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Matrix-model description of $\mathcal{N}=2$ gauge theories with non-hyperelliptic Seiberg–Witten curves

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Abstract

Using matrix-model methods we study three different $=2$ models: $U(N)\times U(N)$ with matter in the bifundamental representation, $U(N)$ with matter in the symmetric representation, and $U(N)$ with matter in the antisymmetric representation. We find that the (singular) cubic Seiberg–Witten curves (and associated Seiberg–Witten differentials) implied by the matrix models, although of a different form from the ones previously proposed using M-theory, can be transformed into the latter and are thus physically equivalent. We also calculate the one-instanton corrections to the gauge-coupling matrix using the perturbative expansion of the matrix model. For the $U(N)$ theories with symmetric or antisymmetric matter we use the modified matrix-model prescription for the gauge-coupling matrix discussed in our paper: Cubic curves from matrix models and generalized Konishi anomalies (hep-th/0303268). Moreover, in the matrix model for the $U(N)$ theory with antisymmetric matter, one is required to expand around a different vacuum than one would naively have anticipated. With these modifications of the matrix-model prescription, the results of this paper are in complete agreement with those of Seiberg–Witten theory obtained using M-theory methods.

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1. Introduction

The study of $\mathcal{N} = 2$ gauge theories using the matrix-model techniques of Dijkgraaf and Vafa was initiated in [2,3] (see [4] for earlier work) and was further developed and studied in [5–10] (see also [11]). In [7] we showed that it is possible to derive all the building blocks of the Seiberg–Witten (SW) solution [12] (i.e., the curve and a preferred meromorphic differential) purely within the matrix-model context. Since the curve is obtained from the large-M solution of the matrix model, one can obtain the SW curve in this manner only when an explicit large-M solution is available. However, as was stressed in [3,5], even when the large-M solution is not available, one can still resort to perturbation theory to derive the prepotential order-by-order without knowledge of the curve or differential. A necessary ingredient for this procedure is the knowledge of the quantum order parameters a_i (periods of the SW curve); in [5] we proposed a perturbative definition of the periods.

In this paper, we extend the matrix-model program to three $\mathcal{N} = 2$ gauge theories whose SW curves are non-hyperelliptic: $U(N) \times U(N)$ with a bifundamental hypermultiplet, $U(N)$ with a symmetric hypermultiplet, and $U(N)$ with an antisymmetric hypermultiplet. In Section 2 we review some previous results for these theories and reformulate them in a way that will facilitate comparison with our later discussion.

The equivalence of the $\mathcal{N} = 1$ versions of the gauge theories considered in this paper and the corresponding matrix models was established, following the approach in [13], in Refs. [1,14] (related earlier and subsequent developments can be found in [15–20]). In principle these general results also show the validity of the matrix-model approach in the $\mathcal{N} = 2$ limit. However, to obtain precise information about the $\mathcal{N} = 2$ gauge theories requires substantial additional work. Furthermore, there are several subtle issues, particularly for the theory with antisymmetric matter, that need to be addressed in order to recover the known results from the matrix models.

In Section 3 we compute the gauge-coupling matrices τ_{ij} using the perturbative expansion of the matrix model. For the $U(N) \times U(N)$ model, the gauge-coupling matrices are given by the second derivative of the free energy of the matrix model [13,15,21]. However, for the $U(N)$ theories with symmetric or antisymmetric hypermultiplets, certain crucial signs must be included among the terms of the second derivative of the free energy to obtain τ_{ij}. The rationale for this was described in Ref. [1], and is implemented in Section 3.2.2.

Furthermore, for the $U(N)$ theory with antisymmetric matter, we will show in Section 3.2 that, in order to obtain a gauge-coupling matrix and $\mathcal{N} = 2$ prepotential that agree with those computed using SW theory, one must perturbatively expand the matrix model around a vacuum different from the one that would have been naively anticipated. In subsequent sections, we will see that this choice of vacuum is similarly required to reproduce the known SW curve and differential. The underlying reason for this choice of vacuum remains somewhat obscure to us.

The matrix models associated with the gauge theories we consider can be solved in the large-M limit, giving rise to cubic algebraic curves [1,2,23–25]

$$u^3 \quad r(z)u \quad s(z) = 0. \tag{1.1}$$

The functions $r(z)$ and $s(z)$ are determined by the matrix-model potential, up to two arbitrary polynomials $r_1(z)$ and $t_1(z)$ (which can be expressed in terms of the eigenvalues of the adjoint field(s) of the matrix model [1,24,25]). To fix the forms of $r_1(z)$ and $t_1(z)$, one must impose an additional criterion, namely, the extremization of the effective superpotential of the associated gauge theory.

In Section 4, we use matrix-model perturbation theory to provide a simple and efficient method for determining the polynomials $r_1(z)$ and $t_1(z)$, and thus the cubic curve, in an expansion in the quantum scale Λ. We show that the cubic curves obtained for each of the theories considered can be transformed into the SW curves of those theories obtained using M-theory (and also from geometric engineering for the $U(N)\times U(N)$ theory).

Also in Section 4, we use extremization of W_{eff} together with the saddle-point solution to derive a condition which implies, via Abel's theorem, the existence of a certain function on the matrix-model curve. We then show that such a function exists on the known (exact) M-theory curves. Assuming uniqueness, this demonstrates that extremization of W_{eff} leads to a matrix-model curve that agrees exactly with the M-theory curve.

The Seiberg–Witten differentials for the gauge theories studied in this paper can also be obtained within the matrix-model framework. In Section 5, we compute these using matrix-model perturbation theory, obtaining agreement with the SW differentials known from M-theory.

Appendix A contain a derivation of the SW curve and differential for the $\mathcal{N}=2$ $U(N)$ theory with fundamental hypermultiplets using methods developed in Sections 4 and 5 of this paper. Appendix B contains some technical details of the calculations of Section 3.

2. Cubic Seiberg–Witten curves from M-theory

The Seiberg–Witten curves and differentials for the $\mathcal{N}=2$ gauge theories considered in this paper were previously obtained using M-theory methods, following the approach of Ref. [26]. (The $U(N)\times U(N)$ curve can also be obtained using geometric engineering [27].)

The SW curves for the theories:

(a) $\mathcal{N}=2$ $U(N)\times U(N)$ with an $\mathcal{N}=2$ bifundamental hypermultiplet [26,28];
(b) $\mathcal{N}=2$ $U(N)$ model with one symmetric ($\square\!\square$) hypermultiplet [29];
(c) $\mathcal{N}=2$ $U(N)$ model with one antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) hypermultiplet [29],

obtained from M-theory considerations are given by

$$y^3+P(z)y^2+\Lambda'^N\widetilde{P}(z)y+\ \Lambda'^2\widetilde{\Lambda}'\big)^N=0, \tag{2.1a}$$

$$y^3+P(z)\,y^2+\Lambda'^{N\ \ 2}z^2P(\ \ z)\,y+\Lambda'^{3N\ \ 6}z^6=0, \tag{2.1b}$$

$$y^3+\left[P(z)+\frac{3\Lambda'^{N+2}}{z^2}\right]y^2+\frac{\Lambda'^{N+2}}{z^2}\left[P(\ \ z)+\frac{3\Lambda'^{N+2}}{z^2}\right]y+\frac{\Lambda'^{3N+6}}{z^6}=0, \tag{2.1c}$$

where $P(z)=\prod_{i=1}^{N}(z-e_i')$ and $\widetilde{P}(z)=\prod_{i=1}^{N}(z-\tilde{e}_i')$. The Seiberg–Witten differential for each of the theories above, obtained from the M-theory setup, is given by [26,29,30]

$$\lambda_{\mathrm{SW}}=z\,\frac{\mathrm{d}y}{y}. \tag{2.2}$$

The map $y\to(\Lambda'\widetilde{\Lambda}')^N/y$ in the curve (2.1a) corresponds to exchanging the two factors of the gauge group, i.e., it can be undone by interchanging $e_i'\leftrightarrow\tilde{e}_i'$, $\widetilde{\Lambda}'\leftrightarrow\Lambda'$, and thus leads to a physically equivalent curve. The two curves (2.1b), (2.1c) are invariant under the involutions $z\to -z$, $y\to\Lambda'^{2N-4}z^4/y$, and $z\to -z$, $y\to\Lambda'^{2N+4}/z^4y$, respectively. The actual SW curves for these theories are the quotients of the curves (2.1b), (2.1c) by the involution. This reflects the presence of the orientifold plane in the type IIA brane configurations that lift to the M-theory configurations leading to these curves.

Using (2.1a)–(2.1c) and (2.2), the leading term in the instanton expansion of the prepotential for each of the theories above was derived in Refs. [31–33]. Recently a more efficient method has been developed [34,35] based on earlier work [36]. (See also [37] for another approach.) In Section 3, we will reproduce these results from a perturbative matrix-model calculation.

For later comparison with matrix model results (Section 4), we need to transform the above curves into another form, which is invariant under the maps discussed above. To do this we define[4]

$$u'=-y-2\Lambda'^{N-2\beta}z^{2\beta}-\frac{\Lambda'^{2N-4\beta}z^{4\beta}}{y}, \tag{2.3}$$

$$w=\frac{1}{2}z^{-\beta}\left[y-\frac{\Lambda'^{2N-4\beta}z^{4\beta}}{y}\right], \tag{2.4}$$

where $\beta=0,1,-1$ for curves (2.1a), (2.1b), and (2.1c) respectively; note that u' is invariant under the maps discussed above. The variables u', w are related via

$$4w^2=u'^2+4\Lambda'^N u', \tag{2.5a}$$

$$4z^2w^2=u'^2+4\Lambda'^{N-2}z^2u', \tag{2.5b}$$

$$4w^2=z^2u'^2+4\Lambda'^{N+2}u'. \tag{2.5c}$$

Using Eq. (2.3) and Eqs. (2.1a)–(2.1c), one may show that

$$y=-\Lambda'^N\left(\frac{u'-\widetilde{P}(z)+3\Lambda'^N}{u'-P(z)+3\Lambda'^N}\right), \tag{2.6a}$$

$$y=-\Lambda'^{N-2}z^2\left(\frac{u'-P(-z)+3\Lambda'^{N-2}z^2}{u'-P(z)+3\Lambda'^{N-2}z^2}\right), \tag{2.6b}$$

$$y=-\frac{\Lambda'^{N+2}}{z^2}\left(\frac{u'-P(-z)}{u'-P(z)}\right). \tag{2.6c}$$

[4] Henceforth, for simplicity, we will set $\widetilde{\Lambda}'=\Lambda'$ in the $U(N)\times U(N)$ model.

Next, use the definition of u' to write $w = z^{-\beta} y + \frac{1}{2} z^{-\beta} u' + \Lambda'^{N-2\beta} z^{\beta}$. Substitute Eqs. (2.6a)–(2.6c) into this equation, square it, and use Eqs. (2.5a)–(2.5c) to find[5]

$$\big(u' - P(z) + 3\Lambda'^N\big)u'\big(u' - \widetilde{P}(z) + 3\Lambda'^N\big) = \Lambda'^N \big(P(z) - \widetilde{P}(z)\big)^2, \tag{2.7a}$$

$$\begin{aligned} &\big(u' - P(z) + 3\Lambda'^{N-2} z^2\big)u'\big(u' - P(-z) + 3\Lambda'^{N-2} z^2\big) \\ &\quad = \Lambda'^{N-2} z^2 \big(P(z) - P(-z)\big)^2, \end{aligned} \tag{2.7b}$$

$$\big(u' - P(z)\big)u'\big(u' - P(-z)\big) = \frac{\Lambda'^{N+2}}{z^2} \big(P(z) - P(-z)\big)^2. \tag{2.7c}$$

In Section 4, we will compare the matrix-model curves to the SW curves written in this form.

On the last two curves, the involution acts as $z \to -z$, with u' invariant. The invariance of the curves under the involution means that the equations can be written in terms of u' and z^2. The actual SW curve (quotient by the involution) is thus a cover of the z^2 plane. The first curve (2.7a) is invariant under the interchange of the two gauge groups.

Were we to reverse the transformation, described in the last paragraph, from the curves (2.1a)–(2.1c) to the ones in (2.7a)–(2.7c), we would obtain two solutions, due to the fact that we squared both sides of an equation in one of the steps above. However, these two solutions are related by the involution in cases (b) and (c). In case (a), the two solutions are related by $e'_i \leftrightarrow \tilde{e}'_i$, i.e., $P \leftrightarrow \widetilde{P}$ and so correspond to exchanging the two $U(N)$ factors. Hence in all cases the two solutions are physically equivalent.

When $\Lambda' \to 0$, the curves (2.7a)–(2.7c) are singular at the roots of $P(z)$, $\widetilde{P}(z)$ (or $P(-z)$), and $P(z) - \widetilde{P}(z)$ (or $P(z) - P(-z)$). (The discriminant has double zeros at those points.) When $\Lambda' \neq 0$, the surface is deformed such that the first two sets of singular points open up into branch cuts, but the singularities at the points z where $P(z) = \widetilde{P}(z)$ (or $P(z) = P(-z)$) remain.[6] One (important) exception occurs for the theory with antisymmetric matter, where the singularity at $z = 0$ also opens up into a branch cut in the curve (2.7c). For the theory with symmetric matter, $z = 0$ remains a singular point in the curve (2.7b).

3. Perturbative approach to the matrix model

As we discussed in [5] (see also [3]), the $\mathcal{N} = 2$ gauge theory prepotential (in an instanton expansion) may be determined using only matrix-model perturbation theory. In this approach, one adds to the $\mathcal{N} = 2$ superpotential an additional piece which freezes the moduli to a generic, but fixed, point on the Coulomb branch of the $\mathcal{N} = 2$ theory, and breaks the $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. After the relevant quantities are computed, the extra piece is removed, restoring $\mathcal{N} = 2$ supersymmetry.

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[5] The form of the curve for $U(N)$ with one antisymmetric hypermultiplet (2.7c) was first obtained in Section 7 of Ref. [29], where the connection to the Atiyah–Hitchin space (2.5c) was also discussed.

[6] This fact is important to get the genus counting to work, cf. Section 4.

This approach was first explored in [3] where the gauge-coupling matrix τ_{ij} was determined for $U(2)$, and was extended to $U(N)$ in [5] where, in particular, a proposal for how to determine the relation between the quantum order parameters a_i and their classical counterparts e_i entirely within the context of matrix-model perturbation theory was put forward. Using this proposal the prepotential $\mathcal{F}(a)$ was calculated to one-instanton level and was shown to agree with the well-known result. In [7] the calculation was extended to include matter in the fundamental representation, and in [9,10] to SO/Sp gauge groups.

In this section, we extend this perturbative matrix-model method to new cases by calculating the one-instanton contribution to the $\mathcal{N}=2$ prepotential in the $U(N)\times U(N)$ gauge theory with a bifundamental hypermultiplet, and the $U(N)$ gauge theory with one symmetric ($\square\!\square$) or antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) hypermultiplet. Besides the additional complication of dealing with two-index matter, and the inclusion of diagrams with the topology of $\mathbb{RP}^2$ (for $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$), there is one significant modification of the procedure developed in Refs. [5,7]: namely, for the models with symmetric or antisymmetric matter, τ_{ij} is no longer given simply by the second derivative of the free energy [1] (see Section 3.2.2). Moreover, in the matrix model for the $U(N)$ theory with antisymmetric matter, one is required to expand around a different vacuum than one would naively have anticipated (this may be related to the results in [22]).

Previously the one-instanton prepotential for these models has been obtained using M-theory methods [31–35] (see also [37] for another approach).

3.1. $U(N)\times U(N)$ with a bifundamental hypermultiplet

Consider the $\mathcal{N}=1$ $U(N)\times U(N)$ supersymmetric gauge theory with the following matter content: two chiral superfields $\phi_i{}^j$, $\tilde{\phi}_{\tilde{\imath}}{}^{\tilde{\jmath}}$ transforming in the adjoint representation of each of the two factors of the gauge group, one chiral superfield $b_i{}^{\tilde{\jmath}}$ transforming in the bifundamental representation $(\square, \overline{\square})$, and one chiral superfield $\tilde{b}_{\tilde{\imath}}{}^j$ transforming in the bifundamental representation $(\overline{\square}, \square)$. The superpotential of this gauge theory is taken to be of the form[7]

$$\mathcal{W}(\phi,\tilde{\phi},b,\tilde{b}) = \mathrm{tr}\big[W(\phi) - \widetilde{W}(\tilde{\phi}) - \tilde{b}\phi b + b\tilde{\phi}\tilde{b}\big], \tag{3.1}$$

where $W(\phi)$ and $\widetilde{W}(\tilde{\phi})$ are polynomials such that

$$W'(z) = \alpha\prod_{j=1}^{N}(z - e_j), \qquad \widetilde{W}'(z) = \alpha\prod_{j=1}^{N}(z - \tilde{e}_j). \tag{3.2}$$

The superpotential (3.1) can be viewed as a deformation of an $\mathcal{N}=2$ theory, which is recovered when $\alpha\to 0$. At the end of our calculation we will take this limit thereby obtaining results valid in the $\mathcal{N}=2$ theory.

[7] A mass term for the bifundamental fields, $m\,\mathrm{tr}(\tilde{b}b)$, can be introduced by shifting $\phi\to\phi - m/2$ ($\tilde{\phi}\to\tilde{\phi}+m/2$) and $e_i\to e_i - m/2$ ($\tilde{e}_i\to\tilde{e}_i+m/2$).

