the presentation more self contained.

6.1. Type IIA configurations

The $\text{SU}(N) \times \text{SU}(N)$ superconformal gauge theory with chiral multiplets in the $2(\ , \bar{\ }) \oplus 2(\bar{\ }, \)$ representations arises in type IIA string theory as the world-volume field theory on D4-branes suspended between two NS5-branes in an elliptic model (i.e., periodic in the x_6 direction) [22]. There are N D4-branes going along half the x_6 circle, and N D4-branes going along the other half; the two stacks of D4-branes give rise to the two factors of the gauge group. If the NS5-branes are parallel, the $\text{SU}(N) \times \text{SU}(N)$ gauge theory has $\mathcal{N} = 2$ supersymmetry; the $\mathcal{N} = 2$ vector multiplet includes a chiral multiplet in the adjoint representation of the gauge group. If the NS5-branes are rotated 90 degrees with respect to one another, the $\text{SU}(N) \times \text{SU}(N)$ gauge theory has only $\mathcal{N} = 1$ supersymmetry [11,12]. Rotating the NS5-branes [23] corresponds field-theoretically to including (opposite sign) masses (4.5) for the adjoint chiral multiplets, which breaks the supersymmetry to $\mathcal{N} = 1$, and integrating them out.

The introduction of a pair of orientifold 6-planes into this configuration results in various $\mathcal{N} = 2$ [24] and $\mathcal{N} = 1$ [6,11,13] world-volume theories on the D4-branes, particular examples of which are the models

$$\begin{aligned} \text{(i)} \quad & \text{Sp}(2N) \times \text{Sp}(2N), \quad \text{with } 2(\ , \bar{\ }) \oplus 4(\ , 1) \oplus 4(1, \bar{\ }), \\ \text{(ii)} \quad & \text{Sp}(2N) \times \text{SO}(2N+2), \quad \text{with } 2(\ , \bar{\ }) \end{aligned} \quad (6.1)$$

whose IIB realizations were already discussed in Section 2. The O6-planes span the 0123789 directions and are separated in the (compact) 6 direction; the two NS5-branes are placed between the O6-planes and are related to each other by the orientifold symmetry. If the NS5-branes are parallel, spanning the 0123 and $v = x_4 + ix_5$ directions, the world-volume field theories have $\mathcal{N} = 2$ supersymmetry and include chiral multiplets in the adjoint representation of the gauge group. The NS5-branes may be rotated (in opposite directions) toward the $u = x_8 + ix_9$ plane (so that one of them spans the $v \cos \alpha + u \sin \alpha$ plane and the other spans the $v \cos \alpha - u \sin \alpha$ plane) while still respecting the orientifold symmetry (which takes $x_6 \rightarrow -x_6$ and $v \rightarrow -v$). When $\alpha = \pi/4$, the NS5-branes become orthogonal, and the world-volume field theory on the D4-branes is given by (6.1). In model (i), both orientifold planes are O6⁻ planes; the configuration also contains 8 D6-branes for cancellation of 6-brane charge. In model (ii), there is one O6⁺ and one O6⁻ plane and no D6-branes.

The form of the \mathbb{Z}_2 orientifold action on the conifold in the type IIB configuration may be determined [6] from the rotated IIA brane configuration described above [11,12]. The D3-branes move in the background $xy = (u \cos \alpha + v \sin \alpha)(u \cos \alpha - v \sin \alpha)$. When $\alpha = \pi/4$, this is just a conifold $xy = wz$, where $w = \frac{1}{\sqrt{2}}(u + v)$ and $z = \frac{1}{\sqrt{2}}(u - v)$. The orientifold action implies $u \rightarrow u, v \rightarrow -v$ so that $w \leftrightarrow z$, as discussed in Section 2.

Generalizations of the orientifolded models described above may be obtained by suspending $2M$ additional D4-branes between the NS5-branes that only go along one of the two halves of the x_6 circle. The extra D4-branes break the superconformal invariance and are the type IIA analog of the fractional D3-branes in the type IIB theory. For recent discussions of cascading theories from the type IIA viewpoint, see [4,25].

6.2. M-theory configurations

Next, we turn to the M-theory lifts of these type IIA brane configurations. First, consider the configuration corresponding to the superconformal $\text{Sp}(2N) \times \text{Sp}(2N)$ gauge theory with two orthogonal NS5-branes (one spanning the z plane and the other the w plane) and $2N$ D4-branes wrapping all the way around the x_6 circle. Because the D4-branes do not end on the NS5-branes, but pass through, they can move transversely away (in the directions z , w , and x_7) from the NS5-branes. The motions of each of the N D4-branes (which are correlated with the motion of the N mirror branes) together with the Wilson loop expectation value around x_6 , gives rise to a six-dimensional moduli space, which is classically a conifold. Since the 2 NS5-branes and the D4-branes can be physically separated, each lifts to a separate M5-brane [12].

