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The $\mathcal{N} = 2$ U(N) gauge theory prepotential and periods from a perturbative matrix model calculation

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Abstract

We perform a completely perturbative matrix model calculation of the physical low-energy quantities of the $\mathcal{N} = 2$ U(N) gauge theory. Within the matrix model framework we propose a perturbative definition of the periods a_i in terms of certain tadpole diagrams, and check our conjecture up to first order in the gauge theory instanton expansion. The prescription does not require knowledge of the Seiberg–Witten differential or curve. We also compute the $\mathcal{N} = 2$ prepotential $\mathcal{F}(a)$ perturbatively up to the first-instanton level, finding agreement with the known result.
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1. Introduction

Dijkgraaf and Vafa, drawing on earlier developments [1–3], have uncovered the surprising result that non-perturbative effective superpotentials for certain $d = 4$ $\mathcal{N} = 1$ supersymmetric gauge theories can be obtained by calculating planar diagrams in a related gauged matrix model [4–7]. In particular, the d-instanton contribution to the effective superpotential can be obtained from the calculation of $(d + 1)$-loop planar diagrams in an associated matrix model. The simplest example is the $\mathcal{N} = 1$ SU(N) gauge theory with an adjoint chiral superfield ϕ and tree-level superpotential $W(\phi)$, for which the instanton corrections can be obtained from the calculation of the planar loop diagrams in a Hermitian

matrix model. This statement has recently been proven [8]. Further work along these lines has been presented in Ref. [9].

The new approach can also be used to study $d = 4$ $\mathcal{N} = 2$ supersymmetric gauge theories, by using $W(\phi)$ to freeze the moduli at an arbitrary point on the Coulomb branch of the $\mathcal{N} = 2$ theory, thereby breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$, and then turning off $W(\phi)$ at the end of the calculation to restore $\mathcal{N} = 2$ supersymmetry [3–6]. The crucial feature that makes this work is that certain quantities are independent of the parameter that goes to zero in the limit when $\mathcal{N} = 2$ supersymmetry is restored, and can thus be calculated for finite values of the parameter.

Even when the matrix model cannot be completely solved, a perturbative diagrammatic expansion of the matrix model can still be used to obtain non-perturbative information about the $\mathcal{N} = 2$ gauge theory. In Ref. [7], the effective gauge coupling matrix τ_{ij} of the unbroken U(1)×U(1) gauge group at an arbitrary point on the Coulomb branch of the $\mathcal{N} = 2$ U(2) gauge theory was computed, as a function of the classical modulus, to several orders in the instanton expansion.[4]

In this paper, we extend this result to the $\mathcal{N} = 2$ U(N) gauge theory, computing the matrix of effective gauge couplings τ_{ij} of the unbroken U(1)^N gauge group as a function of the classical moduli, which we denote by e_i. To explicitly obtain the full low-energy physical content of the model, however, one also needs to determine the relation between the periods a_i and the classical moduli e_i. We argue that a_i can be determined by computing tadpole diagrams in perturbative matrix theory, and verify that this prescription yields the correct results for pure U(N) gauge theory up through one-instanton. Knowing the connection between a_i and e_i enables us to re-express τ_{ij} as a function of a_i. This then allows the relations $\tau_{ij}(a) = \partial a_{D,i}/\partial a_j = \partial^2 \mathcal{F}(a)/\partial a_i \partial a_j$ to be integrated. Thus, we demonstrate that exact non-perturbative quantities in low-energy $\mathcal{N} = 2$ supersymmetric theories, namely, the prepotential $\mathcal{F}(a)$ and the masses of BPS states $|na + ma_D|$, can be computed from a diagrammatic expansion of the matrix model, even in cases when an exact solution of the matrix model is not known.

Solving for the gauge coupling matrix, prepotential and BPS mass spectrum perturbatively, without using the exact solution of the matrix model, is equivalent to deriving these results without knowledge of the Seiberg–Witten curve or differential (although they are known in the particular case we study). Thus the techniques developed here and in Refs. [4–7] could be used to obtain non-perturbative information about $\mathcal{N} = 2$ supersymmetric gauge theories for which the Seiberg–Witten curve is not known.

In Section 2, we review the Seiberg–Witten approach to the calculation of the prepotential, periods, and gauge couplings in $\mathcal{N} = 2$ gauge theories. In Section 3, we describe the matrix model approach to the calculation of the gauge coupling matrix τ_{ij}, and in Section 4 we carry out the calculation of τ_{ij} to one-instanton order for the $\mathcal{N} = 2$ U(N) gauge theory. In Section 5, we present our proposal for computing a_i in the perturbative matrix model, and in Section 6 we compute the relation between a_i and e_i up to one-instanton for U(N). Using this result together with the results of Section 4, we compute

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[4] For this case, the exact all-orders result can be obtained from the known large-M two-cut solution of the matrix model [4,6,7].

the $\mathcal{N} = 2$ prepotential $\mathcal{F}(a)$ to one-instanton level. Finally, in Section 7, we calculate the gauge theory invariants $\langle \mathrm{tr}(\phi^n) \rangle$ perturbatively in the matrix model, finding agreement with known results. In Appendix A, we present an alternative method of computing the relation between a_i and e_i using the relation between the Seiberg–Witten differential and the density of gauge theory eigenvalues in the large-N limit [7].

2. Seiberg–Witten approach to $\mathcal{N} = 2$ gauge theories

The Seiberg–Witten approach to $\mathcal{N} = 2$ supersymmetric gauge theory [10] involves identifying a complex curve Σ and a meromorphic differential λ_{SW} on this curve. For pure SU(N) gauge theory[5] the curve is given by a genus $N-1$ hyperelliptic Riemann surface [11,12]

$$\Sigma: \quad y^2 = P_N(x)^2 - 4\Lambda^{2N},$$

$$P_N(x) = \sum_{\ell=0}^{N} s_{N-\ell}(e)x^\ell = \prod_{i=1}^{N}(x - e_i), \qquad \sum_{i=1}^{N} e_i = 0, \tag{2.1}$$

corresponding to a generic point on the Coulomb branch of the moduli space of vacua, where the gauge symmetry is broken to U(1)$^{N-1}$. In the equation above, $s_m(e)$ is the elementary symmetric polynomial

$$s_m(e) = (-1)^m \sum_{i_1 < i_2 < \cdots < i_m} e_{i_1} e_{i_2} \cdots e_{i_m}, \qquad s_0 = 1. \tag{2.2}$$

Next, one chooses a canonical homology basis of Σ, $\{A_i, B_i\}$, $i = 1, \ldots, N-1$, in terms of which

$$a_i = \frac{1}{2\pi i} \oint_{A_i} \lambda_{SW}, \qquad a_{D,i} = \frac{1}{2\pi i} \oint_{B_i} \lambda_{SW},$$

$$\lambda_{SW} = x \frac{dy}{y} = \frac{x P_N'(x)\,dx}{\sqrt{P_N(x)^2 - 4\Lambda^{2N}}}. \tag{2.3}$$

We will choose A_i, $i = 1, \ldots, N-1$ to be the contour that remains on one sheet of the two-sheeted Riemann surface and encircles the branch cut emanating from e_i [13]. A_N and a_n are defined similarly. However, A_N is not an independent cycle, being equivalent to $-\sum_{i=1}^{N-1} A_i$, and one can show that $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} e_i$ by deforming the contour and evaluating the residue of λ_{SW} at infinity.