The $U(M) \times U(\widetilde{M})$ matrix model associated with this gauge theory [1,2,23,24] has partition function[8]

$$Z = \frac{1}{\operatorname{vol} G} \int \mathrm{d}\Phi \, \mathrm{d}\tilde{\Phi} \, \mathrm{d}B \, \mathrm{d}\widetilde{B} \, \exp\biggl(\quad \frac{1}{g_s} \operatorname{tr}\bigl[W(\Phi) \quad \widetilde{W}(\tilde{\Phi}) \quad \widetilde{B}\Phi B + B\tilde{\Phi}\widetilde{B} \bigr] \biggr), \tag{3.3}$$

where Φ is an $M \times M$ matrix, $\tilde{\Phi}$ is an $\widetilde{M} \times \widetilde{M}$ matrix, B is an $M \times \widetilde{M}$ matrix, and $\widetilde{B}$ is an $\widetilde{M} \times M$ matrix. In the perturbative approach, the matrix integral (3.3) is evaluated about the following extremal point of the potential

$$\Phi_0 = \begin{pmatrix} e_1 \ {}_{M_1} & 0 & \cdots & 0 \\ 0 & e_2 \ {}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \ {}_{M_N} \end{pmatrix},$$

$$\tilde{\Phi}_0 = \begin{pmatrix} \tilde{e}_1 \ {}_{\widetilde{M}_1} & 0 & \cdots & 0 \\ 0 & \tilde{e}_2 \ {}_{\widetilde{M}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{e}_N \ {}_{\widetilde{M}_N} \end{pmatrix},$$

$$B_0 = \widetilde{B}_0 = 0, \tag{3.4}$$

where $\sum_i M_i = M$ and $\sum_i \widetilde{M}_i = \widetilde{M}$. This choice of vacuum breaks the $U(M) \times U(\widetilde{M})$ symmetry to $G = \prod_{i=1}^{N} U(M_i) \times \prod_{i=1}^{N} U(\widetilde{M}_i)$. Writing $\Phi = \Phi_0 + \Psi$ and $\tilde{\Phi} = \tilde{\Phi}_0 + \tilde{\Psi}$, one finds that the off-diagonal fields Ψ_{ij} and $\tilde{\Psi}_{ij}$ $(i \neq j)$ have vanishing contributions to the quadratic part of the action; these fields are zero modes and correspond to gauge degrees of freedom [3]. We fix the gauge $\Psi_{ij} = \tilde{\Psi}_{ij} = 0$ $(i \neq j)$ and introduce Grassmann-odd ghost matrices, exactly as in Refs. [3,5], to which we refer the reader for further details. The bifundamental field B (not to be confused with the ghost field in Refs. [3,5]) can be written

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1N} \\ B_{21} & B_{22} & \cdots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} & B_{N2} & \cdots & B_{NN} \end{pmatrix}, \tag{3.5}$$

where B_{ij} is an $M_i \times \widetilde{M}_j$ matrix, and similarly for $\widetilde{B}$. Expanding about the vacuum (3.4) one finds (in the $\Psi_{ij} = \tilde{\Psi}_{ij} = 0$ $(i \neq j)$ gauge)

$$\operatorname{tr}[\widetilde{B}\Phi B \quad B\tilde{\Phi}\widetilde{B}] = \sum_{i,j} (e_i \quad \tilde{e}_j) \operatorname{tr}(\widetilde{B}_{ji} B_{ij}) + \sum_{i,j} \operatorname{tr}(\widetilde{B}_{ji} \Psi_{ii} B_{ij} \quad B_{ij} \tilde{\Psi}_{jj} \widetilde{B}_{ji}). \tag{3.6}$$

[8] We use capital letters to denote matrix model quantities.

Hence, the matrix integral over the quadratic action contains the e_i-dependent contribution from the bifundamental fields:

$$\prod_{i,j}\left(\frac{1}{e_i \quad \tilde{e}_j}\right)^{M_i\widetilde{M}_j} \tag{3.7}$$

and the trilinear pieces of (3.6) contribute $\widetilde{B}\Psi B$ and $B\tilde{\Psi}\widetilde{B}$ vertices to the Feynman diagrams (in addition to the vertices considered in Ref. [5]).

We are interested in the planar limit of the matrix model, i.e., the limit in which $g_s \to 0$ and $M_i, \widetilde{M}_i \to \infty$, keeping $g_s M_i$ and $g_s \widetilde{M}_i$ fixed. The connected diagrams of the perturbative expansion of $\log Z$ may be organized, using the standard double-line notation, in a topological expansion characterized by the Euler characteristic χ of the surface in which the diagram is embedded [38]

$$\log Z = \sum_{\chi \quad 2} g_s^{\quad \chi} F_\chi(S, \widetilde{S}) \quad \text{where} \quad S_i \equiv g_s M_i,\ \widetilde{S}_i \equiv g_s \widetilde{M}_i, \tag{3.8}$$

where $\chi = 2 \quad 2g$ with g the genus. In the planar limit, the leading contribution

$$F_{\mathrm{s}}(S, \widetilde{S}) \equiv F_{\chi=2}(S, \widetilde{S}) = g_s^2 \log Z|_{\mathrm{sphere}} \tag{3.9}$$

comes from the connected diagrams that can be drawn on the sphere ($\chi = 2$). We will need the contributions to $F_{\mathrm{s}}(S, \widetilde{S})$ up to cubic order in S and $\widetilde{S}$. The explicit formulæ can be found in Eqs. (B.1) and (B.2) in Appendix B.

To relate the matrix model and its free energy to the $\mathcal{N} = 2$ $U(N) \times U(N)$ gauge theory broken to $\prod_i U(N_i) \times \prod_i U(\widetilde{N}_i)$, one introduces, following Dijkgraaf and Vafa,

$$W_{\mathrm{eff}}(S, \widetilde{S}) = \quad \sum_i N_i \frac{\partial}{\partial S_i} F_{\mathrm{s}}(S, \widetilde{S}) \quad \sum_i \widetilde{N}_i \frac{\partial}{\partial \widetilde{S}_i} F_{\mathrm{s}}(S, \widetilde{S}), \tag{3.10}$$

where we have dropped terms linear in S_i and $\widetilde{S}_i$. Since we are examining a generic point on the Coulomb branch of the $\mathcal{N} = 2$ theory, which breaks $U(N) \times U(N)$ to $U(1)^{2N}$, we set $N_i = \widetilde{N}_i = 1$. Next, one extremizes the effective superpotential with respect to S_i and $\widetilde{S}_i$:

$$\left.\frac{\partial W_{\mathrm{eff}}(S, \widetilde{S})}{\partial S_i}\right|_{S_j = \langle S_j\rangle,\ \widetilde{S}_j = \langle \widetilde{S}_j\rangle} = 0, \qquad \left.\frac{\partial W_{\mathrm{eff}}(S, \widetilde{S})}{\partial \widetilde{S}_i}\right|_{S_j = \langle S_j\rangle,\ \widetilde{S}_j = \langle \widetilde{S}_j\rangle} = 0. \tag{3.11}$$

The solutions for $\langle S_i\rangle$, $\langle \widetilde{S}_i\rangle$ can be evaluated in an expansion in Λ. The lowest-order contributions are

$$\langle S_i\rangle = \frac{\alpha T_i}{R_i}\Lambda^N, \qquad \langle \widetilde{S}_i\rangle = \quad \frac{\alpha \widetilde{T}_i}{\widetilde{R}_i}\Lambda^N, \tag{3.12}$$

where

$$T_i = \prod_{j=1}^N (e_i \quad \tilde{e}_j), \qquad \widetilde{T}_i = \prod_{j=1}^N (\tilde{e}_i \quad e_j), \qquad R_i = \prod_{j\neq i} (e_i \quad e_j),$$
$$\widetilde{R}_i = \prod_{j\neq i} (\tilde{e}_i \quad \tilde{e}_j) \tag{3.13}$$

and various constants have been absorbed into a redefinition of the cut-off Λ. In Section 3.1.2, we will also need the next-to-leading-order contributions; these are given in (B.3).

3.1.1. Relation between a_i and e_i

Before computing τ_{ij} and the $\mathcal{N}=2$ prepotential, we must determine the relation between e_i and the periods a_i of the SW differential. In Ref. [5], we proposed a definition of a_i within the context of the perturbation expansion of the matrix model, without referring to the Seiberg–Witten curve or differential. As in Refs. [5,7], a_i and $\tilde{a}_i$ can be determined perturbatively via (setting $N_i = \widetilde{N}_i = 1$)

$$a_i = e_i + \left[\sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \operatorname{tr} \Psi_{ii}\rangle_{S^2} + \sum_{j=1}^{N} \frac{\partial}{\partial \widetilde{S}_j} g_s \langle \operatorname{tr} \Psi_{ii}\rangle_{S^2}\right]_{\langle S\rangle, \langle \widetilde{S}\rangle},$$
$$\tilde{a}_i = \tilde{e}_i + \left[\sum_{j=1}^{N} \frac{\partial}{\partial \widetilde{S}_j} g_s \langle \operatorname{tr} \tilde{\Psi}_{ii}\rangle_{S^2} + \sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \operatorname{tr} \tilde{\Psi}_{ii}\rangle_{S^2}\right]_{\langle S\rangle, \langle \widetilde{S}\rangle}, \tag{3.14}$$

where $\langle \operatorname{tr} \Psi_{ii}\rangle_{S^2}$ ($\langle \operatorname{tr} \tilde{\Psi}_{ii}\rangle_{S^2}$) is obtained by calculating all connected planar tadpole diagrams with an external Ψ_{ii} ($\tilde{\Psi}_{ii}$) leg that can be drawn on a sphere. In addition to the tadpole diagrams discussed in Ref. [5], there are diagrams with B, $\widetilde{B}$ (bifundamental) loops. The total contribution to the tadpole quadratic in S and $\widetilde{S}$ is

$$\langle \operatorname{tr} \Psi_{ii}\rangle_{S^2} = \frac{1}{\alpha g_s}\left[-\sum_{j\neq i} \frac{S_i^2}{R_i e_{ij}} + \sum_{j\neq i} 2\frac{S_i S_j}{R_i e_{ij}} - \sum_{j} \frac{S_i \widetilde{S}_j}{R_i h_{ij}}\right],$$
$$\langle \operatorname{tr} \tilde{\Psi}_{ii}\rangle_{S^2} = -\frac{1}{\alpha g_s}\left[-\sum_{j\neq i} \frac{\widetilde{S}_i^2}{\widetilde{R}_i \tilde{e}_{ij}} + \sum_{j\neq i} 2\frac{\widetilde{S}_i \widetilde{S}_j}{\widetilde{R}_i \tilde{e}_{ij}} - \sum_{j} \frac{\widetilde{S}_i S_j}{\widetilde{R}_i \tilde{h}_{ij}}\right], \tag{3.15}$$

where $e_{ij} = e_i - e_j$, $h_{ij} = e_i - \tilde{e}_j$ and $\tilde{h}_{ij} = \tilde{e}_i - e_j = -h_{ji}$. Inserting these results into Eq. (3.14), and evaluating the resulting expression using Eq. (3.12), one finds

$$a_i = e_i + \Lambda^N\left(\frac{2}{R_i}\sum_{j\neq i}\frac{T_j}{R_j e_{ij}} + \frac{1}{R_i}\sum_j \frac{\widetilde{T}_j}{\widetilde{R}_j h_{ij}} - \frac{T_i}{R_i^2}\sum_j \frac{1}{h_{ij}}\right) + \mathcal{O}(\Lambda^{2N}),$$
$$\tilde{a}_i = \tilde{e}_i + \Lambda^N\left(\frac{2}{\widetilde{R}_i}\sum_{j\neq i}\frac{\widetilde{T}_j}{\widetilde{R}_j \tilde{e}_{ij}} + \frac{1}{\widetilde{R}_i}\sum_j \frac{T_j}{R_j \tilde{h}_{ij}} - \frac{\widetilde{T}_i}{\widetilde{R}_i^2}\sum_j \frac{1}{\tilde{h}_{ij}}\right) + \mathcal{O}(\Lambda^{2N}). \tag{3.16}$$

By using

$$\left[\frac{\widetilde{W}'(z)}{W'(z)} - 1\right] = \sum_i \frac{T_i}{R_i(z - e_i)}, \qquad \left[\frac{W'(z)}{\widetilde{W}'(z)} - 1\right] = \sum_i \frac{\widetilde{T}_i}{\widetilde{R}_i(z - \tilde{e}_i)} \tag{3.17}$$

one may show that

$$\sum_{j\neq i}\frac{T_j}{R_j e_{ij}} = \frac{T_i}{R_i}\left[-\sum_{j\neq i}\frac{1}{e_{ij}} + \sum_j \frac{1}{h_{ij}} - 1\right], \qquad \sum_j \frac{\widetilde{T}_j}{\widetilde{R}_j h_{ij}} = -1 \tag{3.18}$$

so that Eq. (3.16) may be rewritten[9]

$$a_i = e_i + \Lambda^N \left(\quad \frac{2T_i}{R_i^2} \sum_{j\neq i} \frac{1}{e_{ij}} + \frac{T_i}{R_i^2} \sum_j \frac{1}{h_{ij}} \quad \frac{3}{R_i} \right) + \mathcal{O}\; \Lambda^{2N}) \tag{3.19}$$

and similarly for $\tilde{a}_i$. We cannot yet compare this expression with the SW result obtained in [33], because the relation between the roots e_i of $W'(z)$ and the roots e_i' of $P(z)$ (cf. (2.1a)) has not yet been determined. This will be done in Section 4.1.1.

3.1.2. Perturbative calculation of τ_{ij}

Following Dijkgraaf and Vafa, the gauge coupling matrix τ_{ij} is related to the planar free energy F_s of the matrix model by

$$\tau_{ij} = \frac{1}{2\pi i} \frac{\partial^2 F_s}{\partial S_i \partial S_j}\bigg|_{\langle S\rangle, \langle \widetilde{S}\rangle}, \qquad \tau_{i\tilde{j}} = \frac{1}{2\pi i} \frac{\partial^2 F_s}{\partial S_i \partial \widetilde{S}_j}\bigg|_{\langle S\rangle, \langle \widetilde{S}\rangle},$$
$$\tau_{\tilde{i}\tilde{j}} = \frac{1}{2\pi i} \frac{\partial^2 F_s}{\partial \widetilde{S}_i \partial \widetilde{S}_j}\bigg|_{\langle S\rangle, \langle \widetilde{S}\rangle}. \tag{3.20}$$

We may calculate these expressions perturbatively using Eqs. (B.1)–(B.3), and finally, use Eq. (3.16) to re-express the entire expression in terms of a_i rather than e_i. The resulting perturbative and one-instanton contributions to the gauge coupling matrix are given in Eqs. (B.5)–(B.7) in Appendix B. One may verify that Eqs. (B.5)–(B.7) can be written as

$$\tau_{ij} = \frac{\partial^2 \quad (a,\tilde{a})}{\partial a_i \partial a_j}, \qquad \tau_{i\tilde{j}} = \frac{\partial^2 \quad (a,\tilde{a})}{\partial a_i \partial \tilde{a}_j}, \qquad \tau_{\tilde{i}\tilde{j}} = \frac{\partial^2 \quad (a,\tilde{a})}{\partial \tilde{a}_i \partial \tilde{a}_j} \tag{3.21}$$

with (up to a quadratic polynomial)

$$\begin{aligned} 2\pi i \quad (a,\tilde{a}) &= \frac{1}{4}\sum_i \sum_{j\neq i} (a_i \quad a_j)^2 \log\left(\frac{a_i \quad a_j}{\Lambda}\right)^2 \quad \frac{1}{4}\sum_i \sum_{j\neq i} (\tilde{a}_i \quad \tilde{a}_j)^2 \log\left(\frac{\tilde{a}_i \quad \tilde{a}_j}{\Lambda}\right)^2 \\ &+ \frac{1}{4}\sum_{i,j} (a_i \quad \tilde{a}_j)^2 \log\left(\frac{a_i \quad \tilde{a}_j}{\Lambda}\right)^2 + \Lambda^N \sum_j \left[\frac{T_j}{R_j^2} + \frac{\widetilde{T}_j}{\widetilde{R}_j^2}\right] + \mathcal{O}\; \Lambda^{2N}) \end{aligned} \tag{3.22}$$

which agrees perfectly with (version 2 of) Ref. [33].

3.2. $U(N)$ with $\square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

Consider the $\mathcal{N} = 1$ $U(N)$ supersymmetric gauge theory with one chiral superfield $\phi_i{}^j$ transforming in the adjoint representation of the gauge group, one chiral superfield x_{ij}

[9] In previous work [5,7] we used similar identities at intermediate stages of the calculations. However, in the calculation of the = 2 prepotential, it is more efficient to work with the expressions that come naturally out of the matrix-model calculation. To compare with results obtained using M-theory at intermediate stages, identities generally have to be used (as we did to obtain (3.19)).

transforming in either the symmetric ($\square\!\square$) or the antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) representation, and one chiral superfield $\tilde{x}^{ij}$ transforming in the conjugate representation. We treat the cases of the symmetric and antisymmetric representations simultaneously by assuming that x, $\tilde{x}$ satisfy $x^T = \beta x$ and $\tilde{x}^T = \beta\tilde{x}$, where $\beta = 1$ for the symmetric representation and $\beta = \quad 1$ for the antisymmetric representation. The superpotential of the gauge theory is taken to be of the form[10]

$$\mathcal{W}(\phi, x, \tilde{x}) = \mathrm{tr}\big[W(\phi) \quad \tilde{x}\,\phi\, x\big], \tag{3.23}$$

where $W(\phi)$ is a polynomial such that $W'(z) = \alpha \prod_{i=1}^{N}(z \quad e_i)$. This superpotential can be viewed as a deformation of an $\mathcal{N} = 2$ theory, which is recovered when $\alpha \to 0$, restoring $\mathcal{N} = 2$ supersymmetry.

As discussed in [25] there are several classical ground states of the superpotential (3.23). One such ground state is $\phi = e_i \quad$ and $x = \tilde{x} = 0$. Another one is $\phi = 0$, $x = E$ and $\tilde{x} = W'(0)E \quad ^1$, where $E = \quad$ for $\square\!\square$ and $E = J$ for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, where J is the usual Sp-unit. There are also additional ground states as discussed in [25], but these will play no role in our discussion. (Similar extra vacua are also present [23,39] in the $U(N) \times U(N)$ theory discussed above.) A more general vacuum is obtained by combining the above possibilities. In a block-diagonal basis, one ground state is $\phi = \mathrm{diag}(0_{N_0}, e_1 \quad _{N_1}, \ldots, e_k \quad _{N_k})$, with $N = N_0 + \sum_{i=1}^{k} N_i$ (where N_0 is even for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$) and x and $\tilde{x}$ have vanishing entries except for the $N_0 \times N_0$ blocks $x_{00} = E$, $\tilde{x}_{00} = W'(0)E \quad ^1$. Such a vacuum breaks $U(N)$ down to [25] $\mathrm{SO}(N_0) \times \prod_i U(N_i)$ for $\square\!\square$ or $\mathrm{Sp}(N_0) \times \prod_i U(N_i)$ for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$.

