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Level-rank duality of untwisted and twisted D-branes

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Abstract

Level-rank duality of untwisted and twisted D-branes of WZW models is explored. We derive the relation between D0-brane charges of level-rank dual untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{sp}}(n)_k$, and of level-rank dual twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. The analysis of level-rank duality of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$. We also demonstrate level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes in each of these cases.

1. Introduction

D-branes on group manifolds have been the subject of much work, both from the algebraic and geometric points of view [1–26]. (For a review, see Ref. [27].) Algebraically, these D-branes correspond to the allowed boundary conditions for a Wess–Zumino–Witten (WZW) model on a surface with boundary [28].

Much can be learned about D-branes by studying their charges, which are classified by K-theory or, in the presence of a cohomologically non-trivial H-field background, twisted K-theory [29]. The charge group for D-branes on a simply-connected group manifold G with level

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K is given by the twisted K-group [10,12,18,30–32]

$$K^*(G) = \bigoplus_{i=1}^{m} \mathbb{Z}_x, \qquad m = 2^{\operatorname{rank} G - 1}, \tag{1.1}$$

where $\mathbb{Z}_x \equiv \mathbb{Z}/x\mathbb{Z}$ with x an integer depending on G and K. For $\widehat{\mathrm{su}}(N)_K$, for example, x is given by [10]

$$x_{N,K} \equiv \frac{N+K}{\gcd\{N+K, \operatorname{lcm}\{1, \ldots, N-1\}\}}. \tag{1.2}$$

One of the $\mathbb{Z}_x$ factors in the charge group corresponds to the charge of untwisted (symmetry-preserving) D-branes. For su(N) with $N > 2$, another of the $\mathbb{Z}_x$ factors corresponds to D-branes twisted by the charge-conjugation symmetry. For the D-branes corresponding to the remaining factors, see Refs. [10,12,18].

WZW models with classical Lie groups possess an interesting property called level-rank duality: a relationship between various quantities in the $\widehat{\mathrm{su}}(N)_K$, $\widehat{\mathrm{so}}(N)_K$, or $\widehat{\mathrm{sp}}(n)_k$ model, and corresponding quantities in the level-rank dual $\widehat{\mathrm{su}}(K)_N$, $\widehat{\mathrm{so}}(K)_N$, or $\widehat{\mathrm{sp}}(k)_n$ model [33–36]. Implications of level-rank duality for boundary Kazama–Suzuki models were explored in Ref. [24].

In Ref. [37], we began the study of level-rank duality in boundary WZW theories, and in particular the level-rank duality of untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$. In this paper, we extend this work to untwisted D-branes of the $\widehat{\mathrm{sp}}(n)_k$ WZW model, and to twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, which are closely related to the untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$. We focus on two aspects of this duality: the relation between the D0-brane charges of level-rank dual D-branes, and the level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes (i.e., the coefficients of the open-string partition function). For untwisted D-branes, these coefficients are given by the fusion coefficients of the bulk WZW theory [28], so duality of the untwisted open-string partition function follows from the well-known level-rank duality of the fusion rules [33–35]. For twisted D-branes, the open-string partition function coefficients may be calculated in terms of the modular-transformation matrices of twisted affine Lie algebras [4, 14,16]. In this paper, we show that the spectrum of an open string stretched between twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is level-rank dual.

In Section 2, we review some salient features of untwisted D-branes of WZW models. Section 3 describes the level-rank duality of the charges of untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ for all values of N and K (our results in Ref. [37] were restricted to $N + K$ odd), and of the untwisted open-string partition function. Section 4 describes the level-rank duality of the charges of untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$, and of the untwisted open-string partition function. Twisted D-branes of WZW models are reviewed in Section 5, and Section 6 is devoted to demonstrating the level-rank duality of the charges of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, and of the twisted open-string partition function. Concluding remarks constitute Section 7.

2. Untwisted D-branes of WZW models

In this section, we review some salient features of Wess–Zumino–Witten models and their untwisted D-branes.

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted) affine Lie algebra $\hat{g}_K$ at level K. The Dynkin diagram of $\hat{g}_K$ has one more node than that of the associated finite-dimensional Lie algebra g. Let $(m_0, m_1, \ldots, m_n)$ be the dual Coxeter labels of $\hat{g}_K$ (where $n = \operatorname{rank} g$) and $h^\vee = \sum_{i=0}^n m_i$ the dual Coxeter number of g. The Virasoro central charge of the WZW model is then $c = K \dim g/(K + h^\vee)$.

The building blocks of the WZW conformal field theory are integrable highest-weight representations V_λ of $\hat{g}_K$, that is, representations whose highest weight $\lambda \in P_+^K$ has non-negative Dynkin indices $(a_0, a_1, \ldots, a_n)$ satisfying

$$\sum_{i=0}^{n} m_i a_i = K. \tag{2.1}$$

With a slight abuse of notation, we also use λ to denote the highest weight of the irreducible representation of g with Dynkin indices $(a_1, \ldots, a_n)$, which spans the lowest-conformal-weight subspace of V_λ.

For $\widehat{\mathrm{su}}(n+1)_K = (A_n^{(1)})_K$ and $\widehat{\mathrm{sp}}(n)_K = (C_n^{(1)})_K$, the untwisted affine Lie algebras with which we will be principally concerned, we have $m_i = 1$ for $i = 0, \ldots, n$, and $h^\vee = n + 1$. It is often useful to describe irreducible representations of g in terms of Young tableaux. For example, an irreducible representation of $\mathrm{su}(n+1)$ or $\mathrm{sp}(n)$ whose highest weight λ has Dynkin indices a_i corresponds to a Young tableau with n or fewer rows, with row lengths

$$\ell_i = \sum_{j=i}^{n} a_j, \quad i = 1, \ldots, n. \tag{2.2}$$

Let $r(\lambda) = \sum_{i=1}^n \ell_i$ denote the number of boxes of the tableau. Representations λ corresponding to integrable highest-weight representations V_λ of $\widehat{\mathrm{su}}(n+1)_K$ or $\widehat{\mathrm{sp}}(n)_K$ have Young tableaux with K or fewer columns.

We will only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}^{\text{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \bar{V}_{\lambda^*}, \tag{2.3}$$

where $\bar{V}$ represents right-moving states, and λ^* denotes the representation conjugate to λ. The partition function for this theory is

$$Z^{\text{closed}}(\tau) = \sum_{\lambda \in P_+^K} \left|\chi_\lambda(\tau)\right|^2, \tag{2.4}$$

where

$$\chi_\lambda(\tau) = \operatorname{Tr}_{V_\lambda} q^{L_0 - c/24}, \quad q = e^{2\pi i \tau} \tag{2.5}$$

is the affine character of the integrable highest-weight representation V_λ. The affine characters transform linearly under the modular transformation $\tau \to -1/\tau$,

$$\chi_\lambda(-1/\tau) = \sum_{\mu \in P_+^K} S_{\mu\lambda} \chi_\mu(\tau), \tag{2.6}$$

and the unitarity of S ensures the modular invariance of the partition function (2.4).