The A_i-period integral may be inverted to write e_i in terms of a_i, allowing one to express $a_{D,i}$ as a function of a_i. Then, since $\partial a_{D,i}/\partial a_j = \partial a_{D,j}/\partial a_i$, one may write

$$a_{D,i} = \frac{\partial \mathcal{F}(a)}{\partial a_i}, \qquad \mathcal{F}(a) = \mathcal{F}_{\mathrm{pert}}(a, \log \Lambda) + \sum_{d=1}^{\infty} \Lambda^{2Nd} \mathcal{F}^{(d)}(a), \tag{2.4}$$

---

[5] This is the nontrivial piece of the U(N) gauge theory (in later sections we focus on the U(N) theory).

thus defining the $\mathcal{N} = 2$ prepotential $\mathcal{F}(a)$, which can be written as a sum of perturbative and instanton contributions. The masses of the BPS states of the theory can be expressed as $|na + ma_D|$, for integers n, m. Finally,

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}, \tag{2.5}$$

yields the period matrix of Σ, identified with the gauge couplings of the $U(1)^{N-1}$ factors of the unbroken gauge theory.

3. Matrix model approach to $\mathcal{N} = 2$ gauge theories

In this section we describe the matrix model approach to $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory. The first step is to break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by the addition of a tree-level superpotential $W(\phi)$ to the gauge theory. This superpotential is identified with the potential of a chiral matrix model [4–7]. The matrix model thus has the partition function [4–7]

$$Z = \frac{1}{\text{vol}(G)} \int d\Phi \exp\left(-\frac{W(\Phi)}{g_s}\right), \tag{3.1}$$

where the integral is over $M \times M$ matrices Φ (which can be taken to be Hermitian), g_s is a parameter that later will be taken to zero as $M \to \infty$, and G is the unbroken matrix model gauge group. One chooses a superpotential $W(\Phi)$ that freezes the moduli to a generic point on the Coulomb branch of the $\mathcal{N} = 2$ theory:

$$W(\Phi) = \alpha \sum_{\ell=0}^{N} \frac{s_{N-\ell}(e)}{\ell+1} \text{tr} \, \Phi^{\ell+1}) \quad \Rightarrow \quad W'(x) = \alpha \prod_{i=1}^{N} (x - e_i), \tag{3.2}$$

where $s_m(e)$ was defined in Eq. (2.2), and α is a parameter that will be taken to zero at the end of the calculation, restoring $\mathcal{N} = 2$ supersymmetry. The matrix integral (3.1) is evaluated perturbatively about the extremum

$$\Phi_0 = \begin{pmatrix} e_1 \mathbb{1}_{M_1} & 0 & \cdots & 0 \\ 0 & e_2 \mathbb{1}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \mathbb{1}_{M_N} \end{pmatrix}, \quad \text{where } \sum_{i=1}^{N} M_i = M, \tag{3.3}$$

which breaks the $U(M)$ symmetry to $G = \prod_{i=1}^{N} U(M_i)$. (This is the matrix model analog of the gauge theory breaking $U(N) \to U(1)^N$. Note that in the matrix model $M_i \gg 1$ for all i.)

Using the standard double-line notation, the connected diagrams of the perturbative expansion of Z may be organized in an expansion characterized by the genus g of the surface in which the diagram is embedded [14]

$$Z = \exp\left(\sum_{g \geq 0} g_s^{2g-2} F_g(e, S)\right), \quad \text{where } S_i \equiv g_s M_i. \tag{3.4}$$

Evaluating the matrix integral in the $M_i \to \infty$, $g_s \to 0$ limit, with S_i held fixed, is equivalent to retaining only the planar (genus $g = 0$) diagrams. Thus

$$F_0(e, S) = g_s^2 \log Z\big|_{\text{planar}} \tag{3.5}$$

corresponds to the connected planar diagrams of the matrix theory.

To relate this to the $\mathcal{N} = 2$ U(N) gauge theory broken to $\prod_i$ U(N_i), one introduces [2–6]

$$W_{\text{eff}}(e, S) = \sum_i N_i \frac{\partial F_0(e, S)}{\partial S_i} + 2\pi i \tau_0 \sum_i S_i, \tag{3.6}$$

where $\tau_0 = \tau(\Lambda_0)$ is the gauge coupling of the U(N) theory at some scale Λ_0. In this paper, we are interested in breaking U(N) to U(1)^N, so $N_i = 1$ for all i, and i runs from 1 to N. The effective superpotential is extremized with respect to S_i to obtain $\langle S_i \rangle$:

$$\frac{\partial W_{\text{eff}}(e, S)}{\partial S_i}\bigg|_{S_j = \langle S_j \rangle} = 0. \tag{3.7}$$

Finally,

$$\tau_{ij}(e) = \frac{1}{2\pi i} \frac{\partial^2 F_0(e, S)}{\partial S_i \partial S_j}\bigg|_{S_i = \langle S_i \rangle} \tag{3.8}$$

yields the couplings of the unbroken U(1)^N factors of the gauge theory, as a function of e_i. At the end of the matrix model calculation, one must take $\alpha \to 0$ to restore $\mathcal{N} = 2$ supersymmetry, but as will be seen, τ_{ij} is independent of α, and can thus be calculated for any value of α.

In the next section, we will explicitly carry out the procedure outlined above for the pure $\mathcal{N} = 2$ U(N) gauge theory.

Despite the superficial similarity of Eqs. (2.5) and (3.8), the $\mathcal{N} = 2$ gauge theory prepotential $\mathcal{F}(a)$ and the free energy $F_0(e, S)$ of the large M_i matrix model are conceptually distinct. $\mathcal{F}(a)$ is a function of the periods a_i of the Seiberg–Witten differential, whereas $F_0(e, S)$ is a function of the e_i's as well as the auxiliary parameters S_i (which can understood as SU(K) glueball superfields in the related U(NK) $\to$ U(K)^N theory [2]). Although both (2.5) and (3.8) correspond to the same quantity (the period matrix of Σ), they are expressed in terms of different parameters (a_i vs. e_i) on the moduli space.

If we are to use the matrix model result (3.8) to determine the $\mathcal{N} = 2$ prepotential $\mathcal{F}(a)$, we must first express τ_{ij} in terms of a_i. Although the relationship between a_i and e_i is straightforwardly obtained [13] in the Seiberg–Witten approach from the A_i-period integral (2.3), we wish to derive this relationship from within the matrix model, without referring to the Seiberg–Witten curve or differential. After explicitly calculating τ_{ij} for U(N) in the next section, we will turn to a perturbative matrix model calculation of a_i for that same model in Section 5.

4. Calculation of τ_{ij} for U(N) using the matrix model

In this section, we will evaluate the planar free energy $F_0(e, S)$, defined via

$$\exp\left(\frac{1}{g_s^2} F_0(e, S)\right) = \frac{1}{\text{vol}(G)} \int d\Phi \exp\left(-\frac{W(\Phi)}{g_s}\right)\bigg|_{\text{planar}} \tag{4.1}$$

to cubic order in S_i. This will enable us to calculate the gauge coupling matrix τ_{ij} for $\mathcal{N} = 2$ U(N) gauge theory to one-instanton accuracy.