We want to freeze the $\mathcal{N} = 2$ moduli to a generic, but fixed, point on the Coulomb branch of the $\mathcal{N} = 2$ theory. This is accomplished by breaking $U(N)$ down to $U(1)^N$, i.e., choosing $N_i = 1$ and $N_0 = 0$.

The $U(M)$ matrix model associated with this gauge theory [1,25] has partition function[11]

$$Z = \frac{1}{\mathrm{vol}\, G} \int \mathrm{d}\Phi\, \mathrm{d}X\, \mathrm{d}\widetilde{X} \exp\!\left(\quad \frac{1}{g_s} \mathrm{tr}\big[W(\Phi) \quad \widetilde{X}\Phi X\big]\right), \tag{3.24}$$

where $X^T = \beta X$ and $\widetilde{X}^T = \beta\widetilde{X}$. In the perturbative approach, the matrix integral (3.24) is evaluated about an extremal point of $\mathcal{W}(\Phi, X, \widetilde{X})$.

Based on previous experience, it would seem natural to expand around a matrix-model vacuum similar to the gauge theory vacuum but with $N_i = 1$ and $N_0 = 0$ replaced by M_i and M_0 such that $M_i \neq 0$ and $M_0 = 0$. This will indeed turn out to be the right procedure for the $U(N) + \square\!\square$ theory. However, as we will see, it is not the right procedure for the $U(N) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ theory. Instead, for this theory, we will take $M_0 \neq 0$ (in fact we will take the limit $M_0 \to \infty$, with $g_s M_0$ finite), even though $N_0 = 0$. We do not have an a priori reason for making this choice of vacuum. If one does not include the extra $M_0 \times M_0$ block for the matrix model corresponding to the $U(N) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ theory, one still gets an (apparently)

[10] A mass term $m\,\mathrm{tr}(\tilde{x}x)$ for the matter hypermultiplet can be introduced by shifting $\phi \to \phi \quad m$ and $e_i \to e_i \quad m$.

[11] As in the previous section, we use capital letters to denote matrix model quantities. All matrix indices run over M values.

self-consistent result, but one which does not agree with the prepotential, SW curve, or SW differential derived from M-theory [29,31,37] (which have passed several consistency tests). Only if one includes the extra block does one get a result that is in agreement with previous results.[12]

We will decompose all matrices Υ as

$$\Upsilon = \begin{pmatrix} \Upsilon_{00} & \Upsilon_{01} & \cdots & \Upsilon_{0N} \\ \Upsilon_{10} & \Upsilon_{11} & \cdots & \Upsilon_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{N0} & \Upsilon_{N1} & \cdots & \Upsilon_{NN} \end{pmatrix}, \tag{3.25}$$

where Υ_{ij} is an $M_i \times M_j$ matrix, Υ_{i0} is an $M_i \times M_0$ matrix, Υ_{00} is an $M_0 \times M_0$ matrix (where M_0 is even for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$). Throughout we use $i, j = 1, \ldots, N$, displaying the index-0 terms explicitly.

We evaluate the matrix integral (3.24) about the following extremal point of $W(\Phi, X, \widetilde{X})$

$$\Phi_0 = \begin{pmatrix} 0_{M_0} & 0 & \cdots & 0 \\ 0 & e_1 \; _{M_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \; _{M_N} \end{pmatrix}, \tag{3.26}$$

and

$$X_0 = \begin{pmatrix} E & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \widetilde{X}_0 = W'(0) \begin{pmatrix} E^{\;\;1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.27}$$

where $M_0 + \sum_{i=1}^N M_i = M$, and as before $E = \; _{M_0}$ for $\square\square$, and $E = J$ for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, where J is the antisymmetric $\mathrm{Sp}(M_0)$ unit. This choice of vacuum breaks the $U(M)$ symmetry to [25] $G = \mathrm{SO}(M_0) \times \prod_{i=1}^N U(M_i)$ for $\square\square$, and $G = \mathrm{Sp}(M_0) \times \prod_{i=1}^N U(M_i)$ for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$.

The potential in (3.24) is invariant under the gauge symmetry

$$\delta\Phi = [\xi, \Phi], \qquad \delta X = \xi X + X\xi^T, \qquad \delta\widetilde{X} = \quad \xi^T \widetilde{X} \quad \widetilde{X}\xi. \tag{3.28}$$

Writing $\Phi = \Phi_0 + \Psi$ and $X = X_0 + Y$, we fix the gauge $\Psi_{ij} = 0$ $(i \neq j)$, $\Psi_{0i} = 0$, $\Psi_{i0} = 0$, $Y_{00} = 0$. Following [25] we use the BRST approach and introduce the gauge-fixing fermion

$$= \sum_{i \neq j} \mathrm{tr}(\overline{C}_{ij}\Phi_{ji}) + \sum_i \mathrm{tr}(\overline{C}_{i0}\Phi_{0i}) + \sum_i \mathrm{tr}(\overline{C}_{0i}\Phi_{i0}) + \mathrm{tr}\;\; \overline{C}_{00}[X_{00} \quad E]\big) \tag{3.29}$$

where the $\overline{C}$'s are Grassmann-odd. The relevant BRST transformations are [25]

$$s\,\Phi = [C, \Phi], \qquad s\,X = CX + XC^T, \qquad s\,\overline{C} = D \tag{3.30}$$

[12] Although the choice of vacuum appears ad hoc within the purely perturbative framework, it is quite likely that the vacuum is uniquely determined through the exact determination of the curve via Abel's theorem (method II in Section 4.2.2 of this paper), or equivalently, through the condition that all the periods of $T(z)$ are integer-valued [40]. See Refs. [41,42] for the use of the latter criterion to determine the vacua in related theories. (The $M_0 \times M_0$ block must be included whenever an additional cut opens up in the algebraic curve around $z = 0$.)

where C and D are Grassmann-odd matrices. The gauge-fixing action $S_{\mathrm{gf}} = s$ becomes, after using (3.30) and integrating out the D's which act as Lagrange multipliers implementing the gauge choice[13]

$$\sum_{i\neq j} \operatorname{tr}\, \overline{C}_{ij}[C_{ji}\Phi_{ii} \quad \Phi_{jj}C_{ji}]\big) + \sum_i \operatorname{tr}\, \overline{C}_{i0}[C_{0i}\Phi_{ii} \quad \Phi_{00}C_{0i}]\big)$$
$$+ \sum_i \operatorname{tr}\, \overline{C}_{0i}[C_{i0}\Phi_{00} \quad \Phi_{ii}C_{i0}]\big)$$
$$+ \operatorname{tr}\, \overline{C}_{00}\big[C_{00}E + EC_{00}^T\big]\big) + \sum_i \operatorname{tr}\, \overline{C}_{00}\big[C_{0i}Y_{i0} + Y_{0i}C_{i0}^T\big]\big). \tag{3.31}$$

Expanding about the vacuum (3.26), (3.27) using the above gauge one finds (in addition to the quadratic terms in Eq. (3.31)) the quadratic part of the action

$$\operatorname{tr}\big[W(\Phi) \quad \widetilde{X}\Phi X\big]$$
$$= \frac{1}{2}\alpha \sum_{i=1}^{N} R_i \operatorname{tr}\, \Psi_{ii}^2\big) + \frac{1}{2}\alpha R_0 \operatorname{tr}\, \Psi_{00}^2\big) \quad \operatorname{tr}(\widetilde{Y}_{00}\Psi_{00}E)$$
$$\sum_{i=1}^{N} e_i \operatorname{tr}(\widetilde{Y}_{0i}Y_{i0}) \quad \sum_{i=1}^{N} e_i \operatorname{tr}(\widetilde{Y}_{ii}Y_{ii}) \quad \sum_{i<j}(e_i + e_j)\operatorname{tr}(\widetilde{Y}_{ji}Y_{ij}) + \cdots, \tag{3.32}$$

where

$$R_i = \prod_{j\neq i}(e_i \quad e_j), \qquad R_0 = \frac{W''(0)}{\alpha} = \quad \prod_i(\quad e_i)\sum_j \frac{1}{e_j}. \tag{3.33}$$

From this we see [25] that the antisymmetric matrix $\widetilde{Y}_{00}$ acts as a Lagrange multiplier implementing the constraint $\Psi_{00}E + \beta(\Psi_{00}E)^T = \Psi_{00}E + E\Psi_{00}^T = 0$, i.e., $\Psi_{00} \in \mathrm{so}(M_0)$ for $\square\!\square$ and $\Psi_{00} \in \mathrm{sp}(M_0)$ for $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$.

The matrix integral over the quadratic action contains the e_i-dependent contributions

$$\left(\frac{1}{R_0}\right)^{\frac{1}{4}M_0^2 \quad \frac{1}{4}\beta M_0} \prod_i \left[\left(\frac{1}{R_i}\right)^{\frac{1}{2}M_i^2}\left(\frac{1}{e_i}\right)^{\frac{1}{2}M_i(M_i+\beta)} (e_i)^{M_0M_i}\right]$$
$$\times \prod_{i<j}\left[(e_i \quad e_j)^{2M_iM_j}\left(\frac{1}{e_i + e_j}\right)^{M_iM_j}\right]. \tag{3.34}$$

The following terms in the action (in addition to those considered in Ref. [5] and the trilinear terms in Eq. (3.31)) contribute cubic vertices to the Feynman diagrams:

$$\sum_{i=1}^{N}\big[\operatorname{tr}(\widetilde{Y}_{i0}\Psi_{00}Y_{0i}) + \operatorname{tr}(\widetilde{Y}_{0i}\Psi_{ii}Y_{i0}) + \operatorname{tr}(\widetilde{Y}_{ii}\Psi_{ii}Y_{ii})\big]$$
$$\sum_{i<j}\operatorname{tr}(\widetilde{Y}_{ji}\Psi_{ii}Y_{ij} + \widetilde{Y}_{ij}\Psi_{jj}Y_{ji}). \tag{3.35}$$

[13] In Refs. [3,5] the notation B, C was used for the ghost fields instead of $\overline{C}$ and C.

We are interested in the planar limit of the matrix model, i.e., the limit in which $g_s \to 0$ and $M_i, M_0 \to \infty$, keeping $g_s M_i$, $g_s M_0$ fixed. The connected diagrams of the perturbative expansion of $\log Z$ may be organized, using the standard double-line notation, in a topological expansion characterized by the Euler characteristic χ of the surface in which the diagram is embedded [38]

$$\log Z = \sum_{\chi \leq 2} g_s{}^{-\chi} F_\chi(S, \overline{S}), \tag{3.36}$$

where $\chi = 2 - 2g - q$ with g the genus (number of handles) and q the number of crosscaps. In the Feynman diagrams, we generally replace $g_s M_i$ by S_i and $g_s M_0$ by S_0, but for the inner index-loop of an X_{ij}, $\widetilde{X}_{ij}$ loop we write $g_s M_i = \overline{S_i}$ since the arrow on the inner index-loop runs parallel to the outer index-loop, opposite to the direction in which it would run for the adjoint representation This will be important when (but not until) we calculate τ_{ij} in Section 3.2.2.

In the planar limit, the leading contribution to the matrix integral comes from the planar diagrams that can be drawn on the sphere ($\chi = 2$),

$$F_{\mathrm{s}}(S, \overline{S}) \equiv F_{\chi=2}(S, \overline{S}) = g_s^2 \log Z|_{\mathrm{sphere}}. \tag{3.37}$$

The subleading contribution comes from planar diagrams that can be drawn on $\mathbb{RP}^2$, which is a sphere with one cross-cap inserted ($\chi = 1$)

$$F_{\mathrm{rp}}(S) \equiv F_{\chi=1}(S) = g_s \log Z|_{\mathbb{RP}^2}. \tag{3.38}$$

To evaluate the $(1/\operatorname{vol} G)$ prefactor in (3.24) we need (see e.g. [25,43,44])

$$\begin{aligned}
\log \operatorname{vol}\big(\mathrm{SO}(M_0)\big) &= -\frac{1}{4} M_0^2 \log M_0 + \frac{1}{4} M_0 \log M_0 + \cdots, \\
\log \operatorname{vol}\big(\mathrm{Sp}(M_0)\big) &= -\frac{1}{4} M_0^2 \log M_0 - \frac{1}{4} M_0 \log M_0 + \cdots, \\
\log \operatorname{vol}\big(U(M_i)\big) &= -\frac{1}{2} M_i^2 \log M_i + \cdots .
\end{aligned} \tag{3.39}$$

These results together with the integration over the quadratic fields of the matrix-model partition function yields (up to an e_i-independent quadratic monomial in the S's)

$$\begin{aligned}
F_{\mathrm{s}}(S, \overline{S}) = {} & -\sum_{i=1}^{N} S_i W(e_i) + \frac{1}{2} \sum_{i=1}^{N} S_i^2 \log\left(\frac{S_i}{\alpha R_i \Lambda^2}\right) + \sum_{i=1}^{N} \sum_{j \neq i} S_i S_j \log\left(\frac{e_{ij}}{\Lambda}\right) \\
& - \frac{1}{2} \sum_{i=1}^{N} S_i \overline{S_i} \log\left(\frac{e_i}{\Lambda}\right) - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} S_i \overline{S_j} \log\left(\frac{g_{ij}}{\Lambda}\right) + \sum_{i=1}^{N} S_0 S_i \log\left(\frac{e_i}{\Lambda}\right) \\
& + \frac{1}{4} S_0^2 \log\left(\frac{S_0}{\alpha R_0 \Lambda^2}\right) + \sum_{n \geqslant 3} F_{\mathrm{s}}^{(n)}(S, \overline{S}),
\end{aligned} \tag{3.40}$$

where $e_{ij} = e_i - e_j$ and $g_{ij} = e_i + e_j$. The term $F_{\mathrm{s}}^{(n)}(S, \overline{S})$ is an nth order polynomial in S_i and $\overline{S_i}$ arising from planar loop diagrams built from the interaction vertices. We will also need the cubic contribution $F_s^{(3)}(S, \overline{S})$; the explicit expression can be found in Eq. (B.8) in Appendix B.

Next we turn to the sub-leading contributions to the free energy. From Eqs. (3.31), (3.32), (3.39) one finds (up to an e_i-independent part linear in S_i)[14]

$$F_{\mathrm{rp}}(S) = \quad \frac{1}{2}\beta \sum_{i=1}^{N} S_i \log\left(\frac{e_i}{\Lambda}\right) \quad \frac{1}{4}\beta S_0 \log\left(\frac{S_0}{\alpha R_0 \Lambda^2}\right) + \sum_{n\geqslant 2} F_{\mathrm{rp}}^{(n)}(S), \tag{3.41}$$

where $F_{\mathrm{rp}}^{(n)}(S)$ is an nth order polynomial in S_i arising from planar diagrams, built from the interaction vertices, that can be drawn on $\mathbb{R}\;\,^2$. We will need the quadratic contribution $F_{\mathrm{rp}}^{(2)}(S)$; the explicit expression is given in (B.9).

To relate the matrix model to the $\mathcal{N} = 2$ $U(N)$ gauge theory broken to $\prod_i U(N_i)$, one sets $\overline{S} = S$ in the matrix-model free energy [1] and introduces [1,19,25,44–46]

$$W_{\mathrm{eff}}(S) = \quad \sum_i N_i \frac{\partial}{\partial S_i} F_s(S, S) \quad N_0 \frac{\partial}{\partial S_0} F_s(S, S) \quad 4 F_{\mathrm{rp}}(S), \tag{3.42}$$

where we have dropped terms linear in S_i. Since we are examining a generic point on the Coulomb branch of the $\mathcal{N} = 2$ theory, which breaks $U(N)$ to $U(1)^N$, we set $N_i = 1$ and $N_0 = 0$. Next, one extremizes the effective superpotential to obtain $\langle S_i \rangle$ and $\langle S_0 \rangle$:

$$\left.\frac{\partial W_{\mathrm{eff}}(S)}{\partial S_i}\right|_{S_j = \langle S_j \rangle,\, S_0 = \langle S_0 \rangle} = 0, \qquad \left.\frac{\partial W_{\mathrm{eff}}(S)}{\partial S_0}\right|_{S_j = \langle S_j \rangle,\, S_0 = \langle S_0 \rangle} = 0. \tag{3.43}$$

The solution for $\langle S_i \rangle$ and $\langle S_0 \rangle$ can be evaluated in an expansion in Λ. The lowest-order contribution is

$$\langle S_i \rangle = \frac{\alpha G_i}{R_i} \Lambda^{N \quad 2\beta}, \qquad G_i = e_i^{2\beta} \prod_j (e_i + e_j), \qquad R_i = \prod_{j \neq i} (e_i \quad e_j) \tag{3.44}$$

and

$$\langle S_0 \rangle = \quad \alpha(\quad 1)^N \Lambda^{\quad \beta N + 2} W''(0) \left(\frac{W'(0)}{2}\right)^{\beta} \tag{3.45}$$

and constants have been absorbed into a redefinition of the cut-off Λ. The next-to-leading contribution, which will be needed in Section 3.2.2, is given in (B.10).

Now for $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, $\langle S_0 \rangle$ and $\langle S_i \rangle$ are both $\mathcal{O}(\Lambda^{N+2})$ and therefore both need to be included in a perturbative computation to this order. For $\square\square$, however, $\langle S_0 \rangle$ is inversely proportional to Λ, which seems to indicate some inconsistency in perturbing about the vacuum (3.26), (3.27) when $\beta = 1$. Therefore, in the $\square\square$ case, we will simply expand around the vacuum

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[14] We note that the sphere and $\mathbb{R}\;\,^2$ contributions to the free energy obey the relation [44–46]

$$F_{\mathrm{rp}}(S) = \quad \frac{\beta}{2} \frac{\partial}{\partial S_0} F_s(S, S).$$

with $M_0 = 0$ instead; equivalently, we will use

$$\langle S_0 \rangle = 2\alpha\, \delta_{\beta,\ \ 1} \Lambda^{N+2} (\ \ 1)^N \sum_j \frac{1}{e_j} \tag{3.46}$$

in all expressions below.

