10-30-2006

Twisted D-branes of the over(su,ˆ) (N)K WZW model and level-rank duality

Stephen G. Naculich
Bowdoin College

Howard J. Schnitzer
Brandeis University

Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 755 (2006) 164–185
NUCLEAR PHYSICS B

Twisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model and level-rank duality

Stephen G. Naculich [a,*,1], Howard J. Schnitzer [b,2]

[a] *Department of Physics, Bowdoin College, Brunswick, ME 04011, USA*
[b] *Theoretical Physics Group, Martin Fisher School of Physics, Brandeis University, Waltham, MA 02454, USA*

Received 6 July 2006; accepted 9 August 2006

Available online 30 August 2006

Abstract

We analyze the level-rank duality of ${}_c$-twisted D-branes of $\widehat{\mathrm{su}}(N)_K$ (when N and $K > 2$). When N or K is even, the duality map involves $\mathbb{Z}_2$-cominimal equivalence classes of twisted D-branes. We prove the duality of the spectrum of an open string stretched between ${}_c$-twisted D-branes, and ascertain the relation between the charges of level-rank-dual ${}_c$-twisted D-branes.

1. Introduction

Level-rank duality is a relationship between various quantities in bulk Wess–Zumino–Witten models with classical Lie groups [1–3]. It has recently been shown [4,5] that level-rank duality also applies to untwisted and to certain twisted D-branes in the corresponding boundary WZW models [6–31]. (For a review of D-branes on group manifolds, see Ref. [32].) In this paper, we extend this work to include all charge-conjugation-twisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model.

Untwisted (i.e., symmetry-preserving) D-branes of WZW models are labelled by the integrable highest-weight representations V_λ of the affine Lie algebra. For $\widehat{\mathrm{su}}(N)_K$, these representations belong to cominimal equivalence classes generated by the $\mathbb{Z}_N$ simple current of the WZW model, and therefore so do the untwisted D-branes of the model. Level-rank duality is

* Corresponding author.
E-mail addresses: naculich@bowdoin.edu (S.G. Naculich), schnitzr@brandeis.edu (H.J. Schnitzer).
1 Research supported in part by the NSF under grant PHY-0456944.
2 Research supported in part by the DOE under grant DE-FG02-92ER40706.

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2006.08.005

a one-to-one correspondence between cominimal equivalence classes (or simple-current orbits) of integrable representations of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$, and therefore induces a map between cominimal equivalence classes of untwisted D-branes.

The spectrum of an open string stretched between D-branes labelled by α and β is specified by the coefficients of the partition function

$$Z^{\text{open}}_{\alpha\beta}(\tau) = \sum_{\lambda\in P^K_+} n_{\beta\lambda}{}^{\alpha}\chi_\lambda(\tau) \tag{1.1}$$

where $\chi_\lambda(\tau)$ is the affine character of the integrable highest-weight representation V_λ. For untwisted D-branes, the coefficients $n_{\beta\lambda}{}^{\alpha}$ are equal to the fusion coefficients of the bulk WZW theory [33], so the well-known level-rank duality of the fusion rules [1–3] implies the duality of the open-string spectrum between untwisted branes.

Untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ possess a conserved D0-brane charge belonging to $\mathbb{Z}_{x_{N,K}}$:

$$Q_\lambda = (\dim\lambda)_{\mathrm{su}(N)} \bmod x_{N,K} \tag{1.2}$$

where [15,17]

$$x_{N,K} \equiv \frac{N+K}{\gcd\{N+K, \mathrm{lcm}\{1,\ldots,N-1\}\}}. \tag{1.3}$$

The charges of cominimally-equivalent untwisted D-branes are equal up to sign [17]

$$Q_{\sigma(\lambda)} = (-1)^{N-1} Q_\lambda \bmod x_{N,K} \tag{1.4}$$

where σ is the $\mathbb{Z}_N$ simple current of $\widehat{\mathrm{su}}(N)_K$. It was shown in Refs. [4,5] that the charges of level-rank-dual untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$ are related by

$$\tilde{Q}_{\tilde{\lambda}} = \begin{cases} (-1)^{r(\lambda)} Q_\lambda \bmod x & \text{for } N+K \text{ odd},\\ Q_\lambda \bmod x & \text{for } N+K \text{ even (except for } N=K=2^m), \end{cases} \tag{1.5}$$

where $r(\lambda)$ is the number of boxes in the Young tableau associated with the representation λ, where $\tilde{\lambda}$ is the level-rank-dual representation of $\widehat{\mathrm{su}}(K)_N$ associated with the transposed tableau, and $x = \min\{x_{N,K}, x_{K,N}\}$. For the remaining case, it was conjectured that

$$\tilde{Q}_{\tilde{\lambda}} = \left.\begin{matrix} (-1)^{r(\lambda)/N} Q_\lambda \bmod x & \text{when } N \mid r(\lambda) \\ Q_\lambda \bmod x & \text{when } N \quad r(\lambda) \end{matrix}\right\} \quad \text{for } N=K=2^m \tag{1.6}$$

on the basis of numerical evidence.