As described in the previous section, we expand Φ about the following extremum of $W(\Phi)$,

$$\Phi = \Phi_0 + \Psi = \begin{pmatrix} e_1 \, \mathbb{1}_{M_1} & 0 & \cdots & 0 \\ 0 & e_2 \, \mathbb{1}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \, \mathbb{1}_{M_N} \end{pmatrix} + \begin{pmatrix} \Psi_{11} & \Psi_{12} & \cdots & \Psi_{1N} \\ \Psi_{21} & \Psi_{22} & \cdots & \Psi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{N1} & \Psi_{N2} & \cdots & \Psi_{NN} \end{pmatrix}, \tag{4.2}$$

where Ψ_{ij} is an $M_i \times M_j$ matrix. This choice breaks $U(M) \to G = \prod_{i=1}^{N} U(M_i)$.

Expanding $W(\Phi)$ to quadratic order in Ψ, we obtain

$$W(\Phi) = \sum_{i=1}^{N} M_i W(e_i) + \frac{1}{2}\alpha \sum_{i=1}^{N} \left(\sum_{\ell=0}^{N} \ell s_{N-\ell} e_i^{\ell-1}\right) \text{tr} \, \Psi_{ii}^2)$$

$$+ \frac{1}{2}\alpha \sum_{i=1}^{N} \sum_{j\neq i} \left(\sum_{\ell=1}^{N} s_{N-\ell} \sum_{m=0}^{\ell-1} e_i^m e_j^{\ell-m-1}\right) \text{tr}(\Psi_{ij}\Psi_{ji}) + \mathcal{O}(\Psi^3). \tag{4.3}$$

It can be shown that

$$\sum_{\ell=0}^{N} \ell s_{N-\ell} e_i^{\ell-1} = \left[\frac{\partial}{\partial x} \prod_{k=1}^{N}(x - e_k)\right]\bigg|_{x=e_i} = \prod_{k\neq i}(e_i - e_k),$$

$$\sum_{\ell=1}^{N} s_{N-\ell} \sum_{m=0}^{\ell-1} e_i^m e_j^{\ell-m-1} = 0, \tag{4.4}$$

which implies that the coefficient of $\text{tr}(\Psi_{ij}\Psi_{ji})$ vanishes when $i \neq j$. Hence the off-diagonal matrices Ψ_{ij} are zero modes, and correspond to pure gauge degrees of freedom. These zero modes parametrize the coset $U(M)/G = U(\sum_i M_i)/[U(M_1) \times \cdots \times U(M_N)]$. Following Ref. [7], we will fix the gauge $\Psi_{ij} = 0$ $(i \neq j)$ and introduce Grassmann-odd ghost matrices B and C with the action

$$\text{tr} \, B[\Phi, C]) = \sum_{i=1}^{N} \sum_{j\neq i} (e_i - e_j) \text{tr}(B_{ji}C_{ij}) + \sum_{i=1}^{N} \sum_{j\neq i} \text{tr}(B_{ji}\Psi_{ii}C_{ij} - B_{ji}C_{ij}\Psi_{jj}). \tag{4.5}$$

Thus the planar free energy is given in terms of the gauge-fixed integral

$$\exp\left(\frac{1}{g_s^2} F_0(e, S)\right) = \frac{1}{\text{vol}(G)} \exp\left(\frac{1}{g_s} \sum_{i=1}^{N} M_i W(e_i)\right)$$

$$\times \int d\Psi_{ii}\, dB_{ij}\, dC_{ij}\, e^{I_{\text{quad}} + I_{\text{int}}}\Bigg|_{\text{planar}}, \tag{4.6}$$

where the quadratic part of the action is

$$I_{\text{quad}} = \frac{1}{2}\frac{\alpha}{g_s} \sum_{i=1}^{N} R_i \,\text{tr}\; \Psi_{ii}^2) \quad \sum_{i=1}^{N} \sum_{j\neq i} e_{ij}\, \text{tr}(B_{ji} C_{ij}),$$

$$R_i = \prod_{j\neq i} e_{ij}, \qquad e_{ij} = e_i \quad e_j \tag{4.7}$$

and the interaction terms are (after implementing the gauge choice $\Psi_{ij} = 0$)

$$I_{\text{int}} = \frac{\alpha}{g_s} \sum_{i=1}^{N} \sum_{p=3}^{N} \frac{\gamma_{p,i}}{p}\, \text{tr}\; \Psi_{ii}^p) \quad \sum_{i=1}^{N} \sum_{j\neq i} \text{tr}(B_{ji} \Psi_{ii} C_{ij} \quad B_{ji} C_{ij} \Psi_{jj}). \tag{4.8}$$

Here

$$\gamma_{p,i} = \frac{1}{(p \quad 1)!}\left[\left(\frac{\partial}{\partial x}\right)^p \; 1 \prod_{k=1}^{N} (x \quad e_k)\right]\Bigg|_{x=e_i} \tag{4.9}$$

and, in particular, we will need

$$\gamma_{3,i} = R_i \sum_{k\neq i} \frac{1}{e_{ik}}, \qquad \gamma_{4,i} = \frac{1}{2} R_i \sum_{k\neq i} \sum_{\ell\neq i,k} \frac{1}{e_{ik} e_{i\ell}}. \tag{4.10}$$

The Ψ_{ii} and ghost propagators can be derived from Eq. (4.7) and the vertices from Eq. (4.8). Each ghost loop will acquire an additional factor of (2) [7].

For large M_i, the volume prefactor in Eq. (4.6) becomes [15]

$$\frac{1}{\text{vol}(G)} = \exp\left(\frac{1}{2} \sum_{i=1}^{N} M_i^2 \log M_i\right). \tag{4.11}$$

The integral of the quadratic action I_{quad} may be evaluated to give

$$\prod_{i=1}^{N} \left(\frac{g_s}{\alpha R_i}\right)^{\frac{1}{2}M_i^2} \prod_{i=1}^{N} \prod_{j\neq i} (e_{ij})^{M_i M_j} \tag{4.12}$$

up to some multiplicative factors. Thus, setting $S_i = g_s M_i$, the matrix integral (4.6) yields the planar free energy (up to a quadratic monomial in the S_i's)

$$F_0(e, S) = \sum_{i=1}^{N} S_i W(e_i) + \frac{1}{2} \sum_{i=1}^{N} S_i^2 \log\left(\frac{S_i}{\alpha R_i \hat{\,}2}\right)$$

$$+ \sum_{i=1}^{N} \sum_{j\neq i} S_i S_j \log\left(\frac{e_{ij}}{\hat{\,}}\right) + \sum_{n\ 3} F_0^{(n)}(e, S), \tag{4.13}$$
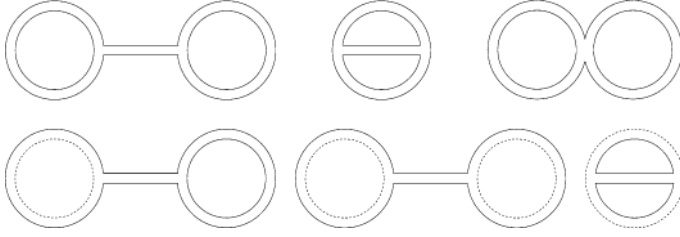
Fig. 1. Diagrams contributing to $F_0(e, S)$ at (S^3). Solid double lines correspond to Ψ_{ii} propagators; solid plus dashed double lines correspond to ghost propagators.

where $F_0^{(n)}(e, S)$ is an nth order polynomial in S_i arising from planar loop diagrams built from the interaction vertices [6]. We have included in Eq. (4.13) a contribution $\sum_{i=1}^{N} S_i)^2 \log \hat{}$ that reflects the ambiguity in the cut-off of the full U(M) gauge group [7]. As we will see below, the first three terms in Eq. (4.13) are already sufficient to give the complete perturbative (from the gauge theory perspective) contribution to τ_{ij}.