3.2.1. Relation between a_i and e_i

Before computing τ_{ij} and the $\mathcal{N} = 2$ prepotential, we must determine the relation between a_i and e_i. As in Refs. [5,7], a_i can be determined perturbatively via (setting $N_i = 1$ and $N_0 = 0$)

$$a_i = e_i + \left[\sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \operatorname{tr} \Psi_{ii} \rangle_{S^2} + 4 \langle \operatorname{tr} \Psi_{ii} \rangle_{\mathbb{R}\ \ 2} \right]_{\langle S \rangle}, \tag{3.47}$$

where $\langle \operatorname{tr} \Psi_{ii} \rangle_{S^2}$ is obtained by calculating all connected planar tadpole diagrams with an external Ψ_{ii} leg that can be drawn on a sphere, and $\langle \operatorname{tr} \Psi_{ii} \rangle_{\mathbb{R}\ \ 2}$ is obtained by computing all connected planar tadpole diagrams with an external Ψ_{ii} leg that can be drawn on $\mathbb{R}\ \ ^2$. (The factor of 4 in Eq. (3.47) arises from the corresponding factor in W_{eff} (3.42), using the arguments in Ref. [5].) The total contribution to the tadpole on the sphere quadratic in S_i is

$$\langle \operatorname{tr} \Psi_{ii} \rangle_{S^2} = \frac{1}{\alpha g_s} \left[\quad \sum_{j \neq i} \frac{S_i^2}{R_i e_{ij}} + \sum_{j \neq i} 2 \frac{S_i S_j}{R_i e_{ij}} \quad \sum_j \frac{S_i S_j}{R_i g_{ij}} + \frac{S_0 S_i}{R_i e_i} \right] \tag{3.48}$$

and contribution to the tadpole on $\mathbb{R}\ \ ^2$ linear in S_i is

$$\langle \operatorname{tr} \Psi_{ii} \rangle_{\mathbb{R}\ \ 2} = \quad \frac{\beta}{\alpha} \frac{S_i}{2 R_i e_i}. \tag{3.49}$$

Inserting these results into Eq. (3.47), and evaluating the resulting expression using Eqs. (3.44) and (3.46), one finds

$$\begin{aligned} a_i = e_i + \Lambda^{N\ \ 2\beta} \left(\frac{2}{R_i} \sum_{j \neq i} \frac{G_j}{R_j e_{ij}} \quad \frac{1}{R_i} \sum_j \frac{G_j}{R_j g_{ij}} \quad \frac{G_i}{R_i^2} \sum_j \frac{1}{g_{ij}} \quad \frac{2\beta G_i}{R_i^2 e_i} \right) \\ + \delta_{\beta,\ \ 1} \Lambda^{N+2} \frac{2(\ \ )^N}{R_i e_i} \sum_j \frac{1}{e_j}. \end{aligned} \tag{3.50}$$

To compare these results with those found using SW theory, we must consider the symmetric and antisymmetric cases separately.

(i) $U(N) + \square\square$ $(\beta = 1)$

Consider the function

$$g(z) = z^2 \prod_{j=1}^{N} \left(\frac{z + e_j}{z \quad e_j} \right) \quad z^2 \quad 2\sigma_1 z \quad 2\sigma_1^2, \tag{3.51}$$

where $\sigma_1 = \sum_{i=1}^{N} e_i$, and the last three terms remove the non-negative powers of the Laurent expansion of $g(z)$. The function $g(z)$ has only simple poles at $z = e_i$ and so can be written in terms of its residues as, cf. (3.44),

$$g(z) = \sum_i \frac{G_i}{R_i(z \quad e_i)}. \tag{3.52}$$

Using Eqs. (3.51) and (3.52), one may show that

$$\begin{aligned} \sum_{j \neq i} \frac{G_j}{R_j e_{ij}} &= \frac{G_i}{R_i} \left[\quad \sum_{j \neq i} \frac{1}{e_{ij}} + \sum_j \frac{1}{g_{ij}} + \frac{2}{e_i} \right] \quad e_i^2 \quad 2\sigma_1 e_i \quad 2\sigma_1^2, \\ \sum_j \frac{G_j}{R_j g_{ij}} &= e_i^2 \quad 2\sigma_1 e_i + 2\sigma_1^2, \end{aligned} \tag{3.53}$$

so that using Eq. (3.50) we get

$$\begin{aligned} a_i = e_i &+ \frac{\Lambda^{N \quad 2}}{R_i} \left[\quad 3e_i^2 \quad 2\sigma_1 e_i \quad 6\sigma_1^2 \right] \\ &+ \Lambda^{N \quad 2} \frac{G_i}{R_i^2} \left[\quad \sum_{j \neq i} \frac{2}{e_{ij}} + \sum_j \frac{1}{g_{ij}} + \frac{2}{e_i} \right] + \mathcal{O}\left(\Lambda^{2N \quad 4}\right). \end{aligned} \tag{3.54}$$

After determining, in Section 4.2.1, the relation between the roots e_i of $W'(z)$ and the roots e_i' of $P(z)$ in (2.1b), we will be able to compare this result with that obtained from SW theory.

(ii) $U(N) + \boxed{}$ $(\beta = \quad 1)$

Consider the function

$$h(z) = (\quad 1)^N \frac{W'(\quad z)}{W'(z)} = \prod_{j=1}^{N} \left(\frac{z + e_j}{z \quad e_j} \right). \tag{3.55}$$

Now $h(z)/z^2$ has a double pole at $z = 0$, which we may remove by writing

$$H(z) = \frac{h(z) \quad h(0) \quad zh'(0)}{z^2} = \frac{(\quad 1)^N}{z^2} \left[\frac{W'(\quad z)}{W'(z)} \quad 1 \quad 2z \sum_{i=1}^{N} \frac{1}{e_i} \right]. \tag{3.56}$$

The function $H(z)$ has only simple poles at $z = e_i$ and so may be written

$$H(z) = \sum_i \frac{G_i}{R_i(z \quad e_i)}. \tag{3.57}$$

Using Eqs. (3.56) and (3.57), one may show that

$$\begin{aligned} \sum_{j \neq i} \frac{G_j}{R_j e_{ij}} &= \frac{G_i}{R_i} \left[\quad \sum_{j \neq i} \frac{1}{e_{ij}} + \sum_j \frac{1}{g_{ij}} \quad \frac{2}{e_i} \right] \quad \frac{(\quad)^N}{e_i^2} \left[1 + 2e_i \sum_j \frac{1}{e_j} \right], \\ \sum_j \frac{G_j}{R_j g_{ij}} &= \frac{(\quad)^N}{e_i^2} \left[1 \quad 2e_i \sum_j \frac{1}{e_j} \right], \end{aligned} \tag{3.58}$$

so that using Eq. (3.50) we get

$$a_i = e_i \quad \frac{3(\ )^N \Lambda^{N+2}}{R_i e_i^2} + \Lambda^{N+2} \frac{G_i}{R_i^2} \left[\quad \sum_{j \neq i} \frac{2}{e_{ij}} + \sum_j \frac{1}{g_{ij}} \quad \frac{2}{e_i} \right] \\ + \mathcal{O} \quad \Lambda^{2N+4}). \tag{3.59}$$

This equation, obtained entirely using matrix model methods, precisely agrees (after letting $\Lambda = \quad \Lambda'$, cf. Eq. (4.43)), with Eq. (4.2) in Ref. [31], obtained using the Seiberg–Witten procedure, because, as we will see in Section 4.2.1, the roots e_i of $W'(z)$ and the roots e_i' of $P(z)$ in (2.1c) coincide to this order in Λ.

3.2.2. Perturbative calculation of τ_{ij}

For these models, in contrast to models containing only adjoint and fundamental representations [13,15] or bifundamental representations (Section 3.1), the gauge coupling matrix τ_{ij} is *not* given by the second derivative of the planar free energy F_s of the matrix model:

$$\tau_{ij} \neq \frac{1}{2\pi i} \frac{\partial^2 F_s}{\partial S_i \partial S_j} \bigg|_{S_k = \langle S_k \rangle}. \tag{3.60}$$

Nevertheless, using a diagrammatic argument, a perturbative prescription for τ_{ij} can be given [1]

$$\tau_{ij} = \frac{1}{2\pi i} \left(\frac{\partial}{\partial S_i} \quad \frac{\partial}{\partial \overline{S_i}} \right) \left(\frac{\partial}{\partial S_j} \quad \frac{\partial}{\partial \overline{S_j}} \right) F_s(S, \overline{S}) \bigg|_{S_k = \overline{S}_k = \langle S_k \rangle}. \tag{3.61}$$

(It is unclear what, if any, the physical meaning of derivatives w.r.t. S_0 is.)

In Appendix B we evaluate the expression (3.61) perturbatively up to one-instanton order, expressing the result in terms of a_i rather than e_i. The result can be written as $\tau_{ij} = \partial^2 \quad (a)/\partial a_i \partial a_j$ with (up to a quadratic polynomial)

$$2\pi i \quad (a) = \quad \frac{1}{4} \sum_i \sum_{j \neq i} (a_i \quad a_j)^2 \log \left(\frac{a_i \quad a_j}{\Lambda} \right)^2 \\ + \frac{1}{8} \sum_i \sum_{j \neq i} (a_i + a_j)^2 \log \left(\frac{a_i + a_j}{\Lambda} \right)^2 \\ + \frac{1}{2}(1+\beta) \sum_i a_i^2 \log \left(\frac{a_i}{\Lambda} \right)^2 + \Lambda^{N \quad 2\beta} \left[\sum_i \frac{G_i}{R_i^2} \quad \frac{2\delta_{\beta, \quad 1}}{\prod_i a_i} \right] \\ + \mathcal{O} \quad \Lambda^{2N \quad 4\beta}). \tag{3.62}$$

Later (cf. Eqs. (4.37), (4.43)) we will see that $\Lambda = \quad \Lambda'$, where Λ' is the quantum scale used in the M-theory curves (2.1b), (2.1c). Taking this into account, Eq. (3.62) precisely agrees with the calculations of the prepotential in Refs. [31,32] which utilize the SW curves for these theories derived from M-theory [29].

4. Cubic matrix-model curves

In this section we study the algebraic curves that arise from the planar solution of the matrix models and show how to obtain from these the SW curves of the $\mathcal{N}=2$ gauge theories discussed in Section 2.

4.1. $U(M)\times U(\widetilde{M})$ matrix model with bifundamental matter

The large-M planar solution of the $U(M)\times U(\widetilde{M})$ matrix model described in Section 3.1 was discussed in [1,24] (several of the results can also be found in [2,23,47]). In this approach, one defines the resolvents

$$\omega(z)=g_s\left\langle \mathrm{tr}\left(\frac{1}{z\quad \Phi}\right)\right\rangle = g_s\sum_{n=0}^{\infty} z^{\quad n\quad 1}\left\langle \mathrm{tr}\,\Phi^n\right\rangle,$$

$$\tilde{\omega}(z)=g_s\left\langle \mathrm{tr}\left(\frac{1}{z\quad \tilde{\Phi}}\right)\right\rangle = g_s\sum_{n=0}^{\infty} z^{\quad n\quad 1}\left\langle \mathrm{tr}\,\tilde{\Phi}^n\right\rangle \tag{4.1}$$

where matrix-model expectation values are defined via

$$\langle \mathcal{O}(\Phi,\tilde{\Phi},B,\widetilde{B})\rangle \equiv \frac{1}{Z}\int \mathrm{d}\Phi\,\mathrm{d}\tilde{\Phi}\,\mathrm{d}B\,\mathrm{d}\widetilde{B}\,\mathcal{O}(\Phi,\tilde{\Phi},B,\widetilde{B})\,\mathrm{e}^{\quad \frac{1}{g_s}\mathrm{tr}[W(\Phi)\quad \widetilde{W}(\tilde{\Phi})\quad \widetilde{B}\Phi B+B\tilde{\Phi}\widetilde{B}]}. \tag{4.2}$$

It may be shown that

$$u_1(z)=\quad \omega(z)+W'(z),\qquad u_2(z)=\omega(z)\quad \tilde{\omega}(z),\qquad u_3(z)=\tilde{\omega}(z)+\widetilde{W}'(z) \tag{4.3}$$

(where $\omega(z)$ is the leading (sphere) part of the resolvent) are the values of a variable u on the three sheets of a Riemann surface defined by[15]

$$u\quad W'(z)\big)u\ u\quad \widetilde{W}'(z)\big)=r_1(z)u\quad t_1(z). \tag{4.4}$$

The coefficients of the cubic curve (4.4) are given by [1,24]

$$r_1(z)=\quad g_s\left\langle \mathrm{tr}\left(\frac{W'(z)\quad W'(\Phi)}{z\quad \Phi}\right)\right\rangle + g_s\left\langle \mathrm{tr}\left(\frac{\widetilde{W}'(z)\quad \widetilde{W}'(\tilde{\Phi})}{z\quad \tilde{\Phi}}\right)\right\rangle \tag{4.5}$$

and

$$t_1(z)=\quad g_s\widetilde{W}'(z)\left\langle \mathrm{tr}\left(\frac{W'(z)\quad W'(\Phi)}{z\quad \Phi}\right)\right\rangle + g_s\,W'(z)\left\langle \mathrm{tr}\left(\frac{\widetilde{W}'(z)\quad \widetilde{W}'(\tilde{\Phi})}{z\quad \tilde{\Phi}}\right)\right\rangle$$
$$\quad g_s^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{W'(z)\quad W'(\Phi)}{z\quad \Phi}\right)\right]\right\rangle \quad g_s^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{\Phi}}\left(\frac{\widetilde{W}'(z)\quad \widetilde{W}'(\tilde{\Phi})}{z\quad \tilde{\Phi}}\right)\right]\right\rangle$$

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15 This equation may be obtained, e.g., from Eq. (2.22) in Ref. [1] by redefining $u\to\quad u+\frac{1}{3}(W'(z)+\widetilde{W}'(z))$ and setting $t_1(z)=s_1(z)+\frac{1}{3}(W'(z)+\widetilde{W}'(z))r_1(z)$.

$$+ g_s\left\langle \mathrm{tr}\left(\frac{W'(z) - W'(\Phi)}{z - \Phi} W'(\Phi)\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\frac{\widetilde{W}'(z) - \widetilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \widetilde{W}'(\tilde{\Phi})\right)\right\rangle \tag{4.6}$$

from which one can see, using Eq. (3.2), that $r_1(z)$ and $t_1(z)$ are polynomials of degree at most $N - 1$ and $2N - 1$, respectively, whose coefficients depend on the vevs $\langle \mathrm{tr}(\Phi^k)\rangle$ and $\langle \mathrm{tr}(\tilde{\Phi}^k)\rangle$ with $k \leqslant 2N - 1$. At this point these vevs, and therefore $r_1(z)$ and $t_1(z)$, are undetermined. We would now like to connect the above general curve to the cubic Seiberg–Witten curve (2.7a) for the $\mathcal{N} = 2$ theory. We will discuss two methods.

4.1.1. Method I: perturbative determination of the curve

The planar solution to the matrix model yields a curve (4.4) dependent on arbitrary polynomials $r_1(z)$ and $t_1(z)$. As we saw above, the coefficients of these polynomials depend on $\langle \mathrm{tr}(\Phi^k)\rangle$ and $\langle \mathrm{tr}(\tilde{\Phi}^k)\rangle$, which at this stage are arbitrary. An additional condition is necessary to fix these polynomials, namely, the extremization of W_{eff}. This will determine $\langle \mathrm{tr}(\Phi^k)\rangle$ and $\langle \mathrm{tr}(\tilde{\Phi}^k)\rangle$, and thus lead to a specific form of the cubic curve. Only then can the matrix-model curve be compared with the Seiberg–Witten curve obtained from M-theory (see Section 2).

One method of using the extremization of W_{eff} to determine the curve employs Abel's theorem, and was described in Section 7 of Ref. [7] (see also [4,9]). This approach will be discussed below in Section 4.1.2.

However, the method using Abel's theorem is difficult to apply in some cases, so in this section we will present an alternative approach that is more straightforward to implement. This method is to evaluate $\langle \mathrm{tr}(\Phi^n)\rangle$ perturbatively in powers of Λ, and use the result to determine $r_1(z)$ and $t_1(z)$, and therefore the form of the cubic curve, order-by-order in perturbation theory. Although this method does not yield the exact form of the curve, it is a quick and efficient way of determining the form of the curve to lowest order in Λ.

Expanding $\Phi = \Phi_0 + \Psi$, where Φ_0 is given by (3.4) one easily sees that to lowest order in perturbation theory the matrix model expectation values $g_s\langle \mathrm{tr}(\Phi^n)\rangle$ are given by $\sum_i \langle S_i\rangle e_i^n$. Thus, writing

$$\frac{W'(z) - W'(\Phi)}{z - \Phi} = \sum_{n=0}^{N-1} c_n(z)\Phi^n \tag{4.7}$$

and using $W'(e_i) = 0$, we have

$$\begin{aligned} g_s\left\langle \mathrm{tr}\left(\frac{W'(z) - W'(\Phi)}{z - \Phi}\right)\right\rangle &= \sum_n c_n(z) g_s\langle \mathrm{tr}\ \Phi^n)\rangle \\ &= \sum_i \langle S_i\rangle \sum_n c_n(z) e_i^n = W'(z)\sum_i \frac{\langle S_i\rangle}{z - e_i}, \end{aligned} \tag{4.8}$$

where the second equality only holds to lowest order. If one is only interested in the lowest-order contribution, one can drop the last four terms in $t_1(z)$ (4.6). (The terms on the second line of (4.6) are double-trace terms; since they contain products of at least two S's, they are at least second order in Λ^N. The terms on the last line of (4.6) vanish to lowest order since $W'(e_i) = \widetilde{W}'(\tilde{e}_i) = 0$.) Using (4.8), one finds (to lowest order)

$$r_1(z) = -W'(z)\sum_i \frac{\langle S_i\rangle}{z-e_i} + \widetilde{W}'(z)\sum_i \frac{\langle \widetilde{S}_i\rangle}{z-\tilde{e}_i},$$
$$t_1(z) = -\widetilde{W}'(z)W'(z)\sum_i \frac{\langle S_i\rangle}{z-e_i} + W'(z)\widetilde{W}'(z)\sum_i \frac{\langle \widetilde{S}_i\rangle}{z-\tilde{e}_i}. \tag{4.9}$$

Using Eq. (3.12) we find, again to lowest order,

$$\sum_i \frac{\langle S_i\rangle}{z-e_i} = \alpha\Lambda^N \sum_i \frac{T_i}{R_i(z-e_i)}, \qquad \sum_i \frac{\langle \widetilde{S}_i\rangle}{z-\tilde{e}_i} = -\alpha\Lambda^N \sum_i \frac{\widetilde{T}_i}{\widetilde{R}_i(z-\tilde{e}_i)}. \tag{4.10}$$

Inserting Eq. (4.10) into Eq. (4.9) and using Eq. (3.17), one obtains

$$r_1(z) = 0 + \mathcal{O}\big(\Lambda^{2N}\big), \qquad t_1(z) = -\alpha\Lambda^N\big[W'(z) - \widetilde{W}'(z)\big]^2 + \mathcal{O}\big(\Lambda^{2N}\big) \tag{4.11}$$

hence the matrix model curve is (to first order in Λ^N)

$$\big(u - W'(z)\big)\big(u - \widetilde{W}'(z)\big) = \alpha\Lambda^N\big[W'(z) - \widetilde{W}'(z)\big]^2 + \mathcal{O}\big(\Lambda^{2N}\big). \tag{4.12}$$

This curve is identical to the (transformed) M-theory curve (2.7a) provided that

$$W'(z) = \alpha\big(P(z) - 3\Lambda'^N\big), \quad u = \alpha u',$$
$$\widetilde{W}'(z) = \alpha\big(\widetilde{P}(z) - 3\Lambda'^N\big), \quad \Lambda = \Lambda'. \tag{4.13}$$

In summary, the matrix model curve (4.4), together with the extremization of W_{eff}, which gives (3.12) and therefore (4.11), leads to the $U(N)\times U(N)$ SW curve (2.7a).