In addition to untwisted D-branes, most WZW models contain twisted D-branes, whose charges also belong to $\mathbb{Z}_{x_{N,K}}$ [23,34–36]. The coefficients $n_{\beta\lambda}{}^{\alpha}$ of the partition function (1.1) of an open string stretched between twisted D-branes α and β are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu\in} \frac{\psi^*_{\alpha\mu} S_{\lambda\mu} \psi_{\beta\mu}}{S_{0\mu}} \tag{1.7}$$

where $\psi_{\alpha\mu}$ is the modular-transformation matrix of the associated twisted affine Lie algebra.

One such class of D-branes for $\widehat{\mathrm{su}}(N)_K$ are those twisted by the charge-conjugation symmetry $_c$, which exist for all $N > 2$. This paper will analyze the level-rank duality of $_c$-twisted

D-branes of $\widehat{\mathrm{su}}(N)_K$ (for N and $K > 2$), and in particular, the relationship between the open-string partition function coefficients (1.7), and between the D-brane charges. (In Ref. [5], level-rank duality of $\ _c$-twisted D-branes was examined in the special case that N and K were both odd.)

As shown in Ref. [21], and reviewed in Sections 4 and 5, the $\ _c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ (respectively $\widehat{\mathrm{su}}(2n+1)_K$) are labelled by a subset of integrable highest-weight representations of $\widehat{\mathrm{so}}(2n+1)_{K+1}$ (respectively $\widehat{\mathrm{so}}(2n+1)_{K+2}$), or alternatively, by a subset of integrable highest-weight representations of $\widehat{\mathrm{sp}}(n)_{K+n-1}$ (respectively $\widehat{\mathrm{sp}}(n)_{K+n}$). In Section 4, we show that, like untwisted D-branes, $\ _c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ belong to cominimal equivalence classes, but now generated by the $\mathbb{Z}_2$ simple current of $\widehat{\mathrm{so}}(2n+1)_{K+1}$. As shown in Section 7, cominimally-equivalent $\ _c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ have equal and opposite charges $(\mathrm{mod}\, x_{2n,K})$.

In Section 6, we describe a one-to-one map $\alpha \quad \hat{\alpha}$ between the $\ _c$-twisted D-branes (or cominimal equivalence classes of branes) of $\widehat{\mathrm{su}}(N)_K$ and the $\ _c$-twisted D-branes (or cominimal equivalence classes of branes) of $\widehat{\mathrm{su}}(K)_N$. The exact form of the level-rank map depends on whether N and K are even or odd. We then show the equality of the open string partition function coefficients (1.7) for level-rank-dual $\ _c$-twisted D-branes. Because the level-rank map involves cominimal equivalence classes in the case of $\widehat{\mathrm{su}}(2n)_K$, the natural quantity to consider in that case is

$$s_{\beta\lambda}{}^{\alpha} = \left(\frac{1}{2}\right)^{\frac{1}{2}[t(\alpha)+t(\beta)]+1} \left[n_{\beta\lambda}{}^{\alpha} + n_{\beta\lambda}{}^{\sigma(\alpha)} + n_{\sigma(\beta)\lambda}{}^{\alpha} + n_{\sigma(\beta)\lambda}{}^{\sigma(\alpha)}\right] \tag{1.8}$$

where σ is the $\mathbb{Z}_2$ simple-current symmetry of $\widehat{\mathrm{so}}(2n+1)_{K+1}$, and $t(\alpha)$ is defined in Eq. (4.3).

In Section 7, we ascertain the relationship between the charges of level-rank-dual $\ _c$-twisted D-branes.

Sections 2 and 3 contain some necessary background material on twisted states in WZW models and on integrable representations of $\widehat{\mathrm{so}}(2n+1)_{K'}$, and concluding remarks comprise Section 8.

2. Twisted D-branes of WZW models

In this section, we review some aspects of twisted D-branes of WZW models and their relation to the twisted Cardy and twisted Ishibashi states of the closed-string sector, drawing on Refs. [7–9,19,21].

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted) affine Lie algebra $\hat{g}_K$ at level K. We only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}^{\mathrm{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \bar{V}_{\lambda^*} \tag{2.1}$$

where V and $\bar{V}$ represent left- and right-moving states respectively, and λ^* denotes the representation conjugate to λ. $V_\lambda \in P_+^K$ are integrable highest-weight representations of $\hat{g}_K$, whose highest weight λ has non-negative Dynkin indices $(a_0, a_1, \ldots, a_n)$ satisfying $\sum_{i=0}^{n} m_i a_i = K$ (where $n = \mathrm{rank}\, g$ and $(m_0, m_1, \ldots, m_n)$ are the dual Coxeter labels of $\hat{g}_K$).

D-branes of the WZW model may be studied algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We label the allowed boundary conditions (and therefore the D-branes) by $\alpha, \beta, \ldots$.