Next we turn to consider D-branes in the WZW model [1–26]. These D-branes may be studied algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We consider boundary conditions that leave unbroken the $\hat{g}_K$

symmetry, as well as the conformal symmetry, of the theory, and we label the allowed boundary conditions (and therefore the D-branes) by $\alpha, \beta, \ldots$. The partition function on a cylinder, with boundary conditions α and β on the two boundary components, is then given as a linear combination of affine characters of $\hat{g}_K$ [28]

$$Z^{\text{open}}_{\alpha\beta}(\tau) = \sum_{\lambda \in P^K_+} n_{\beta\lambda}{}^{\alpha} \chi_\lambda(\tau). \tag{2.7}$$

This describes the spectrum of an open string stretched between D-branes labelled by α and β.

In this section, we consider a special class of boundary conditions, called *untwisted* (or *symmetry-preserving*), that result from imposing the restriction

$$\left[J^a(z) - \bar{J}^a(\bar{z})\right]\Big|_{z=\bar{z}} = 0 \tag{2.8}$$

on the currents of the affine Lie algebra on the boundary $z = \bar{z}$ of the open string world-sheet, which has been conformally transformed to the upper half plane. Open-closed string duality allows one to correlate the boundary conditions (2.8) of the boundary WZW model with coherent states $|B\rangle\rangle \in \mathcal{H}^{\text{closed}}$ of the bulk WZW model satisfying

$$\left[J^a_m + \bar{J}^a_{-m}\right]|B\rangle\rangle = 0, \quad m \in \mathbb{Z}, \tag{2.9}$$

where J^a_m are the modes of the affine Lie algebra generators. Solutions of Eq. (2.9) that belong to a single sector $V_\mu \otimes \bar{V}_{\mu^*}$ of the bulk WZW theory are known as Ishibashi states $|\mu\rangle\rangle_I$ [38], and are normalized such that

$$_I\langle\langle\mu|q^H|\nu\rangle\rangle_I = \delta_{\mu\nu}\chi_\mu(\tau), \quad q = e^{2\pi i\tau} \tag{2.10}$$

where $H = \frac{1}{2}(L_0 + \bar{L}_0 - \frac{1}{12}c)$ is the closed-string Hamiltonian. For the diagonal theory (2.3), Ishibashi states exist for all integrable highest-weight representations $\mu \in P^K_+$ of $\hat{g}_K$.

A coherent state $|B\rangle\rangle$ that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions [28], among which are that the coefficients $n_{\beta\lambda}{}^{\alpha}$ in Eq. (2.7) must be non-negative integers. Solutions to these conditions are labelled by integrable highest-weight representations $\lambda \in P^K_+$ of the untwisted affine Lie algebra $\hat{g}_K$, and are known as (untwisted) Cardy states $|\lambda\rangle\rangle_C$. The Cardy states may be expressed as linear combinations of Ishibashi states

$$|\lambda\rangle\rangle_C = \sum_{\mu \in P^K_+} \frac{S_{\lambda\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\rangle_I \tag{2.11}$$

where $S_{\lambda\mu}$ is the modular transformation matrix given by Eq. (2.6), and 0 denotes the identity representation. Untwisted D-branes of $\hat{g}_K$ correspond to $|\lambda\rangle\rangle_C$ and are therefore also labelled by $\lambda \in P^K_+$.

The partition function of open strings stretched between untwisted D-branes λ and μ

$$Z^{\text{open}}_{\lambda\mu}(\tau) = \sum_{\nu \in P^K_+} n_{\mu\nu}{}^{\lambda} \chi_\nu(\tau) \tag{2.12}$$

may alternatively be calculated as the closed-string propagator between untwisted Cardy states [28]

$$Z^{\text{open}}_{\lambda\mu}(\tau) = {}_C\langle\langle\lambda|\tilde{q}^H|\mu\rangle\rangle_C, \quad \tilde{q} = e^{2\pi i(-1/\tau)}. \tag{2.13}$$

Combining Eqs. (2.13), (2.11), (2.10), (2.6), and the Verlinde formula [39], we find

$$Z^{\text{open}}_{\lambda\mu}(\tau) = \sum_{\rho\in P_+^K} \frac{S^*_{\lambda\rho} S_{\mu\rho}}{S_{0\rho}} \chi_\rho(-1/\tau) = \sum_{\nu\in P_+^K} \sum_{\rho\in P_+^K} \frac{S_{\mu\rho} S_{\nu\rho} S^*_{\lambda\rho}}{S_{0\rho}} \chi_\nu(\tau) = \sum_{\nu\in P_+^K} N_{\mu\nu}{}^\lambda \chi_\nu(\tau). \tag{2.14}$$

Hence, the coefficients $n_{\mu\nu}{}^\lambda$ in the open-string partition function (2.12) are simply given by the fusion coefficients $N_{\mu\nu}{}^\lambda$ of the bulk WZW model.

Finally, an untwisted D-brane labelled by $\lambda \in P_+^K$ can be considered a bound state of D0-branes [5,8–10,12,40]. It possesses a conserved D0-brane charge Q_λ given by $(\dim\lambda)_g$, but the charge is only defined modulo some integer [9,10,12,21]. For D-branes of $\widehat{\text{su}}(N)_K$, for example, this integer is given by Eq. (1.2), thus

$$Q_\lambda = (\dim\lambda)_{\text{su}(N)} \bmod x_{N,K} \quad \text{for } \widehat{\text{su}}(N)_K \tag{2.15}$$

is the charge of the untwisted D-brane labelled by λ.

3. Level-rank duality of untwisted D-branes of $\widehat{\text{su}}(N)_K$

In Ref. [37], the relation between the charges of untwisted D-branes of the $\widehat{\text{su}}(N)_K$ model and those of the level-rank-dual $\widehat{\text{su}}(K)_N$ model was ascertained for odd values of $N+K$. In this section, we extend these results to all values of N and K.

Since charges of $\widehat{\text{su}}(N)_K$ D-branes are only defined modulo $x_{N,K}$, and those of $\widehat{\text{su}}(K)_N$ D-branes modulo $x_{K,N}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\gcd\{x_{N,K}, x_{K,N}\}$. Without loss of generality we will henceforth assume that $N \geqslant K$, in which case $\gcd\{x_{N,K}, x_{K,N}\} = x_{N,K}$.

3.1. Level-rank duality of untwisted D-brane charges

Given a Young tableau λ corresponding to an integrable highest-weight representation of $\widehat{\text{su}}(N)_K$ (with $N-1$ or fewer rows, and K or fewer columns), its transpose $\tilde\lambda$ corresponds to an integrable highest-weight representation of $\widehat{\text{su}}(K)_N$. (The map between representations of $\widehat{\text{su}}(N)_K$ and $\widehat{\text{su}}(K)_N$ is not one-to-one, but the map between cominimal equivalence classes—or simple-current orbits—of representations is. These equivalence classes are generated by the simple-current symmetry σ of $\widehat{\text{su}}(N)_K$, which takes λ into $\lambda' = \sigma(\lambda)$, where the Dynkin indices of λ' are $a'_i = a_{i-1}$ for $i = 1, \ldots, N-1$, and $a'_0 = a_{N-1}$.)