Next consider the case $2N_1 > 2N_2$, in which superconformal symmetry is broken. $2N_2$ of the branes still wrap all the way around the x_6 circle, and can move transversely away from the other branes; the classical moduli space of these branes is, as before, the conifold. These branes lift to a “toroidal” M5-brane which is wrapped in the x_6 and x_{10} directions.

There are $2N_1 - 2N_2$ additional D4-branes that wrap only half-way around the circle. These D4-branes end on the two NS5-branes (which have $w = x_7 = 0$ and $z = x_7 = 0$, respectively) and are therefore pinned in the z , w , and x_7 directions. The two NS5-branes and the D4-branes connecting them lift to a single M5-brane [26]. This M5-brane should be similar to the “MQCD” brane that occurs in the (non-elliptic) type IIA model with O6-planes [27,28] which gives rise to the $\mathcal{N} = 1$ $\text{Sp}(2N_c)$ model; in the limit where the x_6 -periodicity becomes large, they should become identical.

We briefly describe the form of the MQCD brane in the $\mathcal{N} = 1$ $\text{Sp}(2N_c)$ model obtained in a model with O6-planes, following Refs. [27,29]. Begin with a $\mathcal{N} = 2$ $\text{Sp}(2N_c)$ model with $2N_f > 0$ massless fundamentals which arises from a IIA configuration with parallel NS5-branes extended in the v direction. This configuration lifts to an M5-brane whose embedding is given by the Seiberg–Witten curve [30]

$$t_+ + t_- = C(v^2) = v^{2N_c} + \dots, \tag{6.2}$$

$$t_+ t_- = \sum_{i=2}^{4N_c+4} \binom{2N_f}{i} v^{2N_f - i}. \tag{6.3}$$

(A possible v^{-2} term on the right-hand side of the first equation vanishes because of the masslessness of the fundamental fields.) To obtain the curve for the $\mathcal{N} = 1$ theory, we must relatively rotate the NS5-branes, as described above. This is possible only if the curve (6.2) degenerates to genus zero, in which case the coefficients of $C(v^2)$ are fixed. Rotating the NS5-branes through an angle $\alpha = \arctan(\tilde{\mu})$ (where $\tilde{\mu}$ is proportional to the adjoint mass μ) in the $v - u$ hyperplane, we obtain a curve whose projection onto the v plane is

still given by (6.2), but with asymptotic behavior

$$\begin{aligned} x_6 \rightarrow \infty, \quad u \rightarrow \tilde{\mu}v, \quad v \rightarrow \infty, \quad t_+ \rightarrow v^{2N_c}, \\ x_6 \rightarrow +\infty, \quad u \rightarrow \tilde{\mu}v, \quad v \rightarrow \infty, \quad t \rightarrow v^{2N_c}. \end{aligned} \tag{6.4}$$

The resulting genus zero curve may be parametrized in terms of either $w_+ = u + \tilde{\mu}v$ or $w = u - \tilde{\mu}v$. Letting

$$v = P(w_+), \quad t_+ = Q(w_+), \tag{6.5}$$

the orientifold symmetry $t_+ \leftrightarrow t, u \rightarrow u, v \rightarrow -v$ implies

$$v = P(w), \quad t = Q(w). \tag{6.6}$$

The asymptotic conditions (6.4) then imply

$$P(w_+) = \frac{1}{2\tilde{\mu}} \left(w_+ - \frac{w_0^2}{w_+} \right), \tag{6.7}$$

$$w_+ w = w_0^2 \tag{6.8}$$

for some w_0 . Eq. (6.3) yields

$$Q(w_+) = \frac{1}{(2\tilde{\mu})^{2N_c}} w_+^{2N_c+4} w_+^{2N_f} w_+^2 w_0^{2N_f-2}, \tag{6.9}$$

where

$$w_0 = 2\tilde{\mu} w_+^2 \tag{6.10}$$

up to a complex phase. Following the argument of Ref. [29], the parameter w_0 is proportional to the eigenvalue of the meson matrix constructed from the fundamental fields.

6.3. Moduli space

We will now establish the connection between the configuration of two disconnected M5-branes described above and the moduli space of the $Sp(2N_1) \times Sp(2)$ gauge theory as described in Section 5. The motion of the toroidal M5-brane, which is the lift of 2 D4-branes that wrap x_6 , is described by the 2×2 matrices N_{ij} . The MQCD brane configuration is described by M_I^J , or equivalently V^{IJ} .

Case I

In case I, N_{ij} and M_I^J are unrelated, which reflects the independence of the 2 M5-branes. Classically, the moduli space of the toroidal M5-brane is the conifold (5.5). The ADS superpotential modifies the classical geometry to the deformed conifold (5.31).