We consider boundary conditions on the currents of the affine Lie algebra of the form

$$\left[J^a(z) - \quad \bar{J}^a(\bar{z})\right]\Big|_{z=\bar{z}} = 0 \tag{2.2}$$

where   is an automorphism of the Lie algebra g. These boundary conditions leave unbroken the $\hat{g}_K$ symmetry, as well as the conformal symmetry, of the theory. Untwisted D-branes correspond to $= 1$. Open-closed string duality allows one to correlate the boundary conditions (2.2) of the boundary WZW model with coherent states $|B\rangle\rangle \; \in \mathcal{H}^{\text{closed}}$ of the bulk WZW model satisfying

$$\left[J^a_m + \quad \bar{J}^a_{-m}\right]|B\rangle\rangle \quad = 0, \qquad m \in \mathbb{Z} \tag{2.3}$$

where J^a_m are the modes of the affine Lie algebra generators.

Solutions of Eq. (2.3) that belong to a single sector $V_\mu \otimes \bar{V}_{\ (\mu)^*}$ of the bulk WZW theory are known as -twisted Ishibashi states $|\mu\rangle\rangle_I$. (Solutions corresponding to $= 1$ are the ordinary untwisted Ishibashi states [37].) Since we are considering the diagonal closed-string theory (2.1), these states only exist when $\mu = \ (\mu)$, so the -twisted Ishibashi states are labelled by $\mu \in \mathcal{E}$, where $\mathcal{E} \ \subset P_+^K$ are the integrable highest-weight representations of $\hat{g}_K$ that satisfy $(\mu) = \mu$. Equivalently, μ corresponds to an integrable highest-weight representation of $\breve{g}$, the orbit Lie algebra [38] associated with $\hat{g}_K$.

A coherent state $|B\rangle\rangle$ that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions [33]. Solutions of Eq. (2.3) that also satisfy the Cardy conditions are denoted -twisted Cardy states $|\alpha\rangle\rangle_C$, where the labels α take values in some set . The -twisted D-branes of $\hat{g}_K$ correspond to $|\alpha\rangle\rangle_C$ and are therefore also labelled by $\alpha \in$. These states correspond [9] to integrable highest-weight representations of the -twisted affine Lie algebra $\hat{g}_K$ (but see Ref. [24]).

The -twisted Cardy states may be expressed as linear combinations of -twisted Ishibashi states

$$|\alpha\rangle\rangle_C = \sum_{\mu \in} \frac{\psi_{\alpha\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\rangle_I \tag{2.4}$$

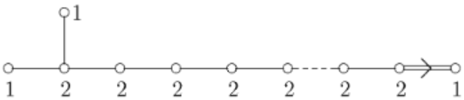
where $S_{\lambda\mu}$ is the modular transformation matrix of $\hat{g}_K$, 0 denotes the identity representation, and the coefficients $\psi_{\alpha\mu}$ may be identified [9] with the modular transformation matrices of characters of the twisted affine Lie algebra $\hat{g}_K$ [39], as may be seen, for example, by examining the partition function of an open string stretched between an -twisted and an untwisted D-brane [19,21]. Using arguments presented, e.g., in Ref. [21], the coefficients of the open string partition function (1.1) may be expressed as

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu \in} \frac{\psi^*_{\alpha\mu} S_{\lambda\mu} \psi_{\beta\mu}}{S_{0\mu}}. \tag{2.5}$$

3. Integrable representations of so $2n + 1)_K$

This section presents details about integrable highest-weight representations of $\widehat{\text{so}}(2n+1)_{K'}$ that will be needed for the discussion of $_c$-twisted states of the $\widehat{\text{su}}(N)_K$ WZW model.

Integrable representations of $\widehat{\mathrm{so}}(2n+1)_{K'}$ have Dynkin indices $(a_0, a_1, \ldots, a_n)$ that satisfy $\sum_{i=0}^{n} m_i a_i = K'$, where m_i are the dual Coxeter labels of the extended Dynkin diagram for so$(2n+1)$



(with the dual Coxeter labels shown adjacent to each node), that is,[3]

$$a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n = K'. \tag{3.1}$$

An even or odd value of a_n corresponds to a tensor or spinor representation respectively. With each tensor representation of so$(2n+1)$ may be associated a Young tableau whose row lengths ℓ_i are given by

$$\ell_i = \begin{cases} \frac{1}{2}a_n + \sum_{j=i}^{n-1} a_j & \text{for } 1 \quad i \quad n-1, \\ \frac{1}{2}a_n & \text{for } i = n, \end{cases} \tag{3.2}$$

with total number of boxes $r = \sum_{i=1}^{n} \ell_i$. We also formally use Eq. (3.2) to define row lengths for a spinor representation. These row lengths are all half-integers, and correspond to a "Young tableau" with a column of "half-boxes." The integrability condition (3.1) corresponds to the constraint $\ell_1 + \ell_2 \quad K'$ on the row lengths of the tableau.