For odd $N+K$, the relation between Q_λ, the charge of the untwisted $\widehat{\text{su}}(N)_K$ D-brane labelled by λ, and $\tilde{Q}_{\tilde\lambda}$, the charge of the level-rank-dual $\widehat{\text{su}}(K)_N$ D-brane labelled by $\tilde\lambda$, was shown to be [37]

$$\tilde{Q}_{\tilde\lambda} = (-1)^{r(\lambda)} Q_\lambda \bmod x_{N,K}, \quad \text{for } N+K \text{ odd}. \tag{3.1}$$

where $r(\lambda)$ is the number of boxes in the tableau λ. In this section, we show that for the case of even $N+K$, the charges obey

$$\tilde{Q}_{\tilde\lambda} = Q_\lambda \bmod x_{N,K}, \quad \text{for } N+K \text{ even } \big(\text{except for } N=K=2^m\big). \tag{3.2}$$

In the remaining case, we conjecture the relation

$$\tilde{Q}_{\tilde\lambda} = \left\{ \begin{array}{ll} (-1)^{r(\lambda)/N} Q_\lambda \bmod x_{N,N}, & \text{when } N|r(\lambda) \\ Q_\lambda \bmod x_{N,N}, & \text{when } N \not| r(\lambda) \end{array} \right\} \quad \text{for } N=K=2^m \tag{3.3}$$

for which we have numerical evidence, but (as of yet) no complete proof.

Proof of Eq. (3.2). We proceed as in Ref. [37] by writing the dimension of an arbitrary irreducible representation λ of $\mathrm{su}(N)$ (with row lengths ℓ_i and column lengths k_i) as the determinant of an $\ell_1 \times \ell_1$ matrix (Eq. (A.6) of Ref. [43])

$$(\dim \lambda)_{\mathrm{su}(N)} = \left|(\dim \Lambda_{k_i+j-i})_{\mathrm{su}(N)}\right|, \quad i, j = 1, \ldots, \ell_1 \tag{3.4}$$

where Λ_s is the completely *antisymmetric* representation of $\mathrm{su}(N)$, whose Young tableau is $\boxminus\}s$. The maximum value of s appearing in Eq. (3.4) is $k_1 + \ell_1 - 1$, which is bounded by $N + K - 2$ for integrable highest-weight representations of $\widehat{\mathrm{su}}(N)_K$. The representations Λ_0 and Λ_N both correspond to the identity representation, with dimension 1. For $1 \leqslant s \leqslant N - 1$, Λ_s are the fundamental representations of $\mathrm{su}(N)$, with $(\dim \Lambda_s)_{\mathrm{su}(N)} = \binom{N}{s}$. We define $\dim \Lambda_s = 0$ for $s < 0$ and for $s > N$.

In Ref. [37], we showed that

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = \begin{cases} (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \bmod x_{N,K}, \\ \quad \text{for } s \leqslant N + K - 2, \text{ except } s = N \\ (-1)^{K-1} (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \bmod x_{N,K}, \\ \quad \text{for } s = N \end{cases} \tag{3.5}$$

where $\tilde{\Lambda}_s$ is the completely *symmetric* representation of $\mathrm{su}(K)$, whose Young tableau is $\underbrace{\square\square\square}_{s}$. (We define $\dim \tilde{\Lambda}_s = 0$ for $s < 0$.) When $N + K$ is odd, Eq. (3.5) becomes simply $(\dim \Lambda_s)_{\mathrm{su}(N)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \bmod x_{N,K}$ for all $s \leqslant N + K - 2$. This was used in Ref. [37] to yield Eq. (3.1).

Now we turn to the case of even $N + K$, first considering $N > K$. In Eq. (1.2), the factor $\mathrm{lcm}\{1, \ldots, N - 1\}$ then contains $(N + K)/2$, so $x_{N,K}$ is at most 2. It is easy to see that $x_{N,K} = 2$ if $N + K = 2^m$, and $x_{N,K} = 1$ otherwise. For $x_{N,K} \leqslant 2$, the minus signs in Eq. (3.5) are irrelevant (since $n = -n \bmod 2$), so we may simply write

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \bmod x_{N,K},$$
$$\text{for } s \leqslant N + K - 2, \text{ with } N + K \text{ even and } N > K. \tag{3.6}$$

We will use this below.

Next we consider $N = K$. We begin by observing that if N is a power of a prime p, then $x_{N,N} = 4$ if $p = 2$, and $x_{N,N} = p$ if $p > 2$. If N contains more than one prime factor, then $x_{N,N} = 1$. In the latter case, Eq. (3.2) is trivially satisfied, so we need only consider $N = K = p^m$, where p is prime. Let us obtain the relation between $(\dim \Lambda_s)_{\mathrm{su}(p^m)}$ and $(\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)}$ by considering three separate cases:

- $0 \leqslant s \leqslant N - 1$:
  By examining the factors of p (prime) in the numerator and denominator of $(\dim \Lambda_s)_{\mathrm{su}(p^m)} = \binom{p^m}{s}$, one can establish that if p^{l-1} divides s but p^l does not (for any $l \leqslant m$), then p^{m-l+1} divides $\binom{p^m}{s}$. Thus $(\dim \Lambda_s)_{\mathrm{su}(p^m)} = 0 \bmod p$ for $1 \leqslant s \leqslant N - 1$. Combining this with Eq. (3.5), we have

$$(\dim \Lambda_s)_{\mathrm{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)} \bmod x_{N,N}, \quad \text{for } 1 \leqslant s \leqslant N - 1. \tag{3.7}$$

  This is trivially extended to $s = 0$.

- $s < 0$, or $N+1 \leqslant s \leqslant 2N-2$:
  In this case,

$$(\dim \Lambda_s)_{\mathrm{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)} \bmod x_{N,N}, \quad \text{for } s<0, \text{ or } N+1 \leqslant s \leqslant 2N-2, \tag{3.8}$$

  is valid because the l.h.s. vanishes, and so, by Eq. (3.5), the r.h.s. either vanishes or is a multiple of $x_{N,N}$.
- $s = N$:
  The remaining case yields [37]

$$(\dim \Lambda_N)_{\mathrm{su}(p^m)} = (-1)^{N-1}(\dim \tilde{\Lambda}_N)_{\mathrm{su}(p^m)} \bmod x_{N,N} \tag{3.9}$$

  which is in accord with the other cases when p is a prime other than 2.

We combine these results with Eq. (3.6) to write

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \bmod x_{N,K}, \quad \text{for } s \leqslant N+K-2, \text{ for } N+K \text{ even } \big(\text{except } N=K=2^m\big). \tag{3.10}$$

Inserting this in Eq. (3.4), we find

$$(\dim \lambda)_{\mathrm{su}(N)} = \big|(\dim \tilde{\Lambda}_{k_i+j-i})_{\mathrm{su}(K)}\big| \bmod x_{N,K}, \quad \text{for } N+K \text{ even } \big(\text{except } N=K=2^m\big). \tag{3.11}$$

By an alternative formula for the dimension of a representation (Eq. (A.5) of Ref. [43]), the r.h.s. is the dimension of a representation of su(K) with row lengths k_i and column lengths ℓ_i,[3] that is, the transpose representation $\tilde{\lambda}$, hence

$$(\dim \lambda)_{\mathrm{su}(N)} = (\dim \tilde{\lambda})_{\mathrm{su}(K)} \bmod x_{N,K}, \quad \text{for } N+K \text{ even } \big(\text{except } N=K=2^m\big). \tag{3.12}$$

from which Eq. (3.2) follows.[4] □

3.2. Level-rank duality of the untwisted open string spectrum

In Refs. [34,35], it was shown that the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ of the bulk $\widehat{\mathrm{su}}(N)_K$ WZW model are related to those of the $\widehat{\mathrm{su}}(K)_N$ WZW model, denoted by $\tilde{N}$, by

$$N_{\mu\nu}{}^{\lambda} = \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\sigma^{\Delta}(\tilde{\lambda})} = \tilde{N}_{\tilde{\mu}\sigma^{-\Delta}(\tilde{\nu})}{}^{\tilde{\lambda}} \tag{3.13}$$

where $\Delta = [r(\mu)+r(\nu)-r(\lambda)]/N$.