The relation (4.13) implies that the roots of the polynomial $P(z) = \prod_{i=1}^N (z - e'_i)$ in the SW curve (2.1a) and the roots e_i of the derivative of the matrix model potential $W'(z) = \alpha\prod_{i=1}^N (z - e_i)$ are equivalent classically, but differ by

$$e'_i = e_i - \frac{3\Lambda^N}{R_i} + \mathcal{O}\big(\Lambda^{2N}\big) \tag{4.14}$$

to first order in Λ^N, and analogously for $\tilde{e}_i$ and $\tilde{e}'_i$. This just amounts to a redefinition of the moduli e_i and $\tilde{e}_i$. In Section 3.1 we determined the relation (3.19) between the SW periods a_i and e_i. Combining (3.19) and (4.14) allows us to write

$$a_i = e'_i + \Lambda'^N \frac{T_i}{R_i^2}\left(-\sum_{j\neq i}\frac{1}{e_{ij}} + \sum_j \frac{1}{h_{ij}}\right) + \mathcal{O}\big(\Lambda'^{2N}\big) \tag{4.15}$$

(and a similar relation between $\tilde{a}_i$ and $\tilde{e}'_i$). This results precisely agrees with Eq. (17) of Ref. [33], obtained using the Seiberg–Witten procedure.

The fact that the first-order curve (4.12) precisely agrees with the M-theory curve (2.7a) (which is believed to be the exact answer) points to the existence of a non-renormalization theorem in the matrix model (which we have not proven). As shown in Appendix A, a similar result holds for $U(N)$ with N_f fundamentals when $N_f < N$ (but not when $N_f \geqslant N$). Thus in some cases (but not always) the perturbative method described above actually gives exact results.

4.1.2. Method II: exact determination of the curve via Abel's theorem

In this section, we will follow the strategy of [7] and discuss, using the saddle-point solution, the condition on the matrix-model curve imposed by extremizing W_{eff}.

The cubic curve (4.4) with the right-hand side set to zero is a singular curve with singularities at $z = e_i$ (the roots of $W'(z)$), at $z = \tilde{e}_i$ (the roots of $\widetilde{W}'(z)$), and at the roots of $W'(z) - \widetilde{W}'(z)$. Turning on the right-hand side *generically* deforms the curve into a three-sheeted Riemann surface with (square-root) branch cuts between sheets one and two located near e_i, branch cuts between sheets two and three located near $\tilde{e}_i$, and branch cuts between sheets one and three located near the roots of $W'(z) - \widetilde{W}'(z)$ (see, e.g., [23] for a picture of the cut structure of this curve). This generic Riemann surface has genus $3N - 2$. If, however, the last described set of cuts does not open up, the curve remains singular, having (geometric) genus $2N - 2$. (This is in fact the case for the SW curve (2.7a), agreeing with the fact that the $\mathcal{N} = 2$ moduli space is $2(N - 1)$-dimensional.)

To impose the extremization of W_{eff} on the matrix model curve (4.4), we begin by expressing the leading (sphere) part of the free-energy of the matrix model in an eigenvalue basis as (cf. [23])

$$F_{\mathrm{s}} = \int \mathrm{d}\lambda\, \mathrm{d}\lambda' \big[\rho(\lambda)\rho(\lambda') \log(\lambda - \lambda') + \tilde{\rho}(\lambda)\tilde{\rho}(\lambda') \log(\lambda - \lambda') - \rho(\lambda)\tilde{\rho}(\lambda') \log(\lambda - \lambda')\big] - \int \mathrm{d}\lambda \big[\rho(\lambda) W(\lambda) - \tilde{\rho}(\lambda) \widetilde{W}(\lambda)\big], \tag{4.16}$$

where $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ are the densities of eigenvalues

$$\rho(\lambda) = g_s \sum_i \delta(\lambda - \lambda_i), \qquad \tilde{\rho}(\lambda) = g_s \sum_i \delta(\lambda - \tilde{\lambda}_i) \tag{4.17}$$

(normalized as $\int \mathrm{d}\lambda\, \rho(\lambda) = g_s M = S$ and $\int \mathrm{d}\lambda\, \tilde{\rho}(\lambda) = g_s \widetilde{M} = \widetilde{S}$), which are related to the resolvents (4.1) via

$$\omega(z) = \int \mathrm{d}\lambda \frac{\rho(\lambda)}{z - \lambda}, \qquad \tilde{\omega}(z) = \int \mathrm{d}\lambda \frac{\tilde{\rho}(\lambda)}{z - \lambda}. \tag{4.18}$$

Next we define[16]

$$S_i = -\frac{1}{2\pi i} \oint_{A_i} u(z)\, \mathrm{d}z, \qquad \widetilde{S}_i = \frac{1}{2\pi i} \oint_{\widetilde{A}_i} u(z)\, \mathrm{d}z \tag{4.19}$$

where A_i and $\widetilde{A}_i$ denote contours around the branch cuts near e_i and $\tilde{e}_i$ on sheets one and three, respectively. Using (4.3) and (4.18) one can show that S_i and $\widetilde{S}_i$ are the integrated densities of eigenvalues along the cuts near e_i and $\tilde{e}_i$. (Thus the definition (4.19) is consistent with the perturbative definition $S_i = g_s M_i$, $\widetilde{S}_i = g_s \widetilde{M}_i$.) Variations in S_i and $\widetilde{S}_i$ can be implemented [7,48] by varying the densities $\delta\rho(\lambda) = \delta S_i\, \delta(\lambda - e_i)$ and $\delta\tilde{\rho}(\lambda) = \delta\widetilde{S}_i\, \delta(\lambda - \tilde{e}_i)$, with e_i and $\tilde{e}_i$ denoting any point along the branch cuts (cf. [40] for an alternative approach). Specifically, (up to terms which will not affect our discussion;

[16] In general there are also corresponding S's for the cuts connecting sheets one and three (see [23] for a discussion). However, these will not affect our discussion so we will suppress them.

see [40] for a discussion of such terms)

$$\begin{aligned}\frac{\partial F_s}{\partial S_i} &= -W(e_i) + \int \mathrm{d}\lambda \big[2\rho(\lambda)\log(\lambda - e_i) - \tilde{\rho}(\lambda)\log(\lambda - e_i)\big] \\ &= \int_{e_i}^{I} \mathrm{d}z\, W'(z) + \int \mathrm{d}\lambda \left[-2\rho(\lambda) \int_{e_i}^{I} \mathrm{d}z \frac{1}{z-\lambda} + \tilde{\rho}(\lambda) \int_{e_i}^{I} \mathrm{d}z \frac{1}{z-\lambda} \right] \\ &= \int_{e_i}^{I} \mathrm{d}z \big[W'(z) - 2\omega(z) + \tilde{\omega}(z)\big] = \int_{e_i}^{I} \mathrm{d}z \big[u_1(z) - u_2(z)\big] = \int_{I_2}^{I_1} u, \end{aligned} \tag{4.20}$$

where we have used (4.3), and the last expression is interpreted as the integral of u from I_2, infinity on the second sheet, to I_1, infinity on the first sheet, along a contour that passes through the cut near e_i. Similarly,

$$\frac{\partial F_s}{\partial \widetilde{S}_i} = \int_{I_3}^{I_2} u, \tag{4.21}$$

where the integral is taken along a contour that passes through the cut near $\tilde{e}_i$. The results (4.20), (4.21) were also obtained in [23].

Next, we wish to extremize the effective superpotential (3.10)

$$W_{\mathrm{eff}} = -\sum_{i=1}^{N} \frac{\partial F_s}{\partial S_i} - \sum_{i=1}^{N} \frac{\partial F_s}{\partial \widetilde{S}_i} \tag{4.22}$$

(setting $N_i = \widetilde{N}_i = 1$) for the $U(N) \times U(N)$ theory broken down to $U(1)^{2N}$. This is accomplished by taking derivatives of W_{eff} w.r.t. the S_i's and $\widetilde{S}_i$'s. However, in analogy with [4,7], one may change variables and instead vary w.r.t. the coefficients of the arbitrary polynomials $r_1(z)$ and $t_1(z)$ in the matrix model curve (4.4). From (4.4), one may check that the derivatives of u w.r.t. the coefficients of $r_1(z)$ and $t_1(z)$ are holomorphic differentials on the Riemann surface.[17] We can change basis to the canonical basis of holomorphic differentials, ζ_k, dual to the homology basis, so that the conditions arising from extremizing W_{eff} may be written as (see Ref. [7] for further details)

$$N \int_{I_3}^{I_1} \zeta_k = 0 \quad \text{(modulo period lattice)}. \tag{4.23}$$

This condition implies, by Abel's theorem, the existence of a function with an Nth order pole at I_1, an Nth order zero at I_3, and regular everywhere else. For a generic choice of $r_1(z)$ and $t_1(z)$ such a function will not exist (by the Weierstrass gap theorem). Thus only in very special circumstances can such a function exist.

[17] Actually not all these differentials are holomorphic; we only consider the holomorphic ones.

Let us first show that the problem has a solution. Consider Eq. (4.4), with $r_1(z) = 0$ and[18] $t_1(z) \propto (W'(z) - \widetilde{W}'(z))^2$, i.e., a curve of the form:

$$\big(u - W'(z)\big)u\big(u - \widetilde{W}'(z)\big) = \text{const} \times \big(W'(z) - \widetilde{W}'(z)\big)^2. \tag{4.24}$$

One can infer the asymptotic behavior of u at I_i, infinity on each of the three sheets

$$I_1\colon\quad u = W'(z) + \mathcal{O}\big(z^{-1}\big), \qquad I_2\colon\quad u = \mathcal{O}\big(z^{-1}\big), \qquad I_3\colon\quad u = \widetilde{W}'(z) + \mathcal{O}\big(z^{-1}\big). \tag{4.25}$$

(Alternatively, this can be deduced from (4.3) together with the asymptotic behavior of the resolvents.)

Now consider the function, defined on the curve (4.24),

$$f(z) = \frac{u - \widetilde{W}'(z)}{u - W'(z)}. \tag{4.26}$$

This function has the right asymptotic properties to satisfy Abel's theorem, but potentially has poles and zeros at finite values z_0. The denominator vanishes at any point z_0 at which $u(z_0) = W'(z_0)$ on one of the sheets. However, using (4.24), one sees that the function is actually regular at these (singular) points. Similarly the potential zeros at the points where $u(z_0) = \widetilde{W}'(z_0)$ are absent. Thus the function (4.26) defined on the curve (4.24) has the divisor implied by (4.23) via Abel's theorem.

Assuming that the solution is unique, we find that the matrix model implies a Seiberg–Witten curve of the form (4.24) (with no terms higher order in Λ). This precisely agrees with the M-theory result (2.7a) after the redefinitions (4.13). Note that after these redefinitions, the function (4.26) is proportional to the variable y (2.6a) appearing in the M-theory curve (2.1a). We have not been able to show uniqueness.

4.2. $U(M)$ matrix model with symmetric or antisymmetric matter

The large-M planar solution of the $U(M)$ matrix models described in Section 3.2, was discussed in Refs. [1,25] (these models are a slight modification of the $O(1)$ model described in Ref. [49]). In this approach, one defines the resolvent

$$\omega(z) = g_s\left\langle \mathrm{tr}\left(\frac{1}{z - \Phi}\right)\right\rangle = g_s \sum_{n=0}^{\infty} z^{-n-1}\big\langle \mathrm{tr}\, \Phi^n\big\rangle, \tag{4.27}$$

where matrix-model expectation values are defined via

$$\big\langle \mathcal{O}(\Phi, X, \widetilde{X})\big\rangle = \frac{1}{Z}\int \mathrm{d}\Phi\, \mathrm{d}X\, \mathrm{d}\widetilde{X}\, \mathcal{O}(\Phi, X, \widetilde{X})\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi) - \widetilde{X}\Phi X]}. \tag{4.28}$$

Then it may be shown that

$$\begin{aligned} u_1(z) &= -\omega_0(z) + W'(z), \qquad u_2(z) = \omega_0(z) + \omega_0(-z), \\ u_3(z) &= -\omega_0(-z) + W'(-z), \end{aligned} \tag{4.29}$$

[18] If one starts with an arbitrary $t_1(z)$ instead, the requirement that the function in Eq. (4.26) below should have the desired properties implies that $t_1(z)$ has to be of this form.

where $\omega_0(z)$ is the leading (sphere) part of the resolvent, are the values of a variable u on the three sheets of a Riemann surface defined by[19]

$$u \quad W'(z)\big)u \;\; u \quad W'(\;\; z)\big) = r_1(z)u \quad t_1(z). \tag{4.30}$$

The coefficients of the cubic curve (4.30) are given by [1,25]

$$r_1(z) = \quad g_s\left\langle \mathrm{tr}\left(\frac{W'(z) \quad W'(\Phi)}{z \quad \Phi}\right)\right\rangle \quad g_s\left\langle \mathrm{tr}\left(\frac{W'(\;\; z) \quad W'(\Phi)}{z \quad \Phi}\right)\right\rangle \tag{4.31}$$

and

$$t_1(z) = \left[\quad g_s W'(\;\; z)\left\langle \mathrm{tr}\left(\frac{W'(z) \quad W'(\Phi)}{z \quad \Phi}\right)\right\rangle \quad g_s^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{W'(z) \quad W'(\Phi)}{z \quad \Phi}\right)\right]\right\rangle \right.$$
$$\left. + g_s\left\langle \mathrm{tr}\left(\frac{W'(z) \quad W'(\Phi)}{z \quad \Phi} W'(\Phi)\right)\right\rangle\right] + (z \leftrightarrow \quad z) \tag{4.32}$$

from which one sees that $r_1(z)$ and $t_1(z)$ are even polynomials of degree at most $N \quad 1$ and $2N \quad 2$, respectively, whose coefficients depend on the vevs $\langle \mathrm{tr}(\Phi^k)\rangle$ with $k \leqslant 2N \quad 1$.

4.2.1. Method I: perturbative determination of the curve

As in Section 4.1.1, we may evaluate the polynomials r_1 and t_1 perturbatively in Λ, and use the result to determine the curve order-by-order in perturbation theory. Expanding $\Phi = \Phi_0 + \Psi$, where Φ_0 is given by (3.26), one sees that, to lowest order in perturbation theory, the matrix model expectation value $g_s\langle \mathrm{tr}(\Phi^n)\rangle$ is given by $\sum_i \langle S_i\rangle e_i^n + \langle S_0\rangle \delta_{n,0}$. Hence, similar to Eq. (4.8), we find

$$\begin{aligned}
r_1(z) &= \quad W'(z)\sum_i \frac{\langle S_i\rangle}{z \quad e_i} \quad W'(z)\frac{\langle S_0\rangle}{z} + (z \to \quad z),\\
t_1(z) &= \quad W'(\;\; z)W'(z)\sum_i \frac{\langle S_i\rangle}{z \quad e_i} \quad W'(\;\; z)W'(z)\frac{\langle S_0\rangle}{z} + (z \to \quad z).
\end{aligned} \tag{4.33}$$

Using the lowest-order perturbative results (3.44), (3.46) we have

$$\begin{aligned}
&\sum_i \frac{\langle S_i\rangle}{z \quad e_i} = \alpha \Lambda^{N \quad 2\beta} \sum_i \frac{G_i}{R_i(z \quad e_i)},\\
&\langle S_0\rangle = 2\alpha\, \delta_{\beta, \quad 1}\Lambda^{N+2}(\quad 1)^N \sum_j \frac{1}{e_j}.
\end{aligned} \tag{4.34}$$

To evaluate $\sum_i [G_i/R_i(z \quad e_i)]$, we must consider the $U(N) + \Box\!\Box$ and $U(N) + \begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$ cases separately.

(i) $U(N) + \Box\!\Box$ $(\beta = 1)$

[19] This may be obtained, e.g., from Eq. (3.20) in Ref. [1], by redefining $u \to \quad u + \frac{1}{3}(W'(z) + W'(\;\; z))$ and setting $t_1(z) = s_1(z) + \frac{1}{3}(W'(z) + W'(\;\; z))r_1(z)$.

Using Eqs. (3.51) and (3.52), we find (to lowest order)

$$\begin{aligned} r_1(z) = {} & \quad \alpha \Lambda^{N \quad 2} W'(z) g(z) + (z \to \quad z) \\ = {} & \alpha \Lambda^{N \quad 2} \; z^2 \quad (\quad 1)^N z^2 + 2\sigma_1^2\big)\big[W'(z) + W'(\quad z)\big] \\ & + 2\alpha \Lambda^{N \quad 2} \sigma_1 z \big[W'(z) \quad W'(\quad z)\big] \end{aligned} \tag{4.35}$$

and (also to lowest order)

$$\begin{aligned} t_1(z) = {} & \quad \alpha \Lambda^{N \quad 2} W'(\quad z) W'(z) g(z) + (z \to \quad z) \\ = {} & \quad \alpha \Lambda^{N \quad 2} (\quad 1)^N z^2 \big[W'(z) \quad W'(\quad z)\big]^2 \\ & + 2\alpha \Lambda^{N \quad 2} \; z^2 \quad (\quad 1)^N z^2 + 2\sigma_1^2\big) W'(z) W'(\quad z). \end{aligned} \tag{4.36}$$

The bottom lines of these equations make manifest that $r_1(z)$ and $t_1(z)$ are (even) polynomials, and the top lines show that they are of degree $N \quad 1$ and $2N \quad 2$, respectively.