To obtain τ_{ij} to one-instanton accuracy in the gauge theory, we need to evaluate the contribution to $F_0(e, S)$ cubic in S_i. The Feynman diagrams that contribute at this order are depicted in Fig. 1.

The six diagrams in Fig. 1 give

$$
\alpha F_0^{(3)}(e, S) = \left(\frac{1}{2} + \frac{1}{6}\right) \sum_i \frac{S_i^3}{R_i} \left(\sum_{k \neq i} \frac{1}{e_{ik}}\right)^2 \quad \frac{1}{4} \sum_i \frac{S_i^3}{R_i} \sum_{k \neq i} \sum_{\ell \neq i, k} \frac{1}{e_{ik} e_{i\ell}}
$$

$$
2 \sum_i \sum_{k \neq i} \frac{S_i^2 S_k}{R_i e_{ik}} \sum_{\ell \neq i} \frac{1}{e_{i\ell}} + 2 \sum_i \sum_{k \neq i} \sum_{\ell \neq i} \frac{S_i S_k S_\ell}{R_i e_{ik} e_{i\ell}}
$$

$$
\sum_i \sum_{k \neq i} \frac{S_i^2 S_k}{R_i e_{ik}^2}. \tag{4.14}
$$

Using Eqs. (4.13) and (4.14) in Eq. (3.6), we obtain

$$
W_{\text{eff}} = \sum_i W(e_i) \quad \sum_i S_i \log\left(\frac{S_i}{\alpha R_i \hat{}^2}\right) \quad 2 \sum_i \sum_{k \neq i} S_k \log\left(\frac{e_{ik}}{\hat{}}\right)
$$

$$
\frac{1}{\alpha}\left[\quad \frac{3}{4} \sum_i \sum_{k \neq i} \sum_{\ell \neq i, k} \frac{S_i^2}{R_i e_{ik} e_{i\ell}} + 2 \sum_i \sum_{k \neq i} \sum_{\ell \neq i, k} \frac{S_k S_\ell}{R_i e_{ik} e_{i\ell}} \right.
$$

$$
\sum_i \sum_{k \neq i} \frac{S_i^2}{R_i e_{ik}^2} \quad 2 \sum_i \sum_{k \neq i} \frac{S_i S_k}{R_i e_{ik}^2} + 2 \sum_i \sum_{k \neq i} \frac{S_i^2}{R_k e_{ik}^2} \left.\right]
$$

$$
+ (2\pi i \tau_0 + \text{const}) \sum_i S_i. \tag{4.15}
$$

Extremizing this with respect to S_i yields the equation

$$0 = \log\left(\frac{S_i R_i}{\alpha^{\wedge 2N}}\right) + \frac{1}{\alpha}\left[\frac{3}{2}\frac{S_i}{R_i}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}} \quad 4\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{S_\ell}{R_k e_{ik}e_{k\ell}}\right.$$
$$\left. 2\frac{S_i}{R_i}\sum_{k\neq i}\frac{1}{e_{ik}^2} \quad \frac{2}{R_i}\sum_{k\neq i}\frac{S_k}{e_{ik}^2} \quad 2\sum_{k\neq i}\frac{S_k}{R_k e_{ik}^2} + 4S_i\sum_{k\neq i}\frac{1}{R_k e_{ik}^2}\right] \quad 2\pi i\tau_0 + \text{const}$$

(4.16)

whose solution, to $\mathcal{O}(\ ^{4N})$, is

$$\langle S_i\rangle = \frac{\alpha}{R_i}\ ^{2N} + \frac{\alpha}{R_i}\ ^{4N}\left[\frac{3}{2R_i^2}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}} + 4\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{R_k R_\ell e_{ik}e_{k\ell}}\right.$$
$$\left. + \frac{2}{R_i^2}\sum_{k\neq i}\frac{1}{e_{ik}^2} \quad \frac{2}{R_i}\sum_{k\neq i}\frac{1}{R_k e_{ik}^2} + 2\sum_{k\neq i}\frac{1}{R_k^2 e_{ik}^2}\right] + \mathcal{O}\big(\ ^{6N}\big),$$

(4.17)

where τ_0 and the other constants in Eq. (4.16) have been absorbed into a redefinition of the cut-off $= \text{const} \times\ ^{\wedge}e^{\pi i\tau_0/N}$. (This definition of corresponds to that used in the Seiberg–Witten curve (2.1).)

Although we are primarily interested in the $\mathcal{N}=2$ limit in this paper, the $\mathcal{N}=1$ effective superpotential may be easily computed by substituting Eq. (4.17) into Eq. (4.15).

We can now evaluate

$$\tau_{ij}(e) = \frac{1}{2\pi i}\frac{\partial^2 F_0(e,S)}{\partial S_i\partial S_j}\bigg|_{S_i=\langle S_i\rangle} = \tau_{ij}^{\text{pert}}(e) + \sum_{d=1}^{\infty}\ ^{2Nd}\tau_{ij}^{(d)}(e)$$

(4.18)

to obtain the perturbative contribution

$$2\pi i\tau_{ij}^{\text{pert}}(e) = \delta_{ij}\left[\text{const} \quad \sum_{k\neq i}\log\left(\frac{e_{ik}}{}\right)^2\right] + (1 \quad \delta_{ij})\left[\text{const} + \log\left(\frac{e_{ij}}{}\right)^2\right]$$

(4.19)

and the one-instanton contribution

$$2\pi i\tau_{ij}^{(1)}(e) = \delta_{ij}\left[\frac{8}{R_i^2}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}} \quad 4\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{R_k^2 e_{ik}e_{k\ell}}\right.$$
$$\left. + \frac{10}{R_i^2}\sum_{k\neq i}\frac{1}{e_{ik}^2} + 10\sum_{k\neq i}\frac{1}{R_k^2 e_{ik}^2}\right]$$
$$+ (1 \quad \delta_{ij})\left[\quad \frac{8}{R_i^2}\sum_{k\neq i,j}\frac{1}{e_{ij}e_{ik}} \quad \frac{8}{R_j^2}\sum_{k\neq i,j}\frac{1}{e_{ji}e_{jk}}\right.$$
$$\left. + 4\sum_{k\neq i,j}\frac{1}{R_k^2 e_{ik}e_{jk}} \quad \frac{10}{R_i^2 e_{ij}^2} \quad \frac{10}{R_j^2 e_{ij}^2}\right]$$

(4.20)

to the gauge coupling matrix. We have repeatedly used the identity

$$\sum_{k\neq i}\frac{1}{R_k e_{ik}} = \quad \frac{1}{R_i}\sum_{k\neq i}\frac{1}{e_{ik}}$$

(4.21)

which can be derived by taking the $z \to e_i$ limit of both sides of

$$\prod_{k=1}^{N} \frac{1}{z - e_k} \frac{1}{R_i(z - e_i)} = \sum_{k \neq i} \frac{1}{R_k(z - e_k)}. \tag{4.22}$$

Finally, we take the limit $\alpha \to 0$ to restore $\mathcal{N} = 2$ supersymmetry, but this has no effect on τ_{ij}, which is independent of α.