Since by Eq. (2.14) the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ are equal to the coefficients $n_{\mu\nu}{}^{\lambda}$ of the open-string partition function (2.12), it follows that if the spectrum of an $\widehat{\mathrm{su}}(N)_K$ open string stretched between untwisted D-branes λ and μ contains $n_{\mu\nu}{}^{\lambda}$ copies of the highest-weight representation

[3] If λ has $\ell_1 = K$, then the transpose $\tilde{\lambda}$ contains leading columns of K boxes. In that case, one can apply the formula [12] $Q_{\sigma(\lambda)} = (-1)^{N-1} Q_\lambda \bmod x_{N,K}$ several times to relate λ to a tableau with no rows of length K before using Eq. (3.4). The minus sign is irrelevant when $x_{N,K} \leqslant 2$, and vanishes when N is an odd prime.

[4] Since minus signs are irrelevant when $x_{N,K} \leqslant 2$, Eq. (3.1) actually holds for all $N \neq K$, not just odd $N+K$. Eq. (3.1) is not valid, however, when $N=K$. This is most easily seen by considering representations of $\widehat{\mathrm{su}}(N)_N$ whose tableaux are invariant under transposition, and whose dimensions are not multiples of x, such as the adjoint of $\widehat{\mathrm{su}}(3)_3$.

V_ν of $\widehat{\mathrm{su}}(N)_K$, then the spectrum of an $\widehat{\mathrm{su}}(K)_N$ open string stretched between untwisted D-branes $\tilde{\lambda}$ and $\tilde{\mu}$ contains an equal number of copies of the highest-weight representation $V_{\sigma^{-\Delta}(\tilde{\nu})}$ of $\widehat{\mathrm{su}}(K)_N$.

4. Level-rank duality of untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$

In this section, we examine the relation between untwisted D-branes of the $\widehat{\mathrm{sp}}(n)_k$ model and those of the level-rank-dual $\widehat{\mathrm{sp}}(k)_n$ model.

Untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$ are labelled by integrable highest-weight representations V_λ of $\widehat{\mathrm{sp}}(n)_k = (C_n^{(1)})_k$. The D0-brane charge of D-branes of $\widehat{\mathrm{sp}}(n)_k$ are defined modulo the integer [17,21]

$$
\begin{aligned}
x &= \frac{n+k+1}{\gcd\{n+k+1, \mathrm{lcm}\{1,2,3,\ldots,n,1,3,5,\ldots,2n-1\}\}} \\
&= \frac{n+k+1}{\gcd\{n+k+1, \frac{1}{2}\mathrm{lcm}\{1,2,\ldots,2n\}\}} = \frac{2(n+k+1)}{\gcd\{2(n+k+1), \mathrm{lcm}\{1,2,\ldots,2n\}\}} \\
&= x_{2n+1,2k+1},
\end{aligned}
\tag{4.1}
$$

where $x_{2n+1,2k+1}$ is given by Eq. (1.2). That is,

$$
Q_\lambda = (\dim\lambda)_{\mathrm{sp}(n)} \bmod x_{2n+1,2k+1} \quad \text{for } \widehat{\mathrm{sp}}(n)_k \tag{4.2}
$$

is the charge of the untwisted $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by λ, where $(\dim\lambda)_{\mathrm{sp}(n)}$ is the dimension of the $\mathrm{sp}(n)$ representation λ. As we showed in the previous section, for $n \neq k$, we have $x_{2n+1,2k+1} = 2$ if $n+k+1 = 2^m$, and $x_{2n+1,2k+1} = 1$ otherwise. For $n = k$, we have $x_{2n+1,2n+1} = p$ if $2n+1 = p^m$, and $x_{2n+1,2n+1} = 1$ if $2n+1$ contains more than one prime factor.

Since charges of $\widehat{\mathrm{sp}}(n)_k$ D-branes are only defined modulo $x_{2n+1,2k+1}$, and those of $\widehat{\mathrm{sp}}(k)_n$ D-branes modulo $x_{2k+1,2n+1}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\}$. Without loss of generality we henceforth assume that $n \geqslant k$, in which case $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\} = x_{2n+1,2k+1}$.

4.1. Level-rank duality of untwisted D-brane charges

Given a Young tableau λ corresponding to an integrable highest-weight representation of $\widehat{\mathrm{sp}}(n)_k$ (with n or fewer rows and k or fewer columns), its transpose $\tilde{\lambda}$ corresponds to an integrable highest-weight representation of $\widehat{\mathrm{sp}}(k)_n$. The mapping between representations is one-to-one, in contrast to the case of $\widehat{\mathrm{su}}(N)_K$.

We will show that the relation between Q_λ, the charge of the $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by λ, and $\tilde{Q}_{\tilde{\lambda}}$, the charge of the level-rank-dual $\widehat{\mathrm{sp}}(k)_n$ D-brane labelled by $\tilde{\lambda}$, is given by

$$
\tilde{Q}_{\tilde{\lambda}} = Q_\lambda \bmod x_{2n+1,2k+1}. \tag{4.3}
$$

The relation (4.3) is non-trivial only when $x_{2n+1,2k+1} > 1$, that is, when $n \neq k$ with $n+k+1 = 2^m$, or when $n = k$ with $2n+1 = p^m$.

Proof of Eq. (4.3). We may write the dimension of an arbitrary irreducible representation λ of $\mathrm{sp}(n)$ as the determinant of an $\ell_1 \times \ell_1$ matrix (Proposition (A.44) of Ref. [43]; see also Ref. [36])

$$(\dim\lambda)_{\mathrm{sp}(n)} = \begin{vmatrix} \chi_{k_1} & (\chi_{k_1+1}+\chi_{k_1-1}) & \cdots & (\chi_{k_1+\ell_1-1}+\chi_{k_1-\ell_1+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{k_i-i+1} & (\chi_{k_i-i+2}+\chi_{k_i-i}) & \cdots & (\chi_{k_1+\ell_1-i}+\chi_{k_1-\ell_1-i+2}) \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix},$$

$$i, j = 1, \ldots, \ell_1, \tag{4.4}$$

where $\chi_s = (\dim \Lambda_s)_{\mathrm{sp}(n)}$, with Λ_s the completely *antisymmetric* representation of $\mathrm{sp}(n)$, whose Young tableau is $\big\}s$. The maximum value of s appearing in Eq. (4.4) is $k_1 + \ell_1 - 1$, which is bounded by $n+k-1$ for integrable highest-weight representations of $\widehat{\mathrm{sp}}(n)_k$. The representation Λ_0 corresponds to the identity representation with dimension 1. For $1 \leqslant s \leqslant n$, Λ_s are the fundamental representations of $\mathrm{sp}(n)$. (We define $(\dim\Lambda_s)_{\mathrm{sp}(n)} = 0$ for $s<0$ and for $s>n$.) Also, let $\tilde{\Lambda}_s$ be the completely *symmetric* representation of $\mathrm{sp}(k)$, whose Young tableau is $\underbrace{\square\square\square}_{s}$. (We define $(\dim\tilde{\Lambda}_s)_{\mathrm{sp}(k)} = 0$ for $s<0$.)