The cubic equation (4.30), with $r_1(z)$ and $t_1(z)$ as above, may be considerably simplified by defining

$$\begin{aligned} & W'(z) = \alpha\big[P(z) + \Lambda^{N \quad 2} \quad 3z^2 \quad 2\sigma_1 z \quad 6\sigma_1^2\big)\big] + \mathcal{O} \;\; \Lambda^{2N+4}\big), \\ & u = \alpha\big[u' \quad 2\Lambda^{N \quad 2} \; z^2 \quad (\quad 1)^N z^2 + 2\sigma_1^2\big)\big] + \mathcal{O} \;\; \Lambda^{2N+4}\big), \\ & \Lambda = \quad \Lambda'. \end{aligned} \tag{4.37}$$

Then

$$\begin{aligned} & u' \quad P(z) + 3\Lambda'^{N \quad 2} z^2\big) u' \; u' \quad P(\quad z) + 3\Lambda'^{N \quad 2} z^2\big) \\ & = \Lambda'^{N \quad 2} z^2 \; P(z) \quad P(\quad z)\big)^2 + \mathcal{O} \;\; \Lambda'^{2N \quad 4}\big) \end{aligned} \tag{4.38}$$

in agreement with the transformed M-theory curve (2.7b).

The relation (4.37) implies that the roots of the polynomial $P(z) = \prod_{i=1}^{N}(x \quad e_i')$ in the SW curve (2.1b) and those of the derivative of the matrix-model potential $W'(z) = \prod_{i=1}^{N}(x \quad e_i)$ are equivalent classically, and are related by

$$e_i' = e_i + \frac{\Lambda^{N \quad 2}}{R_i} \quad 3e_i^2 \quad 2\sigma_1 e_i \quad 6\sigma_1^2\big) + \mathcal{O} \;\; \Lambda^{2N \quad 4}\big) \tag{4.39}$$

at the one-instanton level. Combining this result with the relation (3.54) between a_i and e_i, we find

$$a_i = e_i' + (\quad)^N \Lambda'^{N \quad 2} \frac{G_i}{R_i^2}\left[\quad \sum_{j \neq i} \frac{2}{e_{ij}} + \sum_j \frac{1}{g_{ij}} + \frac{2}{e_i}\right] + \mathcal{O} \;\; \Lambda'^{2N \quad 4}\big). \tag{4.40}$$

This equation precisely agrees with Eq. (26) of Ref. [32], obtained using the Seiberg–Witten procedure.

(ii) $U(N) + \boxminus \; (\beta = \quad 1)$

Using Eqs. (3.56) and (3.57), one finds (to lowest order)

$$r_1(z) = \quad \alpha \Lambda^{N+2} W'(z)\left[H(z) + 2(\quad 1)^N \frac{1}{z} \sum_i \frac{1}{e_i}\right] + (z \to \quad z) = 0 \tag{4.41}$$

and

$$
\begin{aligned}
t_1(z) &= \quad \alpha \Lambda^{N+2} W'(\;\;z) W'(z) \left[H(z) + 2(\;\;1)^N \frac{1}{z} \sum_i \frac{1}{e_i} \right] + (z \to \quad z) \\
&= \quad \alpha(\;\;1)^N \Lambda^{N+2} \left[\frac{W'(z) \quad W'(\;\;z)}{z} \right]^2 .
\end{aligned}
\tag{4.42}
$$

Defining

$$
W'(z) = \alpha P(z), \qquad u = \alpha u', \qquad \Lambda = \quad \Lambda', \tag{4.43}
$$

Eq. (4.30) may be rewritten as

$$
u' \quad P(z)\big) u' \; u' \quad P(\;\;z)\big) = \frac{\Lambda'^{N+2}}{z^2} \; P(z) \quad P(\;\;z)\big)^2 + \mathcal{O} \;\; \Lambda'^{2N+4}\big) \tag{4.44}
$$

which is in agreement with the (transformed) M-theory curve (2.7c).

Note that (4.43) implies that the roots of the polynomial $P(z) = \prod_{i=1}^{N} (x \quad e_i')$ in the SW curve and those of the derivative of the matrix model potential $W'(z)$ coincide to the order we have calculated, i.e., $e_i' = e_i + \mathcal{O}(\Lambda^{2N+4})$. This result was used in Section 3.2 to compare the relation between a_i and e_i obtained from the matrix model with the one derived in Ref. [31] using the M-theory curve (2.1c). The two results agree (see Section 3.2).

4.2.2. Method II: exact determination of the curve via Abel's theorem

The condition imposed on the matrix-model curve by extremizing W_{eff} can also be discussed using the saddle-point solution, as in [7] and in Section 4.1.2.

The Riemann sheet structure of the generic cubic curve (4.30) is similar to that for the $U(N) \times U(N)$ model discussed in Section 4.1.2, except that we are interested in the quotient of this curve by the involution $z \to \quad z$.

In an eigenvalue basis the leading (sphere) part of the free-energy can be written

$$
\begin{aligned}
F_{\text{s}} = \quad & \int \mathrm{d}\lambda \, \rho(\lambda) W(\lambda) \\
& + \int \mathrm{d}\lambda \, \mathrm{d}\lambda' \left[\rho(\lambda)\rho(\lambda') \log(\lambda \quad \lambda') \quad \frac{1}{2} \rho(\lambda)\rho(\lambda') \log(\lambda + \lambda') \right]
\end{aligned}
\tag{4.45}
$$

and the subleading ($\mathbb{R} \;\; ^2$) part is

$$
F_{\text{rp}} = \quad \frac{1}{2} \beta \int \mathrm{d}\lambda \, \rho(\lambda) \log(\lambda), \tag{4.46}
$$

where

$$
\rho(\lambda) = g_s \sum_i \delta(\lambda \quad \lambda_i), \qquad \omega_0(z) = \int \mathrm{d}\lambda \frac{\rho(\lambda)}{z \quad \lambda}. \tag{4.47}
$$

We define[20]

$$S_i = \frac{1}{2\pi i}\oint_{A_i} u(z)\,\mathrm{d}z, \tag{4.48}$$

where A_i denote contours around the branch cuts near e_i on sheet one. As before one can show that S_i is the integrated density of eigenvalues along the cut near e_i (so (4.48) is consistent with the perturbative definition $S_i = g_s M_i$) and $\delta\rho(\lambda) = \delta S_i\,\delta(\lambda \quad e_i)$. As in Section 4.1.2 (up to terms which will not affect our discussion)

$$\begin{aligned}
\frac{\partial F_{\mathrm{s}}}{\partial S_i} &= \int_{I_2}^{I_1} u = \int_{I_3}^{I_2} u = \frac{1}{2}\int_{I_3}^{I_1} u,\\
F_{\mathrm{rp}} &= \quad \frac{\beta}{2}\int_0^I u_1 = +\frac{\beta}{2}\int_0^I u_3 = \quad \frac{\beta}{4}\left[\int_{0_1}^{I_1} u \quad \int_{0_3}^{I_3} u\right],
\end{aligned} \tag{4.49}$$

where I_i denotes infinity on the ith sheet, and 0_i is the point $z = 0$ on sheet i.

Next, we wish to extremize the effective superpotential (3.42)

$$W_{\mathrm{eff}} = \quad \sum_{i=1}^{N}\frac{\partial F_{\mathrm{s}}}{\partial S_i} \quad 4F_{\mathrm{rp}} \tag{4.50}$$

(setting $N_i = 1$ and $N_0 = 0$). By changing basis as in Section 4.1.2 and varying w.r.t. the coefficients of the arbitrary polynomials $r_1(z)$ and $t_1(z)$ in the matrix model curve (4.30), one obtains

$$0 = (N \quad 2\beta)\int_{p_0}^{I_1}\zeta_k \quad (N \quad 2\beta)\int_{p_0}^{I_3}\zeta_k + 2\beta\int_{p_0}^{0_1}\zeta_k \quad 2\beta\int_{p_0}^{0_3}\zeta_k = 0$$
$$\text{(modulo period lattice).} \tag{4.51}$$

This condition seemingly implies, by Abel's theorem, the existence of a function with a pole of order $N \quad 2\beta$ at I_1, a zero of order $N \quad 2\beta$ at I_3, a pole of order 2β at 0_1, and a zero of order 2β at 0_3, and regular everywhere else. However, there is one important caveat. In the undeformed ($r_1(z) = t_1(z) = 0$) curve (4.30), $z = 0$ is a singular double-point. If a cut opens up between sheets one and three when $r_1(z)$ and $t_1(z)$ are turned on, then the points 0_1 and 0_3 will be identical and the last two terms in (4.51) will not contribute, and the function will be regular at $z = 0$ on all the sheets.

[20] As in the $U(N)\times U(N)$ theory, there are in general additional S's corresponding to the other (possible) cuts of the curve. In particular, there is a variable S_0 corresponding to the (possible) cut between sheets one and three around the point $z = 0$

$$S_0 = \quad \frac{1}{2\pi i}\oint_{0_1} u(z)\,\mathrm{d}z.$$

(i) $U(N) + \square\!\square$ $(\beta = 1)$

We will now show that this problem has a solution: that for some choice of $r_1(z)$ and $t_1(z)$ in the matrix-model curve (4.30), there exists a function $f(z)$ on this curve with divisor implied by (4.51) via Abel's theorem. For simplicity, we restrict ourselves to the special case in which N is even and $\sigma_1 = 0$; these conditions are such as to make $W'(z) \quad W'(\ z)$ a polynomial of order $N \quad 3$. Consider $r_1(z) = 0$ and $t_1(z) \propto z^2(W'(z) \quad W'(\ z))^2$, i.e., the matrix-model curve[21]

$$u \quad W'(z)\big)u \ u \quad W'(\ z)\big) = \mathrm{const} \times z^2 \ W'(z) \quad W'(\ z)\big)^2. \tag{4.52}$$

The function

$$f(z) = \frac{u \quad W'(\ z)}{u \quad W'(z)} \tag{4.53}$$

defined on this curve, has the following asymptotic behavior near I_i, infinity on the three sheets:

$$I_1\colon \quad f(z) \sim z^{N \ \ 2}, \qquad I_2\colon \quad f(z) \sim \mathrm{const}, \qquad I_3\colon \quad f(z) \sim z^{\ \ N+2} \tag{4.54}$$

and the following behavior near 0_i, $z = 0$ on the three sheets:

$$0_1\colon \quad f(z) \sim z^{\ \ 2}, \qquad 0_2\colon \quad f(z) \sim \mathrm{const}, \qquad 0_3\colon \quad f(z) \sim z^2. \tag{4.55}$$

As in Section 4.1.2 one can show that $f(z)$ is regular everywhere else, and so satisfies the conditions implied by (4.51) and Abel's theorem. Assuming that the solution is unique (which we have not been able to prove), we find that extremization of W_{eff} (via Abel's theorem) implies a matrix-model curve of the form (4.52). Upon redefining $W'(z) \propto P(z) \quad 3\Lambda'^{N \ \ 2} z^2$, this curve precisely agrees with the M-theory curve (2.7b), and the function (4.53) is proportional, up to a factor of z^2, to y (2.6b).

(ii) $U(N) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(\beta = \quad 1)$

To show that the problem has a solution, consider $r_1(z) = 0$ and $t_1(z) \propto z^{\ \ 2}(W'(z) \quad W'(\ z))^2$, i.e., the matrix-model curve

$$u \quad W'(z)\big)u \ u \quad W'(\ z)\big) = \mathrm{const} \times \left(\frac{W'(z) \quad W'(\ z)}{z}\right)^2. \tag{4.56}$$

For this curve, there is a cut opening up at $z = 0$ between sheets 1 and 3, so that the last two terms in (4.51) do not contribute. The function

$$f(z) = \frac{u \quad W'(\ z)}{u \quad W'(z)} \tag{4.57}$$

on the curve (4.56) has the following asymptotic behavior near I_i, infinity on the three sheets:

$$I_1\colon \quad f(z) \sim z^{N+2}, \qquad I_2\colon \quad f(z) \sim \mathrm{const}, \qquad I_3\colon \quad f(z) \sim z^{\ \ N \ \ 2} \tag{4.58}$$

[21] The polynomial $t_1(z)$ is of degree $2N \quad 2$ or less only when N is even and $\sigma_1 = 0$. In the more general case, a more complicated choice of $r_1(z)$ and $t_1(z)$ would be necessary.

and is regular near $z = 0$ on all three sheets. As in Section 4.1.2 one can show that $f(z)$ is also regular everywhere else. Thus, the function (4.57) on the Riemann surface (4.56) has precisely the divisor specified by (4.51). Assuming that the solution is unique (which we have not shown), we find that Abel's theorem implies a Seiberg–Witten curve of the form (4.56). Setting $W'(z) \propto P(z)$, Eq. (4.56) precisely agrees with the M-theory curve (2.7c), and the function (4.57) is proportional, up to a factor of z^2, to y (2.6c).

5. Seiberg–Witten differential from the matrix model

In this section we will discuss the derivation of the Seiberg–Witten differentials for the $\mathcal{N} = 2$ gauge theories described in Section 2 from the matrix model point of view.

As is the case for the SW curve, the SW differential can be determined order-by-order in Λ by using perturbative matrix model calculations. We will illustrate this in the first part of this section, after which we will very briefly discuss some other approaches, such as the one pursued in (version 3 of) Ref. [7].

5.1. *Method I: perturbative determination of* λ_{SW}

On the first sheet of the Riemann surface, we have (for all the models) [3,4,13,21]

$$\lambda_{\mathrm{SW}} = z\,T(z)\,\mathrm{d}z, \quad T(z) = \left\langle \mathrm{tr}\left(\frac{1}{z-\phi}\right)\right\rangle = \sum_{n=0}^{\infty} z^{-n-1}\langle \mathrm{tr}\,\phi^n\rangle, \tag{5.1}$$

where $\langle \mathrm{tr}\,\phi^n\rangle$ is the gauge-theory vev of the adjoint field and $T(z)$ is sometimes called $h(z)$. The relation between gauge-theory and matrix-model vevs can be obtained using the methods in Ref. [5]

$$\langle \mathrm{tr}\,\phi^n\rangle = \sum_{i=1}^{N}\left[\frac{\partial}{\partial S_i} + \frac{\partial}{\partial \widetilde{S}_i}\right] g_s \langle \mathrm{tr}\,\Phi^n\rangle_{S^2}, \tag{5.2a}$$

$$\langle \mathrm{tr}\,\phi^n\rangle = \sum_{i=1}^{N}\frac{\partial}{\partial S_i} g_s \langle \mathrm{tr}\,\Phi^n\rangle_{S^2} + 4\langle \mathrm{tr}\,\Phi^n\rangle_{\mathbb{RP}^2}, \tag{5.2b,c}$$

where the first equation is for model (a), $U(N)\times U(N)$ with a bifundamental hypermultiplet, and the second equation is for models (b) or (c), $U(N)$ with a symmetric or antisymmetric hypermultiplet, respectively. It is understood that the rhs is evaluated at $S_i = \langle S_i\rangle$ (as well as $\widetilde{S}_i = \langle \widetilde{S}_i\rangle$ for model (a) and $S_0 = \langle S_0\rangle$ for model (c)). Since we are only interested in the first two orders in perturbation theory, we may write [5]

$$\langle \mathrm{tr}\,\Phi^n\rangle = \sum_{i=1}^{N}\left[M_i e_i^n + n\,e_i^{n-1}\langle \mathrm{tr}\,\Psi_{ii}\rangle + \frac{1}{2}n(n-1)e_i^{n-2}\langle \mathrm{tr}\,\Psi_{ii}^2\rangle + \cdots\right]. \tag{5.3}$$

For model (c) we also have the extra terms[22]

$$M_0\delta_{n,0} + \langle \operatorname{tr}\Psi_{00}^2\rangle\delta_{n,2} + \cdots. \tag{5.4}$$

It follows from these expressions that for all three models, the leading term in the perturbative expansion for $T(z)$ is given by

$$T(z)_{\text{pert}} = \sum_{i=1}^{N}\frac{1}{z \quad e_i}. \tag{5.5}$$

The matrix model expectation values $\langle \operatorname{tr}\Psi_{ii}\rangle_{S^2}$ and $\langle \operatorname{tr}\Psi_{ii}\rangle_{\mathbb{R}\;\,^2}$ are given in Eqs. (3.15), (3.48), (3.49) of Section 3, and for all three models, we have

$$\langle \operatorname{tr}\Psi_{ii}^2\rangle_{S^2} = \frac{g_s}{\alpha}\sum_{i=1}^{N}\frac{M_i^2}{R_i}, \qquad \langle \operatorname{tr}\Psi_{ii}^2\rangle_{\mathbb{R}\;\,^2} = 0. \tag{5.6}$$

In addition, for model (c), we will need[23]

$$\langle \operatorname{tr}\Psi_{00}^2\rangle_{\mathbb{R}\;\,^2} = \frac{1}{2\alpha}\frac{S_0}{R_0}. \tag{5.7}$$

Using these results together with the above formulæ leads to the following expressions for the one-instanton corrections (see Section 3 for details about the notation)

$$\begin{aligned} T(z)_{1-\text{inst}} = \Lambda^N \sum_{i=1}^{N}\Bigg\{ &\frac{1}{(z \quad e_i)^2}\left[\frac{2}{R_i}\sum_{j\neq i}\frac{T_j}{R_j e_{ij}} + \frac{1}{R_i}\sum_j \frac{\widetilde{T}_j}{\widetilde{R}_j h_{ij}} \quad \frac{T_i}{R_i^2}\sum_j \frac{1}{h_{ij}}\right] \\ &+ \frac{2}{(z \quad e_i)^3}\frac{T_i}{R_i^2}\Bigg\}, \end{aligned} \tag{5.8a}$$

$$\begin{aligned} T(z)_{1-\text{inst}} = \Lambda^{N \quad 2}\sum_{i=1}^{N}\Bigg\{ &\frac{1}{(z \quad e_i)^2}\left[\frac{2}{R_i}\sum_{j\neq i}\frac{G_j}{R_j e_{ij}} \quad \frac{1}{R_i}\sum_j \frac{G_j}{R_j g_{ij}} \right. \\ &\left. \frac{G_i}{R_i^2}\sum_j \frac{1}{g_{ij}} \quad \frac{2G_i}{R_i^2 e_i}\right] + \frac{2}{(z \quad e_i)^3}\frac{G_i}{R_i^2}\Bigg\}, \end{aligned} \tag{5.8b}$$

$$\begin{aligned} T(z)_{1-\text{inst}} = \Lambda^{N+2}\sum_{i=1}^{N}\Bigg\{ &\frac{1}{(z \quad e_i)^2}\left[\frac{2}{R_i}\sum_{j\neq i}\frac{G_j}{R_j e_{ij}} \quad \frac{1}{R_i}\sum_j \frac{G_j}{R_j g_{ij}} \quad \frac{G_i}{R_i^2}\sum_j \frac{1}{g_{ij}} \right. \\ &\left. + \frac{2G_i}{R_i^2 e_i} + \frac{2(\quad 1)^N}{R_i e_i}\sum_k \frac{1}{e_k}\right] \\ &+ \frac{2}{(z \quad e_i)^3}\frac{G_i}{R_i^2} \quad \frac{4}{z^3}\frac{1}{\prod_j e_j}\Bigg\}, \end{aligned} \tag{5.8c}$$

---

[22] Note that, for this model, $\langle \operatorname{tr}\Psi_{00}\rangle \equiv 0$ because of the Sp-condition on Φ_{00}.