The logarithmic terms in Eq. (4.19) reflect the running of the coupling constants of this asymptotically free theory.

The gauge couplings τ_{ij} are usually written in terms of the periods a_i, which are related to the e_i's by $a_i = e_i + \mathcal{O}(\Lambda^{2N})$. From this it can be seen that one may write the perturbative contribution to the gauge couplings as

$$2\pi i \tau_{ij}^{\text{pert}}(a) = \delta_{ij}\left[\text{const} - \sum_{k \neq i}\log\left(\frac{a_i - a_k}{\Lambda}\right)^2\right]$$

$$+ (1 - \delta_{ij})\left[\text{const} + \log\left(\frac{a_i - a_j}{\Lambda}\right)^2\right] \tag{4.23}$$

which implies that the perturbative prepotential is

$$2\pi i \mathcal{F}_{\text{pert}}(a) = -\frac{1}{4}\sum_i \sum_{j \neq i}(a_i - a_j)^2 \log\left(\frac{a_i - a_j}{\text{const} \times \Lambda}\right)^2 \tag{4.24}$$

in agreement with the well-known result. To obtain the one-instanton contribution to the prepotential, $\mathcal{F}^{(1)}(a)$, from perturbative matrix theory, however, one needs to know the $\mathcal{O}(\Lambda^{2N})$ correction to the relation between a_i and e_i. We turn to this question in the next section, and then return to the computation of $\mathcal{F}^{(1)}(a)$ in Section 6.

5. Determination of a_i within the matrix model

In Seiberg–Witten theory, a_i is the A_i-period integral of λ_{SW}. How is a_i defined in the context of the perturbative matrix model?

To motivative the conjecture below, we first consider

$$u_n = \frac{1}{n}\text{tr}\,\phi^n), \tag{5.1}$$

where ϕ is the scalar component of the adjoint $\mathcal{N} = 1$ chiral superfield of the $\mathcal{N} = 2$ vector multiplet. In the Seiberg–Witten approach, the vevs of these operators may be written in terms of integrals over the A_i cycles [7]:

$$\langle u_n \rangle = \frac{1}{2\pi i n}\sum_{i=1}^{N}\oint_{A_i} x^{n-1}\lambda_{\text{SW}}. \tag{5.2}$$

On the matrix model side, $\langle u_n \rangle$ may be computed via[6]

$$\langle u_n \rangle = \left. \frac{\partial \widetilde{W}_{\text{eff}}(e, \langle \widetilde{S} \rangle, \epsilon)}{\partial \epsilon} \right|_{\epsilon \to 0}, \tag{5.3}$$

where $\widetilde{W}_{\text{eff}}(e, S, \epsilon)$ is the effective superpotential that one obtains by considering the matrix model with action $\widetilde{W}(\Phi) = W(\Phi) + \epsilon (1/n) \operatorname{tr}(\Phi^n)$.[7] Spelling this out more explicitly, one considers

$$\begin{aligned} \widetilde{Z} &= \exp\left(\frac{1}{g_s^2} \widetilde{F}_0(e, S, \epsilon) \right) \\ &= \frac{1}{\operatorname{vol}(G)} \int d\Phi \exp\left(\frac{1}{g_s} \left[W(\Phi) + \epsilon \frac{1}{n} \operatorname{tr} \Phi^n \right] \right) \Big|_{\text{planar}}. \end{aligned} \tag{5.5}$$

Then, writing $\widetilde{F}_0(e, S, \epsilon) = F_0(e, S) + \epsilon \delta F$, one computes

$$\begin{aligned} \widetilde{W}_{\text{eff}}(e, S, \epsilon) &= \sum_{i=1}^{N} N_i \frac{\partial \widetilde{F}_0(e, S, \epsilon)}{\partial S_i} + 2\pi i \tau_0 \sum_{i=1}^{N} S_i \\ &= W_{\text{eff}}(e, S) \; \epsilon \sum_{i=1}^{N} N_i \frac{\partial}{\partial S_i} \delta F. \end{aligned} \tag{5.6}$$

Extremizing $\widetilde{W}_{\text{eff}}(e, S, \epsilon)$ with respect to S gives $\langle \widetilde{S}_i \rangle = \langle S_i \rangle + \epsilon \delta S_i + \mathcal{O}(\epsilon^2)$. Substituting $\langle \widetilde{S} \rangle$ into Eq. (5.6), one obtains

$$\widetilde{W}_{\text{eff}} \; e, \langle \widetilde{S} \rangle, \epsilon) = W_{\text{eff}} \; e, \langle S \rangle) + \epsilon \sum_{i=1}^{N} \delta S_i \left. \frac{\partial W_{\text{eff}}}{\partial S_i} \right|_{\langle S \rangle} \; \epsilon \sum_{i=1}^{N} N_i \frac{\partial}{\partial S_i} \delta F \Big|_{\langle S \rangle} + \mathcal{O} \; \epsilon^2). \tag{5.7}$$

The second term vanishes by the definition of $\langle S \rangle$. Finally, using Eq. (5.3), one obtains

$$\langle u_n \rangle = \sum_{i=1}^{N} N_i \frac{\partial}{\partial S_i} \delta F \Big|_{\langle S \rangle}. \tag{5.8}$$

Now observing that to first order in ϵ

$$\begin{aligned} \widetilde{Z} &= \frac{1}{\operatorname{vol}(G)} \int d\Phi \exp\left(\frac{W(\Phi)}{g_s} \right) \Big|_{\text{planar}} \\ &+ \frac{1}{\operatorname{vol}(G)} \int d\Phi \left[\frac{\epsilon}{g_s n} \right] \operatorname{tr} \Phi^n) \exp\left(\frac{W(\Phi)}{g_s} \right) \Big|_{\text{planar}} \end{aligned} \tag{5.9}$$

---

[6] We thank Cumrun Vafa for this explanation.

[7] For n $N+1$, $\widetilde{W}(\Phi)$ is equivalent to $W(\Phi)$ (3.2), with $\alpha s_{N+1} \; n \to \alpha s_{N+1} \; n + \epsilon$, so (5.3) becomes [2]

$$\langle u_n \rangle = \frac{1}{\alpha} \frac{\partial W_{\text{eff}}(e, \langle S \rangle)}{\partial s_{N+1} \; n}. \tag{5.4}$$

we see that δF can be obtained by computing the (connected) planar n-point function $\langle \text{tr}(\Phi^n) \rangle$ in the matrix model with action $W(\Phi)$, thus giving the explicit expression

$$\langle u_n \rangle = \sum_i N_i \frac{\partial}{\partial S_i} \frac{g_s}{n} \langle \text{tr } \Phi^n \rangle \Big|_{\langle S \rangle}. \tag{5.10}$$

In Section 7, we will use this expression to compute the one-instanton contribution to $\langle u_n \rangle$.