The extended Dynkin diagram of so$(2n+1)$ has a $\mathbb{Z}_2$ symmetry that interchanges the 0th and 1st nodes. This symmetry induces a simple-current symmetry (denoted by σ) of the $\widehat{\mathrm{so}}(2n+1)_{K'}$ WZW model that pairs integrable representations related by $a_0 \leftrightarrow a_1$, with the other Dynkin indices unchanged. Their respective Young tableaux are related by $\ell_1 \quad K' - \ell_1$. Under σ, tensor representations are mapped to tensors, and spinor representations to spinors, and the modular transformation matrix S' of $\widehat{\mathrm{so}}(2n+1)_{K'}$ obeys [3]

$$S'_{\sigma(\alpha')\mu'} = \pm S'_{\alpha'\mu'} \quad \text{for } \mu' \text{ a } \left.\begin{matrix} \text{tensor} \\ \text{spinor} \end{matrix}\right\} \text{representation.} \tag{3.3}$$

Representations related by $\sigma \in \mathbb{Z}_2$ belong to a simple-current orbit, or cominimal equivalence class.

In this paper, we will refer to representations of $\widehat{\mathrm{so}}(2n+1)_{K'}$ with $\ell_1 < \frac{1}{2}K'$, $\ell_1 = \frac{1}{2}K'$, and $\ell_1 > \frac{1}{2}K'$ as being of types I, II, and III, respectively. Type II representations are cominimally self-equivalent, and are tensors (respectively spinors) when K' is even (respectively odd). Each simple-current orbit of $\widehat{\mathrm{so}}(2n+1)_{K'}$ contains either a type I and type III representation, or a single type II representation.

[3] *Note*: throughout this paper, by $\widehat{\mathrm{so}}(3)_{K'}$ we mean the affine Lie algebra $\widehat{\mathrm{su}}(2)_{2K'}$. Its integrable representations have so(3) Young tableaux that obey $\ell_1 \quad K'$. Since $\ell_1 = \frac{1}{2}a_1$, this means that Eq. (3.1) is replaced with $a_0 + a_1 = 2K'$ when $n = 1$.

4. Twisted states of the su $2n)_K$ model

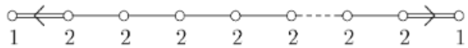
The invariance under reflection of the Dynkin diagram of the finite Lie algebra su(N) gives rise (when $N > 2$) to an order-two automorphism $_c$ of the Lie algebra, under which the Dynkin indices a_i $(i = 1, \ldots, N-1)$ of an irreducible representation are mapped to a_{N-i}, corresponding to charge conjugation. This automorphism lifts to an automorphism of the affine Lie algebra $\widehat{\mathrm{su}}(N)_K$ that leaves the zeroth node of the extended Dynkin diagram invariant. It gives rise (for $N > 2$) to a set of $_c$-twisted Ishibashi states and $_c$-twisted Cardy states of the bulk $\widehat{\mathrm{su}}(N)_K$ WZW model, and a corresponding class of $_c$-twisted D-branes of the boundary model. In this section and the next, we review these twisted states for $\widehat{\mathrm{su}}(2n)_K$ and $\widehat{\mathrm{su}}(2n+1)_K$, respectively. Much of this material is a summary of Ref. [21].

Twisted Ishibashi states of $\widehat{\mathrm{su}}(2n)_K$

Recall from Section 2 that the $_c$-twisted Ishibashi states $|\mu\rangle\rangle_I{}^c$ of the $\widehat{\mathrm{su}}(2n)_K$ WZW model ($n > 1$ is understood throughout this section) are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}\ ^c$ of $\widehat{\mathrm{su}}(2n)_K = (A^{(1)}_{2n-1})_K$. These representations have Dynkin indices $(\mu_0, \mu_1, \ldots, \mu_{n-1}, \mu_n, \mu_{n-1}, \ldots, \mu_1)$ that satisfy

$$\mu_0 + 2(\mu_1 + \cdots + \mu_{n-1}) + \mu_n = K. \tag{4.1}$$

Equivalently, the $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n)_K$ may be characterized [38] by the integrable highest weight representations of the associated orbit Lie algebra $\breve{g} = (D^{(2)}_{n+1})_K$, whose Dynkin diagram is

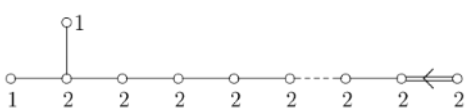


with the integers adjacent to each node indicating the dual Coxeter label m_i. The representation $\mu \in \mathcal{E}\ ^c$ corresponds to the $(D^{(2)}_{n+1})_K$ representation with Dynkin indices $(\mu_0, \mu_1, \ldots, \mu_n)$, whose integrability condition is precisely (4.1).

Each $_c$-twisted Ishibashi state μ of $\widehat{\mathrm{su}}(2n)_K$ may be mapped [21] to an integrable highest-weight representation μ' of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n+1)_{K+1}$ with Dynkin indices $(\mu_0 + \mu_1 + 1, \mu_1, \ldots, \mu_n)$. The constraint (4.1) translates into the constraint $\ell_1(\mu') \quad \frac{1}{2}K$ on the $\widehat{\mathrm{so}}(2n+1)_{K+1}$ Young tableaux. This means that $_c$*-twisted Ishibashi states of* $\widehat{\mathrm{su}}(2n)_K$ *are in one-to-one correspondence with the set of type I tensor and type I spinor representations of* $\widehat{\mathrm{so}}(2n+1)_{K+1}$.