Next, we may use the branching rules $(\Lambda_s)_{\mathrm{su}(2n+1)} = \bigoplus_{t=0}^{s} (\Lambda_t)_{\mathrm{sp}(n)}$ (for $s \leqslant n$) and $(\tilde{\Lambda}_s)_{\mathrm{su}(2n+1)} = \bigoplus_{t=0}^{s} (\tilde{\Lambda}_t)_{\mathrm{sp}(n)}$ of $\mathrm{su}(2n+1) \supset \mathrm{sp}(n)$ to relate the dimensions of the fundamental representations of $\mathrm{sp}(n)$ to those of the fundamental representations of $\mathrm{su}(2n+1)$:

$$(\dim\Lambda_s)_{\mathrm{sp}(n)} = (\dim\Lambda_s)_{\mathrm{su}(2n+1)} - (\dim\Lambda_{s-1})_{\mathrm{su}(2n+1)},$$
$$(\dim\tilde{\Lambda}_s)_{\mathrm{sp}(k)} = (\dim\tilde{\Lambda}_s)_{\mathrm{su}(2k+1)} - (\dim\tilde{\Lambda}_{s-1})_{\mathrm{su}(2k+1)}. \tag{4.5}$$

Using this together with Eq. (3.10), we have

$$(\dim\Lambda_s)_{\mathrm{sp}(n)} = (\dim\tilde{\Lambda}_s)_{\mathrm{sp}(k)} \bmod x_{2n+1,2k+1}, \quad \text{for } s \leqslant 2n+2k. \tag{4.6}$$

We use this in Eq. (4.4) to obtain

$$(\dim\lambda)_{\mathrm{sp}(n)} = \begin{vmatrix} \tilde{\chi}_{k_1} & (\tilde{\chi}_{k_1+1}+\tilde{\chi}_{k_1-1}) & \cdots & (\tilde{\chi}_{k_1+\ell_1-1}+\tilde{\chi}_{k_1-\ell_1+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\chi}_{k_i-i+1} & (\tilde{\chi}_{k_i-i+2}+\tilde{\chi}_{k_i-i}) & \cdots & (\tilde{\chi}_{k_1+\ell_1-i}+\tilde{\chi}_{k_1-\ell_1-i+2}) \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \bmod x_{2n+1,2k+1} \tag{4.7}$$

where $\tilde{\chi}_s = (\dim\tilde{\Lambda}_s)_{\mathrm{sp}(k)}$. By an alternative formula for the dimension of a representation (Proposition (A.50) of Ref. [43]), the r.h.s. is the dimension of a representation of $\mathrm{sp}(k)$ with row lengths k_i and column lengths ℓ_i, that is, the transpose representation $\tilde{\lambda}$, hence

$$(\dim\lambda)_{\mathrm{sp}(n)} = (\dim\tilde{\lambda})_{\mathrm{sp}(k)} \bmod x_{2n+1,2k+1}, \tag{4.8}$$

from which Eq. (4.3) follows. □

4.2. Level-rank duality of the untwisted open string spectrum

In Ref. [35], it was shown that the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ of the bulk $\widehat{\mathrm{sp}}(n)_k$ WZW model are related to those of the $\widehat{\mathrm{sp}}(k)_n$ WZW model by

$$N_{\mu\nu}{}^{\lambda} = \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\tilde{\lambda}}. \tag{4.9}$$

Since the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ are equal to the coefficients $n_{\mu\nu}{}^{\lambda}$ of the open-string partition function, it follows that if the spectrum of an $\widehat{\mathrm{sp}}(n)_k$ open string stretched between untwisted D-branes λ and μ contains $n_{\mu\nu}{}^{\lambda}$ copies of the highest-weight representation V_ν of $\widehat{\mathrm{sp}}(n)_k$, then the spectrum of an $\widehat{\mathrm{sp}}(k)_n$ open string stretched between untwisted D-branes $\tilde{\lambda}$ and $\tilde{\mu}$ contains an equal number of copies of the highest-weight representation $V_{\tilde{\nu}}$ of $\widehat{\mathrm{sp}}(k)_n$.

5. Twisted D-branes of WZW models

In this section we review some aspects of twisted D-branes of the WZW model, drawing on Refs. [2–4,16]. As in Section 2, these D-branes correspond to possible boundary conditions that can imposed on a boundary WZW model.

A boundary condition more general than Eq. (2.8) that still preserves the $\hat{g}_K$ symmetry of the boundary WZW model is

$$\left[J^a(z) - \omega \bar{J}^a(\bar{z})\right]\Big|_{z=\bar{z}} = 0, \tag{5.1}$$

where ω is an automorphism of the Lie algebra g. The boundary conditions (5.1) correspond to coherent states $|B\rangle\rangle^{\omega} \in \mathcal{H}^{\text{closed}}$ of the bulk WZW model that satisfy

$$\left[J^a_m + \omega \bar{J}^a_{-m}\right]|B\rangle\rangle^{\omega} = 0, \quad m \in \mathbb{Z}. \tag{5.2}$$

The ω-twisted Ishibashi states $|\mu\rangle\rangle^{\omega}_I$ are solutions of Eq. (5.2) that belong to a single sector $V_\mu \otimes \bar{V}_{\omega(\mu)^*}$ of the bulk WZW theory, and whose normalization is given by

$${}^{\omega}_{I}\langle\langle\mu|q^H|\nu\rangle\rangle^{\omega}_I = \delta_{\mu\nu}\chi_\mu(\tau), \quad q = \mathrm{e}^{2\pi i\tau}. \tag{5.3}$$

Since we are considering the diagonal closed-string theory (2.3), these states only exist when $\mu = \omega(\mu)$, so the ω-twisted Ishibashi states are labelled by $\mu \in \mathcal{E}^{\omega}$, where $\mathcal{E}^{\omega} \subset P^K_+$ are the integrable highest-weight representations of $\hat{g}_K$ that satisfy $\omega(\mu) = \mu$. Equivalently, μ corresponds to a highest-weight representation, which we denote by $\pi(\mu)$, of $\breve{g}$, the orbit Lie algebra [41] associated with $\hat{g}_K$.

Solutions of Eq. (5.2) that also satisfy the Cardy conditions are denoted ω-twisted Cardy states $|\alpha\rangle\rangle^{\omega}_C$, where the labels α take values in some set $\mathcal{B}^{\omega}$. The ω-twisted Cardy states may be expressed as linear combinations of ω-twisted Ishibashi states

$$|\alpha\rangle\rangle^{\omega}_C = \sum_{\mu\in\mathcal{E}^{\omega}} \frac{\psi_{\alpha\pi(\mu)}}{\sqrt{S_{0\mu}}}|\mu\rangle\rangle^{\omega}_I, \tag{5.4}$$

where $\psi_{\alpha\pi(\mu)}$ are some as-yet-undetermined coefficients. The ω-twisted D-branes of $\hat{g}_K$ correspond to $|\alpha\rangle\rangle^{\omega}_C$ and are therefore also labelled by $\alpha \in \mathcal{B}^{\omega}$. These states (apparently) correspond [4] to integrable highest-weight representations of the ω-twisted affine Lie algebra $\hat{g}^{\omega}_K$ (but see Ref. [19]).