Turning now to a_i, recall that on the gauge theory side,

$$a_i = \frac{1}{2\pi i} \oint_{A_i} \lambda_{\text{SW}}. \tag{5.11}$$

We propose that, just as $\langle u_n \rangle$ is related to $\text{tr}(\Phi^n)$, so a_i is related to $\text{tr}_i(\Phi)$, where in the latter case, we trace only over the ith diagonal block of Φ. This prescription is motivated by the following facts. Whereas the contour in Eq. (5.2) is over the sum of A_i cycles, the contour in Eq. (5.11) is only over a single A_i cycle. It has been observed that when one sums the matrix perturbation series [4], each block of eigenvalues spreads out via eigenvalue repulsion into a distribution along a branch cut of the spectral curve; thus a single block corresponds to a single branch cut.

Considering a generic point in moduli space, where $\text{U}(N) \to \text{U}(1)^N$ (so that $N_i = 1$), we have

$$a_i = \sum_j \frac{\partial}{\partial S_j} g_s \langle \text{tr}_i(\Phi) \rangle \Big|_{\langle S \rangle}. \tag{5.12}$$

Now expanding Φ around the vacuum (4.2), $\text{tr}_i(\Phi) = M_i e_i + \text{tr}(\Psi_{ii})$, and we find

$$a_i = e_i + \sum_j \frac{\partial}{\partial S_j} g_s \langle \text{tr}(\Psi_{ii}) \rangle \Big|_{\langle S \rangle}, \tag{5.13}$$

where $\langle \text{tr}(\Psi_{ii}) \rangle$ is obtained by calculating connected planar tadpole diagrams with an external ii leg in the matrix model.

Eq. (5.13) is our conjectured matrix model definition of a_i. The right-hand side of Eq. (5.13) is independent of α, and thus survives in the $\mathcal{N} = 2$ limit, as required for consistency. One important implication of our conjecture is that we need only evaluate tadpole diagrams in the matrix model to find the relation between a_i and e_i. We stress that this procedure does not require knowledge of the Seiberg–Witten curve or of λ_{SW}, and the calculation can be done order-by-order in the perturbative expansion.

6. Calculation of a_i and $\quad$ (a) for U(N)

We will now test our proposal for the matrix model definition of a_i for the case of $\mathcal{N} = 2$ U(N) gauge theory. The relevant tadpole diagrams to first order in the instanton expansion are displayed in Fig. 2.

Using the Feynman rules derived from the action (4.6), one obtains

$$\langle \text{tr}(\Psi_{ii}) \rangle = \frac{1}{\alpha g_s} \sum_{j \neq i} \left[\frac{S_i^2}{R_i e_{ij}} + 2 \frac{S_i S_j}{R_i e_{ij}} \right]. \tag{6.1}$$
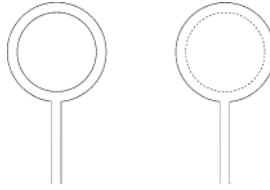
Fig. 2. Tadpole diagrams contributing to the one-instanton contribution to a_i.

Inserting this result into Eq. (5.13), evaluating the resulting expression using Eq. (4.17), and using the identity (4.21), we find (note that this expression is α independent)

$$a_i = e_i \left[\frac{2}{R_i^2} \sum_{j \neq i} \frac{1}{e_{ij}} + \mathcal{O}(\Lambda^{4N}) \right], \tag{6.2}$$

which agrees with the known result [13]. (In Appendix A, we present an alternative derivation of this formula that uses the fact that the Seiberg–Witten differential is related to the density of gauge theory eigenvalues in the large-N limit [7].) Eq. (6.2) implies that

$$\log e_{ij} = \log a_{ij} + \Lambda^{2N} \left[\frac{2}{R_i^2} \sum_{k \neq i,j} \frac{1}{e_{ij} e_{ik}} + \frac{2}{R_j^2} \sum_{k \neq i,j} \frac{1}{e_{ji} e_{jk}} + \frac{2}{R_i^2 e_{ij}^2} + \frac{2}{R_j^2 e_{ij}^2} \right], \tag{6.3}$$

where $a_{ij} = a_i - a_j$. We can now re-express τ_{ij} (4.19), (4.20) in terms of a_i

$$\tau_{ij}(a) = \tau_{ij}^{\text{pert}}(a) + \sum_{d=1}^{\infty} \Lambda^{2Nd} \tau_{ij}^{(d)}(a), \tag{6.4}$$

where the perturbative contribution is as found above (4.23)

$$2\pi i \tau_{ij}^{\text{pert}}(a) = \delta_{ij} \left[\text{const} - \sum_{k \neq i} \log \left(\frac{a_{ik}}{\Lambda} \right)^2 \right]$$

$$+ (1 - \delta_{ij}) \left[\text{const} + \log \left(\frac{a_{ij}}{\Lambda} \right)^2 \right] \tag{6.5}$$

and the one-instanton contribution is

$$2\pi i \tau_{ij}^{(1)}(a) = \delta_{ij} \left[\frac{4}{R_i^2} \sum_{k \neq i} \sum_{\ell \neq i,k} \frac{1}{a_{ik} a_{i\ell}} + \frac{6}{R_i^2} \sum_{k \neq i} \frac{1}{a_{ik}^2} + 6 \sum_{k \neq i} \frac{1}{R_k^2 a_{ik}^2} \right]$$

$$+ (1 - \delta_{ij}) \left[-\frac{4}{R_i^2} \sum_{k \neq i,j} \frac{1}{a_{ij} a_{ik}} - \frac{4}{R_j^2} \sum_{k \neq i,j} \frac{1}{a_{ji} a_{jk}} \right.$$

$$\left. + 4 \sum_{k \neq i,j} \frac{1}{R_k^2 a_{ik} a_{jk}} - \frac{6}{R_i^2 a_{ij}^2} - \frac{6}{R_j^2 a_{ij}^2} \right], \tag{6.6}$$

where now $R_i = \prod_{j\neq i}(a_i - a_j)$. It is readily verified that this can be written as $\tau_{ij} = \partial^2 \mathcal{F}(a)/\partial a_i \partial a_j$ with

$$
2\pi i \, \mathcal{F}(a) = \frac{1}{4}\sum_i \sum_{j\neq i}(a_i - a_j)^2 \log\left(\frac{a_i - a_j}{\text{const}\times\Lambda}\right)^2
$$
$$
+ \, \Lambda^{2N}\sum_i \prod_{j\neq i}\frac{1}{(a_i - a_j)^2} + \mathcal{O}\left(\Lambda^{4N}\right). \tag{6.7}
$$

This precisely agrees with the result obtained in Eq. (4.34) of Ref. [13].