Twisted Cardy states of $\widehat{\mathrm{su}}(2n)_K$

Recall that the $_c$-twisted Cardy states $|\alpha\rangle\rangle_C{}^c$ (and therefore the $_c$-twisted D-branes) of the $\widehat{\mathrm{su}}(2n)_K$ WZW model are labelled [9] by the integrable highest-weight representations $\alpha \in \quad ^c$ of the twisted affine Lie algebra $\hat{g}_K = (A^{(2)}_{2n-1})_K$, whose Dynkin diagram is



The Dynkin indices $(a_0, a_1, \ldots, a_n)$ of the highest weights α thus satisfy

$$a_0 + a_1 + 2(a_2 + \cdots + a_n) = K. \tag{4.2}$$

(For $n = 2$, the twisted affine Lie algebra is instead $D_3^{(2)}$ with nodes 1 and 2 interchanged [21], but the condition (4.2) remains valid.)

The $_c$-twisted Cardy state $\alpha \in$ ^{c} of $\widehat{\text{su}}(2n)_K$ may be associated [21] with an integrable highest-weight *spinor* representation α' of the untwisted affine Lie algebra $\widehat{\text{so}}(2n+1)_{K+1}$ with Dynkin indices $(a_0, a_1, \ldots, a_{n-1}, 2a_n + 1)$. The constraint (4.2) is precisely the condition on integrable representations of $\widehat{\text{so}}(2n+1)_{K+1}$. (In terms of so$(2n+1)$ Young tableaux row lengths, this constraint reads $\ell_1(\alpha') + \ell_2(\alpha')$ $K + 1$.) Therefore, *there is a one-to-one correspondence between the* $_c$*-twisted D-branes of* $\widehat{\text{su}}(2n)_K$ *and integrable spinor representations of* $\widehat{\text{so}}(2n+1)_{K+1}$ *of type I, type II (when K is even), and type III*. For later convenience, we define

$$t(\alpha) = \begin{array}{lll} 0, & \text{if } \ell_1(\alpha') \neq \frac{1}{2}(K+1) & \text{(types I and III)}, \\ 1, & \text{if } \ell_1(\alpha') = \frac{1}{2}(K+1) & \text{(type II)}. \end{array} \tag{4.3}$$

Even though the $_c$-twisted Cardy states and the $_c$-twisted Ishibashi states of $\widehat{\text{su}}(2n)_K$ are characterized differently in terms of integrable representations of $\widehat{\text{so}}(2n+1)_{K+1}$, they are equal in number. The $_c$-twisted Cardy states α may be written as linear combinations of $_c$-twisted Ishibashi states μ, with the transformation coefficients $\psi_{\alpha\mu}$ given by the modular transformation matrix of $(A_{2n-1}^{(2)})_K$. In Ref. [21], it was shown that, for $\widehat{\text{su}}(2n)_K$, these coefficients are proportional to matrix elements of the (real) modular transformation matrix S' of the untwisted affine Lie algebra $\widehat{\text{so}}(2n+1)_{K+1}$:

$$\psi_{\alpha\mu} = \sqrt{2} S'_{\alpha'\mu'} = \sqrt{2} S'^{*}_{\alpha'\mu'} \tag{4.4}$$

where α' and μ' are the $\widehat{\text{so}}(2n+1)_{K+1}$ representations related to α and μ as described above.

Since the finite Lie algebra associated with the twisted affine Lie algebra $(A_{2n-1}^{(2)})_K$ is C_n, the representations of $(A_{2n-1}^{(2)})_K$ form C_n-multiplets at each level. More specifically [21], each $_c$-twisted Cardy state $\alpha \in$ ^{c} of $\widehat{\text{su}}(2n)_K$ may be associated with an integrable highest-weight representation α'' of the untwisted affine Lie algebra[4] $\widehat{\text{sp}}(n)_{K+n-1}$ with (finite) Dynkin indices $(a_1, \ldots, a_n)$. The row lengths of the sp(n) Young tableau associated with α'' are equal to those of the so$(2n+1)$ Young tableau associated with α' reduced by one-half: $\ell_i(\alpha'') = \ell_i(\alpha') - \frac{1}{2}$. Therefore, an alternative characterization of the $_c$-twisted D-branes of $\widehat{\text{su}}(2n)_K$ is as the subset of integrable representations of $\widehat{\text{sp}}(n)_{K+n-1}$ characterized by Young tableaux with row lengths satisfying $\ell_1(\alpha'') + \ell_2(\alpha'')$ K.