The partition function of open strings stretched between ω-twisted D-branes α and β

$$Z^{\text{open}}_{\alpha\beta}(\tau) = \sum_{\lambda\in P^K_+} n_{\beta\lambda}{}^{\alpha}\chi_\lambda(\tau) \tag{5.5}$$

may alternatively be calculated as the closed-string propagator between ω-twisted Cardy states

$$Z^{\text{open}}_{\alpha\beta}(\tau) = {}^{\omega}_{C}\langle\langle\alpha|\tilde{q}^H|\beta\rangle\rangle^{\omega}_C, \quad \tilde{q} = \mathrm{e}^{2\pi i(-1/\tau)}. \tag{5.6}$$

Combining Eqs. (5.6), (5.4), (5.3), and (2.6), we find

$$Z^{\text{open}}_{\alpha\beta}(\tau) = \sum_{\rho\in\mathcal{E}^{\omega}} \frac{\psi^{*}_{\alpha\pi(\rho)}\psi_{\beta\pi(\rho)}}{S_{0\rho}}\chi_{\rho}(-1/\tau) = \sum_{\lambda\in P^{K}_{+}}\sum_{\rho\in\mathcal{E}^{\omega}} \frac{\psi^{*}_{\alpha\pi(\rho)}S_{\lambda\rho}\psi_{\beta\pi(\rho)}}{S_{0\rho}}\chi_{\lambda}(\tau). \tag{5.7}$$

Hence, the coefficients of the open-string partition function (5.5) are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho\in\mathcal{E}^{\omega}} \frac{\psi^{*}_{\alpha\pi(\rho)}S_{\lambda\rho}\psi_{\beta\pi(\rho)}}{S_{0\rho}}. \tag{5.8}$$

Finally, the coefficients $\psi_{\alpha\pi(\rho)}$ relating the ω-twisted Cardy states and ω-twisted Ishibashi states may be identified [4] with the modular transformation matrices of characters of twisted affine Lie algebras [45], as may be seen, for example, by examining the partition function of an open string stretched between an ω-twisted and an untwisted D-brane [14,16].

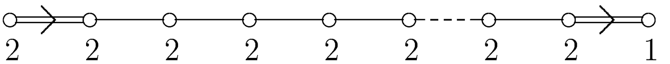
6. Level-rank duality of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$

The finite Lie algebra $\mathrm{su}(N)$ possesses an order-two automorphism ω_c arising from the invariance of its Dynkin diagram under reflection. This automorphism maps the Dynkin indices of an irreducible representation $a_i \to a_{N-i}$, and corresponds to charge conjugation of the representation. This automorphism lifts to an automorphism of the affine Lie algebra $\widehat{\mathrm{su}}(N)_K$, leaving the zeroth node of the extended Dynkin diagram invariant, and gives rise to a class of ω_c-twisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model (for $N > 2$). Since the details of the ω_c-twisted D-branes differ significantly between even and odd N, and we will restrict our attention to the ω_c-twisted D-branes of the $\widehat{\mathrm{su}}(2n+1)_{2k+1} = (A^{(1)}_{2n})_{2k+1}$ WZW model.

First, recall that the ω_c-twisted Ishibashi states $|\mu\rangle\rangle^{\omega_c}_{I}$ are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}^{\omega}$ of $(A^{(1)}_{2n})_{2k+1}$. Eq. (2.1) implies that the Dynkin indices $(a_0, a_1, a_2, \ldots, a_{n-1}, a_n, a_n, a_{n-1}, \ldots, a_1)$ of μ satisfy

$$a_0 + 2(a_1 + \cdots + a_n) = 2k + 1. \tag{6.1}$$

In Ref. [41], it was shown that the self-conjugate highest-weight representations of $(A^{(1)}_{2n})_{2k+1}$ are in one-to-one correspondence with integrable highest weight representations of the associated orbit Lie algebra $\breve{g} = (A^{(2)}_{2n})_{2k+1}$, whose Dynkin diagram is



with the integers indicating the dual Coxeter label m_i of each node. The representation $\mu \in \mathcal{E}^{\omega}$ corresponds to the $(A^{(2)}_{2n})_{2k+1}$ representation $\pi(\mu)$ with Dynkin indices $(a_0, a_1, \ldots, a_n)$. Consistency with Eq. (6.1) requires that the dual Coxeter labels are $(m_0, m_1, \ldots, m_n) = (1, 2, 2, \ldots, 2)$, and hence we must choose as the zeroth node the *right-most* node of the Dynkin diagram above. The finite part of the orbit Lie algebra $\breve{g}$, obtained by omitting the zeroth node, is thus C_n. (Note that C_n is the orbit Lie algebra of the finite Lie algebra A_{2n} [41].)

Observe that, by Eq. (6.1), a_0 must be odd, and that the representation $\pi(\mu)$ of the orbit algebra $\breve{g}$ is in one-to-one correspondence [15,16,41] with the integrable highest-weight representation $\pi(\mu)'$ of the untwisted affine Lie algebra $(C^{(1)}_n)_k$ with Dynkin indices $(a'_0, a'_1, \ldots, a'_n)$, where $a'_0 = \frac{1}{2}(a_0 - 1)$ and $a'_i = a_i$ for $i = 1, \ldots, n$.

Next, the ω_c-twisted Cardy states $|\alpha\rangle\rangle_C^{\omega_c}$ (and therefore the ω_c-twisted D-branes) of the $(A_{2n}^{(1)})_{2k+1}$ WZW model are (apparently) labelled [4] by the integrable highest-weight representations $\alpha \in \mathcal{B}^{\omega_c}$ of the twisted Lie algebra $\hat{g}_{2k+1}^{\omega_c} = (A_{2n}^{(2)})_{2k+1}$ (but see Ref. [19]). We adopt the same convention as above for the labelling of the nodes of the Dynkin diagram (consistent with Refs. [16,45] but differing from Refs. [42,44]). Thus, the Dynkin indices $(a_0, a_1, \ldots, a_n)$ of the highest weights α must also satisfy Eq. (6.1), and the ω_c-twisted D-branes are therefore characterized [16,19] by the irreducible representations of $C_n = \mathrm{sp}(n)$ with Dynkin indices $(a_1, \ldots, a_n)$ (also denoted, with a slight abuse of notation, by α). The charge of the ω_c-twisted D-brane of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ labelled by α is given by [17]

$$Q_\alpha^{\omega_c} = (\dim\alpha)_{\mathrm{sp}(n)} \bmod x_{2n+1,2k+1}, \quad \text{for } \widehat{\mathrm{su}}(2n+1)_{2k+1}. \tag{6.2}$$

The periodicity of the charge is the same as that of all D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$.