To conclude, we have shown that a completely perturbative matrix model calculation, which does not use the Seiberg–Witten curve or differential, gives the correct result for the prepotential to first order in the instanton expansion. Higher-instanton corrections to the prepotential may be obtained by higher-loop contributions to the matrix model free energy and tadpole diagrams.

7. Calculation of ⟨uₙ⟩ in the matrix model

In Section 5, we showed that the gauge theory invariant $\langle u_n \rangle = (1/n)\langle \text{tr}(\phi^n)\rangle$ can be expressed in terms of a matrix model n-point function as

$$
\langle u_n \rangle = \sum_i N_i \frac{\partial}{\partial S_i}\frac{g_s}{n}\langle \text{tr}\,\Phi^n)\rangle\bigg|_{\langle S\rangle}. \tag{7.1}
$$

As a check of Eq. (7.1) we now evaluate this expression to one-instanton order in the $U(N) \to U(1)^N$ theory (so $N_i = 1$).

First one expands Φ around the vacuum (4.2), using $\Psi_{ij} = 0$ $(i \neq j)$

$$
\text{tr}\,\Phi^n) = \sum_{i=1}^{N}\sum_{\ell=0}^{n}\binom{n}{\ell}\text{tr}\,e_i^{n-\ell}\Psi_{ii}^{\ell})
$$
$$
= \sum_{i=1}^{N}\left[M_i e_i^n + n e_i^{n-1}\text{tr}(\Psi_{ii}) + \frac{n(n-1)}{2}e_i^{n-2}\text{tr}\,\Psi_{ii}^2) + \cdots\right]. \tag{7.2}
$$

By counting powers of S_i of the diagrams, it is not hard to see that only the $\ell \leq 2$ terms will contribute to the one-instanton term. The tadpole term $\langle \text{tr}(\Psi_{ii})\rangle$ was already computed in the previous section. To quadratic order, the only diagram contributing to $\langle \text{tr}(\Psi_{ii}^2)\rangle$ is a Ψ_{ii} loop, giving $g_s M_i^2/\alpha R_i$. Thus,

$$
\frac{g_s}{n}\langle \text{tr}\,\Phi^n)\rangle = \frac{1}{n}\sum_{i=1}^{N}S_i e_i^n + \frac{1}{\alpha}\left[\sum_{i=1}^{N}e_i^{n-1}\sum_{j\neq i}\left(-\frac{S_i^2}{R_i e_{ij}} + 2\frac{S_i S_j}{R_i e_{ij}}\right) + \frac{n-1}{2}e_i^{n-2}\frac{S_i^2}{R_i}\right]
$$
$$
+ \mathcal{O}\,S^3). \tag{7.3}
$$

Substituting this into Eq. (7.1), one obtains the α-independent expression

$$\langle u_n \rangle = \frac{1}{n} \sum_{i=1}^{N} e_i^n + \Lambda^{2N} \left[2 \sum_{i=1}^{N} \sum_{j \neq i} \frac{e_i^{n-1}}{R_i R_j e_{ij}} + (n-1) \sum_{i=1}^{N} \frac{e_i^{n-2}}{R_i^2} \right] + \mathcal{O}(\Lambda^{4N}). \quad (7.4)$$

The first term is just the classical vev of u_n. Using the identity (4.21) the term in square brackets can be written

$$2 \sum_{i=1}^{N} \frac{e_i^{n-1}}{R_i^2} \sum_{j \neq i} \frac{1}{e_{ij}} + (n-1) \sum_{i=1}^{N} \frac{e_i^{n-2}}{R_i^2} = \sum_{i=1}^{N} \frac{\partial}{\partial e_i} \frac{e_i^{n-1}}{R_i^2}. \quad (7.5)$$

Now consider $z^{n-1}/\prod_j (z - e_j)^2$. This function has double poles at $z = e_i$. The sum of the residues at these poles is exactly equal to the sum that appear on the right hand of the equality in (7.5). Thus provided that there is no residue at infinity the sum vanishes. This is the case for $n < 2N$; thus, there is no one-instanton correction to $(u_n)_{\rm cl}$ for $n < 2N$. This is consistent with the exact result [3] $\langle u_n \rangle = (u_n)_{\rm cl}$ for $n \leqslant N+1$, which should hold to all orders in matrix model perturbation theory.

For $n \geqslant 2N$, however, the term in square brackets does not vanish, since for this case the residue at infinity is equal to the residue at $w = 0$ of $-w^{2N-n-1}/\prod_j (1 - w e_j)^2$. The sum above is equal to minus the residue at infinity, and hence equals ($m = n - 2N \geqslant 0$)

$$\frac{1}{m!} \left(\frac{\mathrm{d}^m}{\mathrm{d} w^m} \right) \left[\frac{1}{\prod_j (1 - w e_j)^2} \right] \Bigg|_{w=0}. \quad (7.6)$$

For example, it is exactly equal to 1 for $n = 2N$, yielding

$$\langle u_{2N} \rangle = \frac{1}{2N} \sum_{i=1}^{N} e_i^{2N} + \Lambda^{2N} + \mathcal{O}(\Lambda^{4N}). \quad (7.7)$$

It may be readily verified by deforming the contour, and evaluating the residue at $x = \infty$, that the gauge theory expression (5.2), which uses the Seiberg–Witten differential, yields precisely the same result.

Matrix model perturbation theory thus provides an alternative way of evaluating $\langle \mathrm{tr}(\phi^n) \rangle$ in $\mathcal{N} = 2$ U(N) gauge theory.

8. Conclusions

The remarkable results of Dijkgraaf, Vafa, and collaborators indicate that several non-perturbative results in supersymmetric gauge theories can be obtained from perturbative calculations in auxiliary matrix models, without reference to string/M-theory.

The Seiberg–Witten approach to $\mathcal{N} = 2$ gauge theories requires the knowledge of a Seiberg–Witten curve and one-form, where the most general method of obtaining these involves M-theory [16].

By contrast, in the matrix model approach to $\mathcal{N} = 2$ gauge theories, one expects that all the relevant information should be contained within the matrix model itself. Previously,

the only way to obtain the periods a_i has been via the Seiberg–Witten differential, which was obtained from the restriction of a three-form in the Calabi–Yau setup [3]. However, this approach falls outside the spirit of the Dijkgraaf–Vafa program, as all gauge theory quantities should be derivable without reference to string theory. One of the main results of this paper is to provide the missing link that allows us to compute the periods a_i of $\mathcal{N} = 2$ gauge theories by means of a perturbative calculation in the matrix model. We have shown that the prescription reproduces previously known results.

To obtain explicit expressions for the periods a_i, a_{Di}, and the prepotential (a) from knowledge of the curve requires extensive calculations (see, e.g., Ref. [13]). Our computations are somewhat simpler than such calculations. However, it should be emphasized that there are other methods for obtaining the $\mathcal{N} = 2$ instanton expansion. One is via the solution of Picard–Fuchs differential equation [17]; this method quickly becomes cumbersome as the rank of the gauge group increases. A promising technique involves recursion relations relating multi-instanton results to the one-instanton results [18]. Other methods utilize the connection to integrable models [19] and Whitham hierarchies [20]. It would be interesting to see how the above strategies manifest themselves in the matrix model setup. Most importantly, there are other methods which also do not make reference to string/M-theory. In this context we note the beautiful work of Nekrasov [21]. It would be very interesting to connect this approach to that of the matrix model.