[23] Note that $\langle \operatorname{tr}\Psi_{00}^2\rangle_{S^2} \propto S_0^2$ does not contribute to (5.2b,c) because of the absence of a derivative w.r.t. S_0.

where Eqs. (5.8a), (5.8b), and (5.8c) correspond to models (a), (b), and (c), respectively. Next we use the identities (3.18), (3.53), and (3.58), together with the definitions (4.14), (4.39) to obtain

$$T(z)\,\mathrm{d}z = \mathrm{d}\log P(z) \quad \Lambda^{N\;\;2\beta}\,\mathrm{d}K(z) + \mathcal{O}\;\;\Lambda^{2N\;\;4\beta}\big) \tag{5.9}$$

with

$$\beta = 0, \qquad K(z) = \frac{\widetilde{P}(z)}{P(z)^2}, \tag{5.10a}$$

$$\beta = 1, \qquad K(z) = (\;\;1)^N \frac{z^2 P(\;\;z)}{P(z)^2}, \tag{5.10b}$$

$$\beta = \;\;1, \qquad K(z) = (\;\;1)^N \left[\frac{1}{z^2}\frac{P(\;\;z)}{P(z)^2} \quad \frac{3}{z^2}\frac{1}{P(z)}\right] \tag{5.10c}$$

recalling that $P(z) = \prod_{i=1}^{N}(z \quad e_i')$ and $\widetilde{P}(z) = \prod_{i=1}^{N}(z \quad \tilde{e}_i')$.

These results are consistent with, using (5.1), what one obtains by expanding the M-theory result (2.2) on the first sheet of the Riemann surface given by (2.6a)–(2.6c).

5.2. Other methods

For the $U(N) \times U(N)$ gauge theory with a bifundamental hypermultiplet, we may combine Eqs. (5.1), (5.2a), and (4.1) to derive the relation (on sheet one of the Riemann surface)[24]

$$\lambda_{\mathrm{SW}} = z\,T(z)\,\mathrm{d}z, \quad T(z) = \sum_{i=1}^{N}\left[\frac{\partial}{\partial S_i} + \frac{\partial}{\partial \widetilde{S}_i}\right]\omega(z), \tag{5.11}$$

where $\omega(z)$ is the leading (sphere) part of the resolvent. Similarly, for the $U(N)$ gauge theory with one symmetric or antisymmetric hypermultiplet, we may combine Eqs. (5.1), (5.2b,c), and (4.27) to derive (on sheet one)

$$\lambda_{\mathrm{SW}} = zT(z)\,\mathrm{d}z, \quad T(z) = \sum_{i=1}^{N}\frac{\partial}{\partial S_i}\,\omega_0(z) + \omega_{1/2}(z), \tag{5.12}$$

where $\omega_0(z)$ is the leading (sphere) part of the resolvent and $\omega_{1/2}(z)$ is the subleading ($\mathbb{R}\;\;^2$) part. These results appeared (using a different approach) in [1,19]), and earlier for the case of $U(N)$ without [50] and with [7] fundamental hypermultiplets. Now we may use Eq. (4.3) to show, for $U(N) \times U(N)$,

$$\lambda_{\mathrm{SW}} = \quad z\sum_{i=1}^{N}\left[\frac{\partial}{\partial S_i} + \frac{\partial}{\partial \widetilde{S}_i}\right]u\,\mathrm{d}z. \tag{5.13}$$

[24] In the equations in this section, it is understood that, after taking the derivatives, the results are to be evaluated at $S_i = \langle S_i \rangle$ and $\widetilde{S}_i = \langle \widetilde{S}_i \rangle$.

Since both λ_{SW} and u are defined on all the sheets, this equation extends to the entire Riemann surface. A similar equation may be derived for $U(N)$ with a symmetric or antisymmetric hypermultiplet starting from (5.12).

Two methods for computing λ_{SW} (in addition to the approach used in Section 5.1) present themselves. First, one may use perturbation theory to calculate the curve polynomials $r_1(z)$ and $t_1(z)$ as functions of the S's, and then use the curve equation to calculate the derivatives of u in Eq. (5.13). In Appendix A, we use the analog of this approach to determine the one-instanton contribution to $T(z)$ for the $U(N)$ model with $N_f < N$ fundamentals. Obtaining the one-instanton contribution for the theories considered in this paper is straightforward, but somewhat cumbersome, so we will not pursue it here.

A second approach is to investigate the integrals around the A-cycles together with the behavior at infinity, and try to use this information to pin down the function $T(z)$. This method was used in (version 3 of) [7] to determine $T(z)$ for the $U(N)$ model with $N_f < N$ fundamentals. For that case $T(z)$ was given by $\mathrm{d}\psi/\psi$, where $\psi(z)$ was the function arising from Abel's theorem. Such a relation, involving the function $f(z)$ given in (4.26), probably also holds for the $U(N) \times U(N)$ theory. Recalling that $f(z)$ (4.26) is proportional to y (2.6a), using (4.13), this implies $T(z) = \mathrm{d}y/y$, which is consistent with the M-theory result (2.2). For the $U(N)$ models the situation is less clear since $f(z)$ (4.53), (4.57) and y (2.6b), (2.6c) differ by rescalings with z^2.

6. Summary

This paper is part of a larger program aimed at studying the applicability of the Dijkgraaf–Vafa matrix model approach to $\mathcal{N} = 2$ supersymmetric gauge theory theories, i.e., those theories amenable to the methods of Seiberg–Witten theory. Previously it has been shown that one can recover the ingredients of the Seiberg–Witten solution from the matrix model for theories with hyperelliptic Seiberg–Witten curves. In this paper we focused on three different models: (a) $U(N) \times U(N)$ with matter in the bifundamental representation, (b) $U(N)$ with matter in the symmetric representation, and (c) $U(N)$ with matter in the antisymmetric representation. Each of these theories is described by a cubic non-hyperelliptic Seiberg–Witten curve. Our goal was to determine the Seiberg–Witten curve and differential, as well as the order parameters and the prepotential, for the above models, entirely within the context of the matrix model, without reference to string/M-theory. Our results confirm the results previously obtained using M-theory.

For models (a) and (b), a straightforward generalization of our earlier work (including a refinement of the prescription for τ_{ij}, as discussed in [1]) produced expressions in complete agreement with earlier results in the literature. For model (c), $U(N)$ with matter in the antisymmetric representation, the naive extension of earlier work leads to discrepancies with previous results in the literature. The discrepancies occur when one expands around the simplest matrix model vacuum with $\prod_i U(M_i)$ gauge symmetry, where each $M_i \to \infty$ with $S_i = g_s M_i$ fixed. However there are other vacua; one of these has $\mathrm{Sp}(M_0) \times \prod_i U(M_i)$ gauge symmetry [25], and hence an additional parameter $S_0 = g_s M_0$. When one expands around this vacuum, extremization of W_{eff} (with $N_i = 1$ and $N_0 = 0$) leads to non-zero vevs for both S_i and S_0, and all discrepancies are removed. What is missing is a guiding principle for the enlargement of the matrix-model vacua for the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ theory. (A similar enlargement of the vacua of the $\square\!\square$ theory seemingly leads to an inconsistency.)

We close with an empirical observation about the form of the one-instanton contribution to the prepotential. For pure $U(N)$ gauge theory [5], or $U(N)$ with matter in the $\square$ [7] or $\square\!\square$ (Section 3.2.2) representations, one may verify, using our results, that the following equation holds

$$2\pi i \quad {}_{1\text{–inst}} = \sum_i \frac{\langle S_i \rangle}{W''(e_i)}, \tag{6.1}$$

where the sum is over all the extrema of the superpotential $W(z)$. Observe that the expression (6.1) has a finite limit when the coefficient α multiplying the superpotential is taken to zero to restore $\mathcal{N}=2$ supersymmetry. For $U(N) \times U(N)$ with a bifundamental hypermultiplet (Section 3.1.2), the sum also extends over the extrema of $\widetilde{W}(z)$:

$$2\pi i \quad {}_{1\text{–inst}} = \sum_i \frac{\langle S_i \rangle}{W''(e_i)} + \sum_i \frac{\langle \widetilde{S}_i \rangle}{\widetilde{W}''(\tilde{e}_i)}, \tag{6.2}$$

where the relative minus sign is due to the fact that $\widetilde{W}(z)$ enters with a minus sign in the superpotential (3.1). Finally, for $U(N)$ with $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (Section 3.2.2), there is an additional contribution from the vacuum state at $z=0$:

$$2\pi i \quad {}_{1\text{–inst}} = \sum_i \frac{\langle S_i \rangle}{W''(e_i)} + \frac{\langle S_0 \rangle}{W''(0)}. \tag{6.3}$$

However, at present we do not have an understanding of why these results are true.

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Appendix A. U N) N_f curve and T z) from perturbation theory

In Ref. [7], we derived the SW curve for the $\mathcal{N}=2$ $U(N)$ gauge theory with N_f fundamental multiplets from the associated matrix model. Specifically, the saddle-point solution of the matrix model gives rise to the equation for the resolvent[25]

$$\omega^2(z) \quad W'(z)\,\omega(z) + \frac{1}{4} f(z) = 0 \tag{A.1}$$

[25] We have converted to the notation in this paper; to convert back let $\omega(z) \to \quad S\omega(z)$.

where $W'(z) = \alpha \prod_{i=1}^{N}(z - e_i)$ and using (4.7)

$$f(z) = 4g_s\left\langle \mathrm{tr}\left(\frac{W'(z) - W'(\Phi)}{z - \Phi}\right)\right\rangle = 4\sum_{n=0}^{N-1} c_n(z)\, g_s\langle \mathrm{tr}\, \Phi^n\rangle \tag{A.2}$$

is an $(N-1)$th order polynomial. Defining $y(z) = -2\omega(z) + W'(z)$ one may rewrite Eq. (A.1) as a hyperelliptic curve

$$y^2 = W'(z)^2 - f(z). \tag{A.3}$$

In Ref. [7], we determined the polynomial $f(z)$ using Abel's theorem (see also [4]). In this appendix, we will show how this polynomial can be evaluated using the perturbative solution. The lowest order contribution to the matrix model vev is [5,7]

$$g_s\langle \mathrm{tr}\, \Phi^n\rangle = \alpha\Lambda^{2N-N_f}\sum_i \frac{L_i}{R_i} e_i^n + \mathcal{O}\left(\Lambda^{4N-2N_f}\right), \tag{A.4}$$

where $L_i = \prod_{I=1}^{N_f}(e_i + m_I)$ and $R_i = \prod_{j\neq i}(e_i - e_j)$. It follows from (A.4) that

$$f(z) = 4\alpha\,\Lambda^{2N-N_f}\left[W'(z) - W'(e_i)\right]\sum_i \frac{L_i}{R_i(z - e_i)} + \mathcal{O}\left(\Lambda^{4N-2N_f}\right). \tag{A.5}$$

Now using the fact that $W'(e_i) = 0$ together with

$$\sum_i \frac{L_i}{R_i(z - e_i)} = \frac{\prod_{I=1}^{N_f}(z + m_I)}{\prod_{i=1}^{N}(z - e_i)} - \widetilde{T}(z), \tag{A.6}$$

where $\widetilde{T}(z)$ is the polynomial part of $\prod_{I=1}^{N_f}(z + m_I)/\prod_{i=1}^{N}(z - e_i)$, we finally obtain

$$f(z) = 4\alpha^2\Lambda^{2N-N_f}\left(\prod_{I=1}^{N_f}(z + m_I) - \widetilde{T}(z)\prod_{i=1}^{N}(z - e_i)\right) + \mathcal{O}\left(\Lambda^{4N-2N_f}\right) \tag{A.7}$$

which is precisely the result obtained by Abel's theorem in Ref. [7].

When $N_f < N$, the polynomial $\widetilde{T}(z)$ vanishes. Moreover, in that case, Eq. (A.7) is exact to all orders in Λ [7], which points to the existence of a non-renormalization theorem.

Similarly, in (version 3 of) [7] (see also [50]), we derived the expression for the Seiberg–Witten differential for this theory (with $N_f < N$) from the saddle-point approach

$$\lambda_{\mathrm{SW}} = zT(z)\,\mathrm{d}z, \qquad T(z) = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial y}{\partial S_i}\bigg|_{\langle S\rangle} + \frac{y - W'(z)}{2y}\frac{f'(z)}{f(z)}. \tag{A.8}$$

Let us now evaluate this expression using only the perturbative solution. As in Eq. (4.8), we have

$$f(z) = 4W'(z)\sum_i \frac{S_i}{z - e_i} + \mathcal{O}\left(S_i^2\right) \tag{A.9}$$

so that, using Eq. (A.3)

$$\frac{1}{2}\sum_{i=1}^{N}\frac{\partial y}{\partial S_i}=\frac{1}{4y}\sum_{i=1}^{N}\frac{\partial f}{\partial S_i}=\sum_{i=1}^{N}\frac{W'(z)}{y(z-e_i)}+\mathcal{O}(S_i)=\frac{W''(z)}{y}+\mathcal{O}(S_i). \tag{A.10}$$

Evaluating this expression at $S_i=\langle S_i\rangle$, the $\mathcal{O}(S_i)$ term is subleading in Λ, so the leading order contribution to $T(z)$ is

$$T(z)=\frac{1}{y}\left[W''(z)+\frac{1}{2}\left(y-W'(z)\right)\frac{f'}{f}\right]. \tag{A.11}$$

However, as we showed in [7], this expression is exact, again pointing to the existence of a non-renormalization theorem (when $N_f<N$).

Appendix B. Some technical details

In this appendix we collect some technical details of the perturbative calculations performed in Section 3.

B.1. $U(N)\times U(N)$

The integration over the quadratic fields of the matrix-model partition function (3.3) yields (up to an e_i-independent quadratic monomial in the S_i, $\widetilde{S}_i$'s)

$$\begin{aligned}F_{\mathrm{s}}(S,\widetilde{S})={}&-\sum_{i=1}^{N}S_iW(e_i)+\frac{1}{2}\sum_{i=1}^{N}S_i^2\log\left(\frac{S_i}{\alpha R_i\Lambda^2}\right)+\sum_{i=1}^{N}\sum_{j\neq i}S_iS_j\log\left(\frac{e_{ij}}{\Lambda}\right)\\&+\sum_{i=1}^{N}\widetilde{S}_i\widetilde{W}(\tilde{e}_i)+\frac{1}{2}\sum_{i=1}^{N}\widetilde{S}_i^2\log\left(\frac{\widetilde{S}_i}{\alpha\widetilde{R}_i\Lambda^2}\right)+\sum_{i=1}^{N}\sum_{j\neq i}\widetilde{S}_i\widetilde{S}_j\log\left(\frac{\tilde{e}_{ij}}{\Lambda}\right)\\&-\sum_{i}\sum_{j}S_i\widetilde{S}_j\log\left(\frac{h_{ij}}{\Lambda}\right)+\sum_{n\geqslant 3}F_{\mathrm{s}}^{(n)}(S,\widetilde{S})\end{aligned} \tag{B.1}$$

where $e_{ij}=e_i-e_j$, $h_{ij}=e_i-\tilde{e}_j$ and $R_i=\prod_{j\neq i}e_{ij}$. The term $F_{\mathrm{s}}^{(n)}(S,\widetilde{S})$ is an nth order polynomial in S_i and $\widetilde{S}_i$ arising from planar loop diagrams built from the interaction vertices. The contribution to $F_{\mathrm{s}}(S,\widetilde{S})$ cubic in S_i and $\widetilde{S}_i$,

$$\begin{aligned}&\alpha F_{\mathrm{s}}^{(3)}(S,\widetilde{S})\\&\quad=\left[\frac{2}{3}\sum_i\frac{S_i^3}{R_i}\left(\sum_{k\neq i}\frac{1}{e_{ik}}\right)^2-\frac{1}{4}\sum_i\frac{S_i^3}{R_i}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}}\right.\\&\qquad-2\sum_i\sum_{k\neq i}\frac{S_i^2S_k}{R_ie_{ik}}\sum_{\ell\neq i}\frac{1}{e_{i\ell}}+2\sum_i\sum_{k\neq i}\sum_{\ell\neq i}\frac{S_iS_kS_\ell}{R_ie_{ik}e_{i\ell}}-\sum_i\sum_{k\neq i}\frac{S_i^2S_k}{R_ie_{ik}^2}\end{aligned}$$

$$+\sum_i\sum_k \frac{S_i^2\widetilde{S}_k}{R_i h_{ik}}\sum_{\ell\neq i}\frac{1}{e_{i\ell}}+\frac{1}{2}\sum_i\sum_k\frac{S_i^2\widetilde{S}_k}{R_i h_{ik}^2}\quad 2\sum_i\sum_{k\neq i}\sum_\ell\frac{S_iS_k\widetilde{S}_\ell}{R_ie_{ik}h_{i\ell}}$$

$$+\frac{1}{2}\sum_i\sum_k\sum_\ell\frac{S_i\widetilde{S}_k\widetilde{S}_\ell}{R_ih_{ik}h_{i\ell}}\Bigg]\quad [S\leftrightarrow\widetilde{S};\, e_i\leftrightarrow\tilde{e}_i], \tag{B.2}$$

is obtained by adding to the result from Ref. [5] the planar two-loop diagrams containing B loops that can be drawn on a sphere.