Observe also that the ω_c-twisted D-branes $\alpha \in \mathcal{B}^{\omega_c}$ are in one-to-one correspondence with integrable highest-weight representations α' of the untwisted affine Lie algebra $(C_n^{(1)})_k$ with Dynkin indices $(a'_0, a'_1, \ldots, a'_n)$, where $a'_0 = \frac{1}{2}(a_0 - 1)$ and $a'_i = a_i$ for $i = 1, \ldots, n$. That is, both the ω_c-twisted Ishibashi states and the ω_c-twisted Cardy states of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ are classified by integrable representations of $\widehat{\mathrm{sp}}(n)_k$.

Recall from Eq. (5.8) that the coefficients of the partition function of open strings stretched between ω_c-twisted D-branes α and β are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho\in\mathcal{E}^\omega} \frac{\psi^*_{\alpha\pi(\rho)} S_{\lambda\rho} \psi_{\beta\pi(\rho)}}{S_{0\rho}}, \tag{6.3}$$

where $\alpha, \beta \in \mathcal{B}^{\omega_c}$, $\lambda \in P_+^K$, and $\pi(\rho)$ is the representation of the orbit Lie algebra $(A_{2n}^{(2)})_{2k+1}$ that corresponds to the self-conjugate representation ρ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. The coefficients $\psi_{\alpha\pi(\rho)}$ are given [4,14,16] by the modular transformation matrix of the characters of $(A_{2n}^{(2)})_{2k+1}$. These in turn may be identified [15,16,41] with $S'_{\alpha'\pi(\rho)'}$, the modular transformation matrix of $(C_n^{(1)})_k = \widehat{\mathrm{sp}}(n)_k$, so

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho\in\mathcal{E}^\omega} \frac{S'^*_{\alpha'\pi(\rho)'} S_{\lambda\rho} S'_{\beta'\pi(\rho)'}}{S_{0\rho}}. \tag{6.4}$$

We will use this below to demonstrate level-rank duality of $n_{\beta\lambda}{}^{\alpha}$.

6.1. Level-rank duality of twisted D-brane charges

It is now straightforward to show the equality of charges of level-rank-dual ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. As seen above, the ω_c-twisted $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ D-brane labelled by α is in one-to-one correspondence with an integrable highest-weight representation α' of $\widehat{\mathrm{sp}}(n)_k$, and has the same charge (6.2) as the untwisted $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by α' (4.2), including periodicity. The integrable highest-weight representation α' of $\widehat{\mathrm{sp}}(n)_k$ is level-rank-dual to the integrable highest-weight representation $\tilde{\alpha}'$ of $\widehat{\mathrm{sp}}(k)_n$ obtained by transposing the Young tableau corresponding to α', and the charges of the corresponding untwisted D-branes obey

$$(\dim\alpha')_{\mathrm{sp}(n)} = (\dim\tilde{\alpha}')_{\mathrm{sp}(k)} \bmod x_{2n+1,2k+1}, \tag{6.5}$$

as shown in Section 4. Therefore the ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ are in one-to-one correspondence with the ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$, and the charges of level-rank-dual ω_c-twisted D-branes obey

$$Q_\alpha^{\omega_c} = \tilde{Q}_{\tilde{\alpha}}^{\omega_c} \bmod x_{2n+1,2k+1}, \tag{6.6}$$

where the map between ω_c-twisted D-branes is given by transposition of the associated $\widehat{\mathrm{sp}}(n)_k$ tableaux.

6.2. Level-rank duality of the twisted open string spectrum

The coefficients of the partition function of open strings stretched between ω_c-twisted D-branes α and β are real numbers so we may write (6.4) as

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho\in\mathcal{E}^\omega} \frac{S'_{\alpha'\pi(\rho)'} S^*_{\lambda\rho} S'^*_{\beta'\pi(\rho)'}}{S^*_{0\rho}}. \tag{6.7}$$

Under level-rank duality, the $\widehat{\mathrm{su}}(N)_K$ modular transformation matrices transform as [34,35]

$$S_{\lambda\mu} = \sqrt{\frac{K}{N}} \mathrm{e}^{-2\pi i r(\lambda) r(\mu)/NK} \tilde{S}^*_{\tilde{\lambda}\tilde{\mu}} \tag{6.8}$$

and the (real) $\widehat{\mathrm{sp}}(n)_k$ modular transformation matrices transform as [35]

$$S'_{\alpha'\beta'} = \tilde{S}'_{\tilde{\alpha}'\tilde{\beta}'} = \tilde{S}'^*_{\tilde{\alpha}'\tilde{\beta}'} \tag{6.9}$$

where $\tilde{S}$ and $\tilde{S}'$ denote the $\widehat{\mathrm{su}}(K)_N$ and $\widehat{\mathrm{sp}}(k)_n$ modular transformation matrices respectively, $\tilde{\mu}$ is the transpose of the Young tableau corresponding to the $\widehat{\mathrm{su}}(N)_K$ representation μ, and $\tilde{\alpha}'$ is the transpose of the Young tableau corresponding to the $\widehat{\mathrm{sp}}(n)_k$ representation α'. These imply

$$\begin{aligned} n_{\beta\lambda}{}^{\alpha} &= \sum_{\rho\in\mathcal{E}^\omega} \frac{\tilde{S}'^*_{\tilde{\alpha}'\widetilde{\pi(\rho)'}} \tilde{S}_{\tilde{\lambda}\tilde{\rho}} \tilde{S}'_{\tilde{\beta}'\widetilde{\pi(\rho)'}}}{\tilde{S}_{0\tilde{\rho}}} \mathrm{e}^{2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} \\ &= \sum_{\rho\in\mathcal{E}^\omega} \frac{\tilde{\psi}^*_{\tilde{\alpha}\widetilde{\pi(\rho)}} \tilde{S}_{\tilde{\lambda}\tilde{\rho}} \tilde{\psi}_{\tilde{\beta}\widetilde{\pi(\rho)}}}{\tilde{S}_{0\tilde{\rho}}} \mathrm{e}^{2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)}. \end{aligned} \tag{6.10}$$

Let $\hat{\rho}$ be the self-conjugate $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ representation that maps to the $\widehat{\mathrm{sp}}(k)_n$ representation $\widetilde{\pi(\rho)}'$, which is the transpose of the $\widehat{\mathrm{sp}}(n)_k$ representation $\pi(\rho)'$. In other words, the representation $\pi(\hat{\rho})$ of the orbit algebra is identified with $\widetilde{\pi(\rho)}$. Now $\hat{\rho}$ is not equal to $\tilde{\rho}$ (the transpose of ρ), which is generally not a self-conjugate representation, but they are in the same cominimal equivalence class (simple-current orbit),

$$\tilde{\rho} = \sigma^{r(\rho)/(2n+1)}(\hat{\rho}), \tag{6.11}$$

which we prove at the end of this section. Eq. (6.11) implies that [34,35]

$$\tilde{S}_{\tilde{\lambda}\tilde{\rho}} = \mathrm{e}^{-2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} \tilde{S}_{\tilde{\lambda}\hat{\rho}} \tag{6.12}$$

so that Eq. (6.10) becomes

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\hat{\rho}} \frac{\tilde{\psi}^*_{\tilde{\alpha}\pi(\hat{\rho})} \tilde{S}_{\tilde{\lambda}\hat{\rho}} \tilde{\psi}_{\tilde{\beta}\pi(\hat{\rho})}}{\tilde{S}_{0\hat{\rho}}} = \tilde{n}_{\tilde{\beta}\tilde{\lambda}}{}^{\tilde{\alpha}}, \tag{6.13}$$

proving the level-rank duality of the coefficients of the open-string partition function of ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. That is, if the spectrum of an $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ open string stretched between ω_c-untwisted D-branes α and β contains ${n_{\beta\lambda}}^{\alpha}$ copies of the highest-weight representation V_λ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, then the spectrum of an $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ open string stretched between ω_c-twisted D-branes $\tilde{\alpha}$ and $\tilde{\beta}$ contains an equal number of copies of the highest-weight representation $V_{\tilde{\lambda}}$ of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$.