Equivalence classes of $_c$*-twisted D-branes of* $\widehat{\text{su}}(2n)_K$

The $\mathbb{Z}_2$ simple current symmetry σ of $\widehat{\text{so}}(2n+1)_{K+1}$ relates type I and type III representations in pairs. Using the 1–1 correspondence between integrable $\widehat{\text{so}}(2n+1)_{K+1}$ spinor representations and $_c$-twisted Cardy states, we lift the map σ to the twisted D-branes of $\widehat{\text{su}}(2n)_K$, and refer to $\sigma(\alpha)$ as cominimally equivalent to α. (In Section 7, we will show that α and $\sigma(\alpha)$ have equal and opposite D0-brane charges, modulo $x_{2n,K}$.) Therefore, *the cominimal equivalence classes of* $_c$*-twisted D-branes of* $\widehat{\text{su}}(2n)_K$ *are in one-to-one correspondence with the set of*

[4] Throughout this paper, our convention is $\text{sp}(n) = C_n$.

type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{K+1}$ *when* K *is odd, and with type I and type II spinor representations of* $\widehat{\mathrm{so}}(2n+1)_{K+1}$ *when* K *is even.*

Twisted open string partition function of $\widehat{\mathrm{su}}(2n)_K$

The coefficients of the partition function of an open string stretched between $_c$-twisted D-branes α and β of $\widehat{\mathrm{su}}(2n)_K$ are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\{\substack{\text{tensors I}\\ \text{spinors I}}} \frac{2S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} \tag{4.5}$$

using Eqs. (2.5) and (4.4). Since the $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ belong to $\mathbb{Z}_2$-cominimal equivalence classes, we also define the linear combination

$$\begin{aligned} s_{\beta\lambda}{}^{\alpha} &= \left(\frac{1}{2}\right)^{\frac{1}{2}[t(\alpha)+t(\beta)]+1} \left[n_{\beta\lambda}{}^{\alpha} + n_{\beta\lambda}{}^{\sigma(\alpha)} + n_{\sigma(\beta)\lambda}{}^{\alpha} + n_{\sigma(\beta)\lambda}{}^{\sigma(\alpha)}\right] \\ &= \left(\frac{1}{2}\right)^{\frac{1}{2}[t(\alpha)+t(\beta)]-2} \sum_{\mu'=\text{tensors I}} \frac{S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} \end{aligned} \tag{4.6}$$

where, as a result of Eq. (3.3), the sum over spinor representations drops out. (The normalization is chosen so that $s_{\beta\lambda}{}^{\alpha} = n_{\beta\lambda}{}^{\alpha}$ when α and β are both type II, and therefore belong to single-element cominimal equivalence classes.) The quantity $s_{\beta\lambda}{}^{\alpha}$ is the more natural one to consider in the context of level-rank duality.

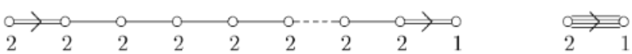
5. Twisted states of the su $2n+1)_K$ model

Twisted Ishibashi states of $\widehat{\mathrm{su}}(2n+1)_K$

Recall from Section 2 that the $_c$-twisted Ishibashi states $|\mu\rangle\rangle_I{}^{c}$ of the $\widehat{\mathrm{su}}(2n+1)_K$ WZW model are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}^{\,c}$ of $\widehat{\mathrm{su}}(2n+1)_K = (A_{2n}^{(1)})_K$. The Dynkin indices $(\mu_0, \mu_1, \ldots, \mu_{n-1}, \mu_n, \mu_n, \mu_{n-1}, \ldots, \mu_1)$ of these representations satisfy

$$\mu_0 + 2(\mu_1 + \cdots + \mu_n) = K. \tag{5.1}$$

Equivalently, the $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n+1)_K$ may be characterized [38] by the integrable highest weight representations of the associated orbit Lie algebra $\breve{g} = (A_{2n}^{(2)})_K$, whose Dynkin diagram is (the right-hand diagram is for $n = 1$)



The representation $\mu \in \mathcal{E}^{\,c}$ corresponds to the $(A_{2n}^{(2)})_K$ representation with Dynkin indices $(\mu_0, \mu_1, \ldots, \mu_n)$. Consistency with Eq. (5.1) requires that the dual Coxeter labels be $(m_0, m_1, \ldots, m_n) = (1, 2, 2, \ldots, 2)$, and hence we must choose as the zeroth node the right-most node of the Dynkin diagrams above (consistent with Ref. [21], but differing from Refs. [40,41]).

Each $_c$-twisted Ishibashi state μ of $\widehat{\mathrm{su}}(2n+1)_K$ may be mapped [21] to an integrable highest-weight *spinor* representation μ' of the untwisted affine Lie algebra[5] $\widehat{\mathrm{so}}(2n+1)_{K+2}$ with Dynkin indices[6] $(\mu_0+\mu_1+1, \mu_1, \ldots, \mu_{n-1}, 2\mu_n+1)$. The constraint (5.1) translates into the constraint $\ell_1(\mu') \quad \frac{1}{2}(K+1)$ on the $\widehat{\mathrm{so}}(2n+1)_{K+2}$ Young tableau. This means that *$_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n+1)_K$ are in one-to-one correspondence with the set of type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{K+2}$*.