Each of the above methods has certain advantages for particular aspects of $\mathcal{N} = 2$ theories. The matrix model approach promises to give a number of new insights into the structure of $\mathcal{N} = 2$ gauge theories and their relations to string theory. In its present form the approach seems to be less computationally efficient than the state-of-the-art methods of Nekrasov [21], although there might be some models for which the matrix model approach offer certain advantages. We should also mention that the matrix model approach to $\mathcal{N} = 2$ theories as presented here is rather roundabout. A more direct approach would be desirable. In Ref. [5], a more direct route was proposed for SU(2) by relating this case to a double scaling limit of a unitary matrix model. It is not obvious to us, however, how to extend this to more general models.

Acknowledgements

Appendix A

In this appendix, we will derive the relation between a_i and e_i in an expansion in purely within the context of Seiberg–Witten theory. This calculation was done in a very elegant fashion in Ref. [13]. We will present an alternative route to the same result, based on the relation between the Seiberg–Witten differential and the density of gauge theory eigenvalues in the large-N limit [7].

Consider the U(KN) theory on the dimension N subspace of the Coulomb branch at which the symmetry is broken only to U(K)^N. The SW curve $\widetilde{\Sigma}$ on this subspace takes the form [2,12]

$$\widetilde{\Sigma}: \quad y^2 = P_{KN}(x)^2 - 4\widetilde{\Lambda}^{2KN},$$

$$P_{KN}(x) = \prod_{i=1}^{N}\prod_{j=1}^{K}(x - \tilde{e}_{ij}) = \widetilde{\Lambda}^{KN} T_K\left(P_N(x)/\Lambda^N\right), \tag{A.1}$$

where

$$T_K(x) = 2\cos\left[K\arccos\left(\frac{1}{2}x\right)\right] \tag{A.2}$$

are the first Chebyshev polynomials. Using $T_K(x)^2 - 4 = (x^2 - 4)U_{K-1}(x)^2$, where $U_K(x)$ are the second Chebyshev polynomials, it follows that

$$y^2 = \widetilde{\Lambda}^{2KN}\left(\Lambda^{-2N}\left[U_{K-1}\left(P_N(x)/\Lambda^N\right)\right]^2\left(P_N(x)^2 - 4\Lambda^{2N}\right)\right) \tag{A.3}$$

that is, $\widetilde{\Sigma}$ shares the $2N$ branch points of Σ, and the other $2N(K-1)$ branch points coalesce in pairs to give $N(K-1)$ double zeros of y^2. In other words, the KN branch cuts of the U(KN) Seiberg–Witten curve at a generic point in its moduli space merge along this subspace of the moduli space to give the N branch cuts of the U(N) theory. Thus, the A_i cycle of Σ is the sum $\sum_{j=1}^{K}\tilde{A}_{ij}$ of cycles of $\widetilde{\Sigma}$.

From Eqs. (A.1) and (A.2), the roots $\tilde{e}_{ij}$ of $P_{KN}(x)$ obey

$$P_N(\tilde{e}_{ij}) = 2\Lambda^N\cos\left[\pi\left(j-\frac{1}{2}\right)\Big/K\right], \quad j = 1,\ldots,K \tag{A.4}$$

that is

$$\tilde{e}_{ij} - e_i = \frac{2\Lambda^N}{\prod_{k\neq i}(\tilde{e}_{ij} - e_k)}\cos\left[\pi\left(j-\frac{1}{2}\right)\Big/K\right]. \tag{A.5}$$

This equation can be iteratively solved for $\tilde{e}_{ij}$ giving

$$\tilde{e}_{ij} = e_i + \frac{2\Lambda^N\cos\left[\pi\left(j-\frac{1}{2}\right)/K\right]}{\prod_{k\neq i}(e_i - e_k)}$$
$$- \frac{4\Lambda^{2N}\cos^2\left[\pi\left(j-\frac{1}{2}\right)/K\right]}{\prod_{k\neq i}(e_i - e_k)^2}\sum_{\ell\neq i}\frac{1}{(e_i - e_\ell)} + \cdots. \tag{A.6}$$

Moreover, using Eq. (A.4), one may calculate that, in the large K limit, the density $\sigma(x)$ of $\tilde{e}_{ij}$ along the branch cut is

$$\sigma(x) = \frac{K}{\pi}\frac{P_N'(x)}{\sqrt{4\Lambda^{2N} - P_N(x)^2}}. \tag{A.7}$$

Comparing this to Eq. (2.3) we see that

$$\sigma(x)\,dx = \frac{K}{\pi i x}\lambda_{SW} \tag{A.8}$$

which is no accident, as we will now see.

Again using Eq. (A.2), one may compute that the SW differential of the U(KN) theory along this subspace of the Coulomb branch is proportional to that of the U(N) theory:

$$\tilde{\lambda}_{\rm SW} = \frac{x\,P'_{KN}(x)\,{\rm d}x}{\sqrt{P^2_{KN}(x) \; 4 \; {}^{\sim 2KN}}} = \frac{K x\,P'_N(x)\,{\rm d}x}{\sqrt{P^2_N(x) \; 4 \; {}^{2N}}} = K\lambda_{\rm SW}. \tag{A.9}$$

Thus

$$a_i = \frac{1}{2\pi i} \oint_{A_i} \lambda_{\rm SW} = \frac{1}{2\pi i K} \sum_{j=1}^{K} \oint_{\tilde{A}_{ij}} \tilde{\lambda}_{\rm SW} = \frac{1}{K} \sum_{j=1}^{K} \tilde{a}_{ij}. \tag{A.10}$$

Also, using Eq. (5.2), it immediately follows from Eq. (A.9) that $\langle \tilde{u}_n \rangle = K \langle u_n \rangle$ for $n = 1, \ldots, N$, where $\langle \tilde{u}_n \rangle$ are the vevs of $(1/n)\,{\rm tr}(\phi^n)$ in SU(KN). Eq. (A.10) holds for all K, so we take K large. In the $K \to \infty$ limit, $\tilde{a}_{ij} = \tilde{e}_{ij}$, thus

$$a_i = \lim_{K \to \infty} \frac{1}{K} \sum_{j=1}^{K} \tilde{e}_{ij}. \tag{A.11}$$

In the large K limit, the sum over j can be replaced with an integral over the density of $\tilde{e}_{ij}$'s along the ith cut

$$a_i = \frac{1}{K} \int_{i\text{th cut}} x\sigma(x)\,{\rm d}x = \frac{1}{\pi i} \int_{i\text{th cut}} \lambda_{\rm SW} \tag{A.12}$$

which is simply our starting point (2.3), since the integral along the cut is exactly half an A_i cycle. Moreover, in the large K limit, we can use Eq. (A.6) in Eq. (A.11) to obtain

$$a_i = e_i \quad \frac{2 \quad {}^{2N}}{\prod_{k \neq i}(e_i \quad e_k)^2} \sum_{\ell \neq i} \frac{1}{(e_i \quad e_l)} + \cdots. \tag{A.13}$$

This agrees with the results of Ref. [13], and with the matrix model calculation presented in the main body of this paper.

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