Proof of Eq. (6.11). Let ρ, a self-conjugate representation of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, have Dynkin indices

$$\rho = (2k+1-2\ell_1, a_1, \ldots, a_n, a_n, \ldots, a_1) \tag{6.14}$$

where $\ell_1 = \sum_{i=1}^n a_i$. The Young tableau for ρ has $r(\rho) = (2n+1)\ell_1$ boxes. The representation $\pi(\rho)'$ of $\widehat{\mathrm{sp}}(n)_k$ that corresponds to ρ has Dynkin indices $(k-\ell_1, a_1, \ldots, a_n)$. Let the transpose representation $\widetilde{\pi(\rho)}'$ of $\widehat{\mathrm{sp}}(k)_n$ have Dynkin indices $(n-\tilde{\ell}_1, \tilde{a}_1, \ldots, \tilde{a}_k)$, with $\tilde{\ell}_1 = \sum_{i=1}^k \tilde{a}_i$. The representation $\hat{\rho}$ of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ that corresponds to $\widetilde{\pi(\rho)}'$ has Dynkin indices $(2n+1-2\tilde{\ell}_1, \tilde{a}_1, \ldots, \tilde{a}_k, \tilde{a}_k, \ldots, \tilde{a}_1)$. Finally, the representation $\sigma_1^{\ell}(\hat{\rho})$ has Dynkin indices

$$\sigma_1^{\ell}(\hat{\rho}) = (\tilde{a}_{\ell_1}, \tilde{a}_{\ell_1-1}, \ldots, \tilde{a}_1, 2n+1-2\tilde{\ell}_1, \tilde{a}_1, \ldots, \tilde{a}_{\ell_1}, 0, \ldots, 0) \tag{6.15}$$

where the last $2(k-\ell_1)$ entries vanish since $\tilde{a}_i = 0$ for $i > \ell_1$.

Since $\pi(\rho)'$ and $\widetilde{\pi(\rho)}'$ are transpose representations, with row lengths $\ell_i = \sum_{j=i}^n a_j$ and $\tilde{\ell}_i = \sum_{j=i}^k \tilde{a}_j$ respectively, their index sets, defined by [34,35]

$$I = \{\ell_i - i + n + 1 \mid 1 \leqslant i \leqslant n\}, \qquad \bar{I} = \{n + i - \tilde{\ell}_i \mid 1 \leqslant i \leqslant \ell_1\} \tag{6.16}$$

satisfy

$$I \cup \bar{I} = \{1, 2, \ldots, n+\ell_1\}, \qquad I \cap \bar{I} = 0, \tag{6.17}$$

where we have used $\tilde{\ell}_i = 0$ for $i > \ell_1$.

To prove that the Young tableau of $\sigma^{\ell_1}(\hat{\rho})$ is the transpose of ρ, we must show that the index sets [34,35]

$$\begin{aligned} J &= \{\lambda_i - i + 2n + 2 \mid 1 \leqslant i \leqslant 2n+1\}, \\ \bar{J} &= \{2n + 1 + i - \hat{\lambda}_i \mid 1 \leqslant i \leqslant 2k+1\} \end{aligned} \tag{6.18}$$

(where λ_i and $\hat{\lambda}_i$ are the row lengths of ρ and $\sigma^{\ell_1}(\hat{\rho})$ respectively, and $\lambda_{2n+1} = \hat{\lambda}_{2k+1} = 0$) satisfy

$$J \cup \bar{J} = \{1, 2, \ldots, 2n+2k+2\}, \qquad J \cap \bar{J} = 0. \tag{6.19}$$

Using Eqs. (6.14) and (6.15), one gets

$$\begin{aligned} J &= J_1 \cup J_2 \cup J_3, \\ \bar{J} &= \bar{J}_1 \cup \bar{J}_2 \cup \bar{J}_3, \\ J_1 &= \{\ell_1 + i - \ell_i \mid 1 \leqslant i \leqslant n\}, \\ \bar{J}_1 &= \{\tilde{\ell}_i - i + \ell_1 + 1 \mid 1 \leqslant i \leqslant \ell_1\}, \\ J_2 &= \{n + \ell_1 + 1\}, \\ \bar{J}_2 &= \{2n + \ell_1 + 1 + i - \tilde{\ell}_i \mid 1 \leqslant i \leqslant \ell_1\}, \end{aligned}$$

$$J_3 = \{2n + 2 + \ell_1 + \ell_i - i \mid 1 \leqslant i \leqslant n\},$$
$$\bar{J}_3 = \{2n + 2\ell_1 + 1 + i \mid 1 \leqslant i \leqslant 2k - 2\ell_1 + 1\}, \tag{6.20}$$

where ℓ_i and $\tilde{\ell}_i$ are the row lengths of the $\widehat{\mathrm{sp}}(n)_k$ and $\widehat{\mathrm{sp}}(k)_n$ representations $\pi(\rho)'$ and $\widetilde{\pi(\rho)}'$. Using Eq. (6.17), one observes that

$$J_1 \cup \bar{J}_1 = \{1, 2, \ldots, n + \ell_1\}, \qquad J_1 \cap \bar{J}_1 = 0,$$
$$J_2 = \{n + \ell_1 + 1\},$$
$$J_3 \cup \bar{J}_2 = \{n + \ell_1 + 2, \ldots, 2n + 2\ell_1 + 1\}, \qquad J_3 \cap \bar{J}_2 = 0,$$
$$\bar{J}_3 = \{2n + 2\ell_1 + 2, \ldots, 2n + 2k + 2\}, \tag{6.21}$$

which establishes Eq. (6.19). □

7. Conclusions

In this paper, we have continued our analysis, begun in Ref. [37], of level-rank duality in boundary WZW models. We examined the relation between the D0-brane charges of level-rank dual untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{sp}}(n)_k$, and of level-rank dual ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. We also demonstrated the level-rank duality of the spectrum of an open string stretched between untwisted or ω_c-twisted D-branes in each of these theories. The analysis of level-rank duality of ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$.

It is expected that level-rank duality will also be present in the boundary WZW models and D-branes of other level-rank dual groups. Also, the level-rank duality of bulk $\widehat{\mathrm{su}}(N)_K$ WZW models presumably has consequences for the twisted D-branes of boundary $\widehat{\mathrm{su}}(N)_K$ models even when N and K are not odd. The level-rank map between the twisted D-branes in these cases is expected, however, to be more complicated than for $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. We leave this to future work.

Further, it would be interesting to derive the level-rank dualities described in this paper directly from K-theory.

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