Twisted Cardy states of $\widehat{\mathrm{su}}(2n+1)_K$

Recall that the $_c$-twisted Cardy states $|\alpha\rangle\rangle_C{}^c$ (and therefore the $_c$-twisted D-branes) of the $\widehat{\mathrm{su}}(2n+1)_K$ WZW model are labelled [9] by the integrable highest-weight representations $\alpha \in \ ^c$ of the twisted affine Lie algebra $\hat{g}_K = (A_{2n}^{(2)})_K$ (but see Ref. [24]). We adopt the same convention as above for the labelling of the nodes of the Dynkin diagram of $(A_{2n}^{(2)})_K$. Thus the Dynkin indices $(a_0, a_1, \ldots, a_n)$ of the highest weights α must satisfy

$$a_0 + 2(a_1 + \cdots + a_n) = K. \tag{5.2}$$

The $_c$-twisted Cardy state $\alpha \in \ ^c$ of $\widehat{\mathrm{su}}(2n+1)_K$ may be associated [21] with an integrable highest-weight *spinor* representation α' of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n+1)_{K+2}$ with Dynkin indices[7] $(a_0+a_1+1, a_1, \ldots, a_{n-1}, 2a_n+1)$. The constraint (5.2) translates into the constraint $\ell_1(\alpha') \quad \frac{1}{2}(K+1)$ on the $\widehat{\mathrm{so}}(2n+1)_{K+2}$ Young tableaux. This means that *$_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_K$ are in one-to-one correspondence with the set of type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{K+2}$*.

Since $_c$-twisted Cardy states of $\widehat{\mathrm{su}}(2n+1)_K$ correspond only to type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{K+2}$, there is no notion of cominimal equivalence of $_c$-twisted Cardy states in this case.

In the case of $\widehat{\mathrm{su}}(2n+1)_K$, the total number of $_c$-twisted Cardy states is manifestly equal to the total number of $_c$-twisted Ishibashi states. The coefficients $\psi_{\alpha\mu}$ relating $_c$-twisted Cardy states α to $_c$-twisted Ishibashi states μ are given by the modular transformation matrix of $(A_{2n}^{(2)})_K$. In Ref. [21], it was shown that, for $\widehat{\mathrm{su}}(2n+1)_K$, these coefficients are proportional to matrix elements of the modular transformation matrix S' of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n+1)_{K+2}$:

$$\psi_{\alpha\mu} = 2S'_{\alpha'\mu'} \tag{5.3}$$

where α' and μ' are the $\widehat{\mathrm{so}}(2n+1)_{K+2}$ representations related to α and μ as described above.

Since the finite Lie algebra associated with the twisted affine Lie algebra $(A_{2n}^{(2)})_K$ is C_n, the representations of $(A_{2n}^{(2)})_K$ form C_n-multiplets at each level. More specifically [21], each $_c$-twisted Cardy state $\alpha \in \ ^c$ of $\widehat{\mathrm{su}}(2n+1)_K$ may be associated with an integrable highest-weight representation α'' of the untwisted affine Lie algebra $\widehat{\mathrm{sp}}(n)_{K+n}$ with (finite) Dynkin indices $(a_1, \ldots, a_n)$. The row lengths of the sp(n) Young tableau associated with α'' are equal to those of the so$(2n+1)$ Young tableau associated with α' reduced by one-half: $\ell_i(\alpha'') = \ell_i(\alpha') - \frac{1}{2}$.

[5] See the note regarding $\widehat{\mathrm{so}}(3)_{K'}$ in footnote 3.
[6] For $n = 1$, μ' has Dynkin indices $(2\mu_0 + 2\mu_1 + 3, 2\mu_1 + 1)$.
[7] For $n = 1$, α' has Dynkin indices $(2a_0 + 2a_1 + 3, 2a_1 + 1)$.

Therefore, an alternative characterization of the $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_K$ is as the subset of integrable representations of $\widehat{\mathrm{sp}}(n)_{K+n}$ characterized by Young tableaux with row lengths satisfying $\ell_1(\alpha') \quad \frac{1}{2}K$.

Twisted open string partition function of $\widehat{\mathrm{su}}(2n+1)_K$

The coefficients of the partition function of an open string stretched between $_c$-twisted D-branes α and β of $\widehat{\mathrm{su}}(2n+1)_K$ are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'}S_{\lambda\mu}S'_{\beta'\mu'}}{S_{0\mu}} \tag{5.4}$$

using Eqs. (2.5) and (5.3).

Special case of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$

Note that in the special case of odd level, the $_c$-twisted Cardy states α and $_c$-twisted Ishibashi states μ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ are in one-to-one correspondence with the integrable representations α'' and μ'' of $\widehat{\mathrm{sp}}(n)_k$ with finite Dynkin indices $(a_1,\ldots,a_n)$ and $(\mu_1,\ldots,\mu_n)$ respectively. Moreover, it was observed [20,21,38] in this case that the Cardy/Ishibashi coefficients may be expressed as

$$\psi_{\alpha\mu} = S''_{\alpha''\mu''} \tag{5.5}$$

where $S''_{\alpha''\mu''}$ are elements of the modular transformation matrix of $\widehat{\mathrm{sp}}(n)_k$.