Using the above expressions to calculate (3.10) and extremizing (3.11) leads to

$$\begin{aligned}
\langle S_i\rangle = \frac{\alpha T_i}{R_i}\Lambda^N\Bigg[1+\Lambda^N\Bigg\{&\sum_\ell\sum_{k\neq i}\Bigg(\quad\frac{2T_i}{R_i^2e_{ik}h_{i\ell}}+\frac{2T_k}{R_iR_ke_{ik}h_{i\ell}}\quad\frac{2T_k}{R_k^2e_{ik}h_{k\ell}}\\
&\qquad+\frac{2\widetilde{T}_\ell}{R_k\widetilde{R}_\ell e_{ik}h_{k\ell}}\Bigg)\\
&+\sum_{k\neq i}\sum_{\ell\neq i,k}\left(\frac{3T_i}{2R_i^2e_{ik}e_{i\ell}}+\frac{4T_\ell}{R_kR_\ell e_{ik}e_{k\ell}}\right)\\
&+\sum_{k\neq i}\left(\frac{2T_i}{R_i^2e_{ik}^2}\quad\frac{4T_i}{R_iR_ke_{ik}^2}+\frac{2T_k}{R_iR_ke_{ik}^2}+\frac{2T_k}{R_k^2e_{ik}^2}\right)\\
&+\sum_\ell\sum_{k\neq\ell}\left(\frac{2\widetilde{T}_k}{\widetilde{R}_k\widetilde{R}_\ell\tilde{e}_{k\ell}h_{i\ell}}\right)\\
&+\sum_k\sum_\ell\left(\frac{\widetilde{T}_k}{R_i\widetilde{R}_kh_{ik}h_{i\ell}}\quad\frac{\widetilde{T}_k}{\widetilde{R}_k^2h_{ik}h_{\ell k}}+\frac{T_k}{R_k\widetilde{R}_\ell h_{i\ell}h_{k\ell}}\right)\\
&+\sum_k\left(\quad\frac{T_i}{R_i^2h_{ik}^2}\quad\frac{\widetilde{T}_k}{\widetilde{R}_k^2h_{ik}^2}+\frac{\widetilde{T}_k}{R_i\widetilde{R}_kh_{ik}^2}\right)\Bigg\}+\mathcal{O}\ \Lambda^{2N})\Bigg],
\end{aligned}$$

$$\langle\widetilde{S}_i\rangle = \quad\frac{\alpha\widetilde{T}_i}{\widetilde{R}_i}\Lambda^N\big[1+\Lambda^N\{\text{above expression with } e_i\leftrightarrow\tilde{e}_i\}+\mathcal{O}\ \Lambda^{2N})\big], \tag{B.3}$$

where $T_i=\prod_{j=1}^N(e_i\quad\tilde{e}_j)$, $\widetilde{T}_i=\prod_{j=1}^N(\tilde{e}_i\quad e_j)$, $\widetilde{R}_i=\prod_{j\neq i}(\tilde{e}_i\quad\tilde{e}_j)$, and various constants have been absorbed into a redefinition of the cut-off Λ.

The gauge-coupling matrix (3.20) can be expanded as

$$\tau_{ij}=\tau_{ij}^{\text{pert}}+\sum_{d=1}^{\infty}\Lambda^{Nd}\tau_{ij}^{(d)} \tag{B.4}$$

and similarly for $\tau_{\tilde{i}\tilde{j}}$ and $\tau_{i\tilde{j}}$. Using (B.1), (B.2), (B.3), and then reexpressing the result in terms of a_i and $\tilde{a}_i$ using (3.16), the first two terms in the expansion (B.4) can be determined.

The perturbative contribution is (up to additive constants)

$$
\begin{aligned}
2\pi i \tau_{ij}^{\text{pert}} = {} & \delta_{ij}\left[-2\sum_{k\neq i}\log\left(\frac{a_i - a_k}{\Lambda}\right) + \sum_k \log\left(\frac{a_i - \tilde a_k}{\Lambda}\right)\right] \\
& + (1-\delta_{ij})\left[2\log\left(\frac{a_i - a_j}{\Lambda}\right)\right]
\end{aligned}
\tag{B.5}
$$

and the one-instanton contribution, after some algebra, is

$$
\begin{aligned}
2\pi i \tau_{ij}^{(1)} = {} & \delta_{ij}\left[\sum_{k\neq i}\left(\frac{6T_k}{R_k^2 a_{ik}^2} + \frac{2T_i}{R_i^2 a_{ik}^2} + \frac{4T_i}{R_i^2 a_{ik}}\sum_{\ell\neq i}\frac{1}{a_{i\ell}}\right)\right. \\
& \left. - \sum_\ell \frac{T_i}{R_i^2 h_{i\ell}}\left(\sum_{k\neq i}\frac{4}{a_{ik}} - \sum_k \frac{1}{h_{ik}} + \frac{1}{h_{i\ell}}\right)\right] \\
& + (1-\delta_{ij})\left[-\sum_{k\neq i,j}\frac{4T_k}{R_k^2 a_{ik} a_{jk}} + \sum_k \frac{\widetilde T_k}{\widetilde R_k^2 h_{ik} h_{jk}}\right. \\
& \left. - \left[\frac{2T_i}{R_i^2 a_{ij}}\left(\sum_{k\neq i}\frac{2}{a_{ik}} - \sum_k \frac{1}{h_{ik}} + \frac{1}{a_{ij}}\right) + (i\leftrightarrow j)\right]\right]
\end{aligned}
\tag{B.6}
$$

where now $a_{ij} = a_i - a_j$, $h_{ij} = a_i - \tilde a_j$, $R_i = \prod_{j\neq i} a_{ij}$, $\widetilde R_i = \prod_{j\neq i}(\tilde a_i - \tilde a_j)$, $T_i = \prod_{j=1}^N (a_i - \tilde a_j)$, and $\widetilde T_i = \prod_{j=1}^N (\tilde a_i - a_j)$. The expressions for $\tau_{\tilde i \tilde j}$ are obtained from those above by letting $a_i \leftrightarrow \tilde a_i$. Finally,

$$
\begin{aligned}
2\pi i \tau_{i\tilde j}^{\text{pert}} = {} & -\log\left(\frac{a_i - \tilde a_j}{\Lambda}\right),
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
2\pi i \tau_{i\tilde j}^{(1)} = {} & \frac{T_i}{R_i^2 h_{ij}}\left(\sum_{k\neq i}\frac{2}{a_{ik}} - \sum_k \frac{1}{h_{ik}} + \frac{1}{h_{ij}}\right) + 2\sum_{k\neq i}\sum_\ell \frac{T_k}{R_k^2 a_{ik} h_{k\ell}} \\
& + (i\leftrightarrow j;\, a_i \leftrightarrow \tilde a_i).
\end{aligned}
$$

These expressions may be expressed succinctly as the second derivative of the prepotential (3.22).

B.2. $U(N)$ with $\square\square$ or $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$

Using the Feynman rules, the cubic contribution to the sphere part of the free energy can be shown to be

$$
\begin{aligned}
\alpha F_{\text{s}}^{(3)}(S,\overline S) = {} & \frac{2}{3}\sum_i \frac{S_i^3}{R_i}\left(\sum_{k\neq i}\frac{1}{e_{ik}}\right)^2 - \frac{1}{4}\sum_i \frac{S_i^3}{R_i}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}} \\
& - 2\sum_i\sum_{k\neq i}\frac{S_i^2 S_k}{R_i e_{ik}}\sum_{\ell\neq i}\frac{1}{e_{i\ell}} + 2\sum_i\sum_{k\neq i}\sum_{\ell\neq i}\frac{S_i S_k S_\ell}{R_i e_{ik} e_{i\ell}} - \sum_i\sum_{k\neq i}\frac{S_i^2 S_k}{R_i e_{ik}^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_i\sum_k \frac{S_i^2\overline{S}_k}{R_i g_{ik}}\sum_{\ell\neq i}\frac{1}{e_{i\ell}} \quad 2\sum_i\sum_{k\neq i}\sum_\ell \frac{S_iS_k\overline{S}_\ell}{R_i e_{ik} g_{i\ell}}\\
&+\frac{1}{2}\sum_i\sum_k\sum_\ell \frac{S_i\overline{S}_k\overline{S}_\ell}{R_i g_{ik} g_{i\ell}}\\
&+\frac{1}{2}\sum_i\sum_k \frac{S_i^2\overline{S}_k}{R_i g_{ik}^2}+2\sum_i\sum_{j\neq i}\frac{S_0S_iS_j}{R_ie_ie_{ij}} \quad \sum_{i,j}\frac{S_0S_i\overline{S}_j}{R_ie_ig_{ij}}\\
&\quad \sum_i\sum_{j\neq i}\frac{S_0S_i^2}{R_ie_ie_{ij}} \quad \frac{1}{2}\sum_i\frac{S_0S_i^2}{R_ie_i^2}+\frac{1}{4}\sum_{i,j}\frac{S_0\overline{S}_i(\overline{S}_j \quad S_j)}{R_0e_ie_j}\\
&+S_0^2S_i \text{ terms}+S_0^3 \text{ terms}
\end{aligned}
\tag{B.8}
$$

where $e_{ij}=e_i \quad e_j$, $g_{ij}=e_i+e_j$, $R_i=\prod_{j\neq i}e_{ij}$, and $R_0= \quad \prod_i(\quad e_i)\sum_j(1/e_j)$. This result was obtained by adding to the result from Ref. [5] the new planar two-loop diagrams that can be drawn on a sphere and contains Y and ghost loops. Similarly the quadratic contribution to the $\mathbb{R}\quad{}^2$ part of the free energy can be shown to be

$$
\begin{aligned}
\alpha F_{\mathrm{rp}}^{(2)}(S)=\beta\sum_i\Bigg[&\frac{1}{2}\frac{S_i^2}{R_ie_i}\sum_{\ell\neq i}\frac{1}{e_{i\ell}} \quad \sum_{j\neq i}\frac{S_iS_j}{R_ie_ie_{ij}}+\frac{1}{2}\sum_j\frac{S_iS_j}{R_ie_ig_{ij}}+\frac{1}{4}\frac{S_i^2}{R_ie_i^2}\\
&\frac{1}{2}\frac{S_iS_0}{R_ie_i^2}+\frac{1}{4}\frac{S_iS_0}{R_0e_i^2}\Bigg]+S_0^2 \text{ terms.}
\end{aligned}
\tag{B.9}
$$

The terms in (B.8) and (B.9) involving R_0 come from diagrams containing Ψ_{00} legs. Since $\Psi_{00}\in \mathrm{sp}(M_0)$ for $\beta= \quad 1$, one must use $(1/2R_0)(\delta_d^a\delta_b^c+J^{ac}J_{bd})$ for the propagator [51]. Observe that Eqs. (B.8) and (B.9) obey the relation given in footnote 14.

Using the above expressions to calculate (3.42) and then (3.43), the solution for $\langle S_i\rangle$ can be evaluated in an expansion in Λ

$$
\begin{aligned}
\langle S_i\rangle=\frac{\alpha G_i}{R_i}\Lambda^{N\quad 2\beta}\Bigg[1+\Lambda^{N\quad 2\beta}\Bigg\{&\sum_{k\neq i}\sum_{\ell\neq i,k}\left(\frac{3G_i}{2R_i^2e_{ik}e_{i\ell}}+\frac{4G_\ell}{R_kR_\ell e_{ik}e_{k\ell}}\right)\\
&\sum_\ell\sum_{k\neq\ell}\left(\frac{2G_k}{R_kR_\ell e_{k\ell}g_{i\ell}}\right)+\sum_{k\neq i}\left(\frac{2G_i}{R_i^2e_{ik}^2} \quad \frac{4G_i}{R_iR_ke_{ik}^2}+\frac{2G_k}{R_iR_ke_{ik}^2}+\frac{2G_k}{R_k^2e_{ik}^2}\right)\\
&\sum_k\left(\frac{G_i}{R_i^2g_{ik}^2}+\frac{G_k}{R_k^2g_{ik}^2}+\frac{G_k}{R_iR_kg_{ik}^2}\right)\\
&+\sum_\ell\sum_{k\neq i}\left(\quad \frac{2G_i}{R_i^2e_{ik}g_{i\ell}}+\frac{2G_k}{R_iR_ke_{ik}g_{i\ell}} \quad \frac{2G_k}{R_k^2e_{ik}g_{k\ell}} \quad \frac{2G_\ell}{R_kR_\ell e_{ik}g_{k\ell}}\right)\\
&\beta\frac{2G_i}{R_i^2e_i^2} \quad \sum_k\sum_\ell\left(\frac{G_k}{R_iR_kg_{ik}g_{i\ell}}+\frac{G_k}{R_k^2g_{ik}g_{\ell k}}+\frac{G_k}{R_kR_\ell g_{i\ell}g_{k\ell}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\beta \sum_k \left(\frac{2G_k}{R_i R_k g_{ik} e_i} + \frac{2G_k}{R_k^2 g_{ik} e_k} \right) + 2\delta_{\beta,-1}(-1)^N \\
& \times \sum_j \frac{1}{e_j} \left(\frac{1}{R_0 e_i^2} - \frac{1}{R_i e_i^2} + \sum_{k \neq i} \frac{2}{R_k e_k e_{ik}} + \sum_k \frac{1}{R_k e_k g_{ik}} + \sum_k \frac{1}{R_i e_i g_{ik}} \right) \\
& + \beta \sum_{k \neq i} \left(-\frac{4G_i}{R_i^2 e_{ik} e_i} + \frac{4G_k}{R_i R_k e_{ik} e_i} - \frac{4G_k}{R_k^2 e_{ik} e_k} \right) \Bigg\} \Bigg]
\end{aligned}
\tag{B.10}
$$

where $G_i = e_i^{2\beta} \prod_j (e_i + e_j)$. To evaluate (3.61) perturbatively as

$$
\tau_{ij} = \tau_{ij}^{\text{pert}} + \sum_{d=1}^{\infty} \Lambda^{(N-2\beta)d} \tau_{ij}^{(d)},
\tag{B.11}
$$

we use Eqs. (3.40) and (B.8) in Eq. (3.61), evaluate the resulting expression using Eqs. (3.46) and (B.10), and then use the results of Section 3.2.1 to re-express the entire expression in terms of a_i. The perturbative contribution is (up to additive constants)

$$
\begin{aligned}
2\pi i \tau_{ij}^{\text{pert}} = {} & \delta_{ij} \left[-2 \sum_{k \neq i} \log\left(\frac{a_i - a_k}{\Lambda} \right) + \sum_{k \neq i} \log\left(\frac{a_i + a_k}{\Lambda} \right) + 2(1+\beta) \log\left(\frac{a_i}{\Lambda} \right) \right] \\
& + (1 - \delta_{ij}) \left[2 \log\left(\frac{a_i - a_j}{\Lambda} \right) + \log\left(\frac{a_i + a_j}{\Lambda} \right) \right]
\end{aligned}
\tag{B.12}
$$

and the one-instanton contribution, after some algebra, is

$$
\begin{aligned}
2\pi i \tau_{ij}^{(1)} = {} & \delta_{ij} \Bigg[\frac{2(\beta - 1)}{a_i^2 \prod_k a_k} + \frac{7}{2} \frac{G_i}{R_i^2 a_i^2} + \sum_{k \neq i} \left(\frac{6G_k}{R_k^2 a_{ik}^2} + \frac{2G_i}{R_i^2 a_{ik}^2} + \frac{4G_i}{R_i^2 a_{ik}} \sum_{\ell \neq i} \frac{1}{a_{i\ell}} \right) \\
& + \sum_k \sum_\ell \frac{G_i}{R_i^2 g_{ik} g_{i\ell}} + \sum_{k \neq i} \left(-\frac{4G_k}{R_k^2 a_{ik} g_{ik}} - \frac{2(4\beta + 1) G_i}{R_i^2 a_i a_{ik}} \right. \\
& \left. - \frac{4G_i}{R_i^2 a_{ik}} \sum_\ell \frac{1}{g_{i\ell}} \right) + \frac{G_i}{R_i^2} \sum_k \left(-\frac{1}{g_{ik}^2} + \frac{(4\beta + 1)}{a_i g_{ik}} \right) \Bigg] \\
& + (1 - \delta_{ij}) \Bigg[\frac{\beta - 1}{a_i a_j \prod_k a_k} + \sum_{k \neq i,j} \frac{G_k}{R_k^2} \left(-\frac{2}{a_{ik}} + \frac{1}{g_{ik}} \right) \left(-\frac{2}{a_{jk}} + \frac{1}{g_{jk}} \right) \\
& + \Bigg\{ \frac{G_i}{R_i^2} \left(\frac{2}{a_{ij}} + \frac{1}{g_{ij}} \right) \left(-\sum_{k \neq i} \frac{2}{a_{ik}} + \sum_k \frac{1}{g_{ik}} + \frac{(2\beta + \frac{1}{2})}{a_i} \right) \\
& + \frac{G_i}{R_i^2} \left(-\frac{2}{a_{ij}^2} - \frac{1}{g_{ij}^2} \right) + (i \leftrightarrow j) \Bigg\} \Bigg]
\end{aligned}
\tag{B.13}
$$

where now $a_{ij} = a_i \quad a_j$, $g_{ij} = a_i + a_j$, $R_i = \prod_{j \neq i} a_{ij}$, and $G_i = a_i^{2\beta} \prod_j (a_i + a_j)$. Eqs. (B.12) and (B.13) may be written succinctly as the second derivative of the prepotential (3.62).

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