The solution for the antisymmetric meson matrix V^{IJ} (5.35) involves a single vev, which by virtue of the relation [20] $\mu_{=1}^{3N_1-1} = \mu_{=2}^{N_1+1} \mu_{=2}^{2N_1-2}$ becomes

$$\tilde{Z} = 2 \mu_{=2}^{\frac{3N_1}{2N_1-2}} \tag{6.11}$$

This is proportional to the parameter w_0 (6.10) of the MQCD brane. This is consistent with our interpretation that M_I^J describes the M5-brane that is the lift of two orthogonal NS5-branes and $2N_1 - 2$ D4-branes.

Case II

In the case II solution, N_{ij} and M_I^J are also unrelated, indicating that the two M5-branes are still disconnected. M_I^J has the same form as in case I, so the MQCD brane is unaltered. In addition to satisfying the deformed conifold constraint, the N_{ij} must also obey $N_{11} + N_{22} = 0$ (5.38). This may be understood geometrically as follows.

Case II represents a Higgs branch of the gauge theory in which the scalar vev Q_2 is non-zero. In the type IIA configuration, this branch corresponds to D4-branes breaking on the D6-branes that lie in the interval between the two NS-branes containing the 2 D4-branes. Thus, only the D4-branes that wrap around the x_6 circle (those which lift to the toroidal M5-brane) can break on the D6-branes. Since the D6-branes are coincident with the O6-plane (the fundamental fields have no bare mass), which is located at $v = 0$ (i.e., $w = z$), the D4-branes can only break on them if they satisfy $w = z$ as well. This then implies that the toroidal M5-brane must satisfy the condition $N_{11} + N_{22} = 0$.

Case III

Case III represents a Higgs branch of the gauge theory in which the scalar vev Q_1 is non-zero. This branch corresponds to D4-branes breaking on the D6-branes that lie in the interval between the two NS5-branes containing the $2N_1$ D4-branes. Since all the D4-branes can now break on the D6-branes, the configurations of both M5-branes, described by M_I^J and N_{ij} , are altered by the Q_1 vevs.

As in case II, the D4-branes can only break on the D6-branes if they satisfy $w = z$, thus the toroidal M5-brane satisfies $N_{11} + N_{22} = 0$ (5.40). The remaining $2N_1 - 2$ D4-branes were already pinned at the D6-brane locus, so there is no additional constraint on M_I^J .

Finally, since the breaking of the D4-branes on the D6-branes allows the entire configuration of D4-branes to be interconnected, the M5-branes to which they lift are no longer disconnected; this is reflected in the fact that N_{ij} and M_I^J are no longer independent, but are related by Eq. (5.41).

7. Summary

In this paper we have presented a description of the moduli space of the $\mathcal{N} = 1$ cascading $\text{Sp}(2N_1) \times \text{Sp}(2N_2)$ gauge theory, and the interpretation of its various branches in terms of both type IIB and type IIA/M-theory brane configurations.

In Section 4 we discussed the (classical) F-term equations appropriate to the generic case, i.e., without restriction to the end of the cascade. When the scalar components of the Q_i 's do not have vevs, we argued that the D3-branes move on the orientifolded conifold. When the vevs of the Q_i 's are no longer zero we found that there are subsectors in which the N_{ij} 's are no longer mutually commuting matrices. In these sectors there does not appear to be a geometric interpretation of the N_{ij} 's as (commuting) coordinates. However, the vevs of these non-commutative N_{ij} 's span (at most) a 4×4 subspace of the $2N_2 \times 2N_2$ matrices N_{ij} , therefore for N_2 large, one intuitively expects them to be only a $1/N_2$ effect.

In Section 5, which is the main part of the paper, we presented an extensive study of the various branches of the moduli space at the end of the cascade. We studied both the

classical and the quantum versions of the moduli space. The structure of the moduli space and the dual type IIB interpretations was summarized in Section 5.3.

In Section 6 we discussed the moduli space from the viewpoint of type IIA brane configurations and their lift to M-theory. The solutions of the quantum F-term equations can be interpreted in terms of the configuration of two M5-branes, with N_{ij} corresponding to a toroidal M5-brane that wraps the x_6 direction, and M_I^J corresponding to an MQCD brane that is the lift of the NS5-branes and D4-branes connecting them. The case III solution in which N_{ij} and M_I^J are related (5.41) corresponds to one of the Higgs branches of the theory in which the two M5-branes are connected.

In a companion paper we will discuss the leading α' -corrections to the supergravity solution for the orientifolded models discussed in this paper (analogous to those considered in Ref. [31] for the supergravity solution of Ref. [3]) and the role of these corrections in the dual field theory.

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