6. Level-rank duality of the twisted D-branes of su $N)_K$

This section is the heart of the paper, in which we present the level-rank map between the $_c$-twisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$. We use this to show the level-rank duality of the spectrum of an open string stretched between $_c$-twisted D-branes.

As in the case of untwisted D-branes, the level-rank correspondence involves cominimal equivalence classes (unless N and K are both odd). The details of the correspondence differ markedly depending on whether N and K are even or odd, so we must treat three cases separately. In Refs. [4,5], the tilde (˜) notation was used to denote the level-rank dual of an untwisted state, because the duality map was given by transposition of the associated Young tableaux. Here, in all cases, we will use the hat (ˆ) notation to denote the level-rank dual of an $_c$-twisted state, but the specific form of the duality map depends on whether N and K are even or odd, and on whether we are considering $_c$-twisted Cardy or $_c$-twisted Ishibashi states.

Duality of twisted states of $\widehat{\mathrm{su}}(2n)_{2k} \leftarrow \quad \widehat{\mathrm{su}}(2k)_{2n}$

As we saw in Section 4, the cominimal equivalence classes of $_c$-twisted Cardy states (and therefore of $_c$-twisted D-branes α) of $\widehat{\mathrm{su}}(2n)_{2k}$ correspond to type I and type II spinor representations α' of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$. The number of equivalence classes of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_{2k}$ is equal to the number of equivalence classes of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2k)_{2n}$, and there is a natural map $\alpha \quad \hat{\alpha}$ between them (when $n, k > 1$). This map is defined in terms of the map $\alpha' \quad \hat{\alpha}'$ between the corresponding spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+1}$, as follows:

- reduce each of the row lengths of α' by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- take the complement with respect to a $k \times n$ rectangle,
- add $\frac{1}{2}$ to each of the row lengths.

(The map $\alpha'' \quad \hat{\alpha}''$ between the corresponding representations of $\widehat{\mathrm{sp}}(n)_{2k+n-1}$ and $\widehat{\mathrm{sp}}(k)_{2n+k-1}$ is given by the middle two steps above.) The map $\alpha' \quad \hat{\alpha}'$ was first described in the appendix of Ref. [3] in the context of level-rank duality of $\widehat{\mathrm{so}}(N)_K$ WZW models. It takes type I (respectively type II) spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ to type II (respectively type I) spinor representations of $\widehat{\mathrm{so}}(2k+1)_{2n+1}$. Hence,

$$t(\alpha) + \tilde{t}(\hat{\alpha}) = 1 \tag{6.1}$$

for all $_c$-twisted Cardy states α of $\widehat{\mathrm{su}}(2n)_{2k}$, where $t(\alpha)$ is defined in Eq. (4.3), and $\tilde{t}(\hat{\alpha})$ is the corresponding quantity in $\widehat{\mathrm{su}}(2k)_{2n}$. The map $\alpha' \quad \hat{\alpha}'$ lifts to a one-to-one map $\alpha \quad \hat{\alpha}$ between cominimal equivalence classes of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_{2k}$ and cominimal equivalence classes of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2k)_{2n}$.

Next, we turn to the level-rank map for $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2k)_{2n}$. As we saw in Section 4, $_c$-twisted Ishibashi states μ of $\widehat{\mathrm{su}}(2n)_{2k}$ correspond to type I tensor and type I spinor representations μ' of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$. The level-rank map $\mu \quad \hat{\mu}$ between $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n)_{2k}$ and those of $\widehat{\mathrm{su}}(2k)_{2n}$ is defined *only* for states that correspond to type I tensor representations. The map between μ' and $\hat{\mu}'$, the corresponding $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+1}$ representations, is simply given by transposition of the tensor tableaux; that is, $\hat{\mu}' = (\tilde{\mu}')$. There is no level-rank map between $_c$-twisted Ishibashi states that correspond to type I spinor representations, for the simple reason that these sets of representations are not equal in number. (Moreover, the map described above for $_c$-twisted Cardy states maps type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ to type II spinor representations of $\widehat{\mathrm{so}}(2k+1)_{2n+1}$, which do not correspond to $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2k)_{2n}$.)

Having defined the level-rank map between μ and $\hat{\mu}$ in terms of the corresponding tensor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$, one may show that

$$\hat{\mu} = \sigma^{-r(\mu)/(2n)}(\tilde{\mu}) \tag{6.2}$$

that is, $\hat{\mu}$ is in the same $\widehat{\mathrm{su}}(2n)_{2k}$ cominimal equivalence class (simple-current orbit) as $\tilde{\mu}$, where $\tilde{\mu}$ is the transpose[8] of the Young tableau of the self-conjugate representation μ of $\widehat{\mathrm{su}}(2n)_{2k}$, and $r(\mu)$ is the number of boxes of this $\widehat{\mathrm{su}}(2n)_{2k}$ tableau. (Note that $\tilde{\mu}$ is, in general, not self-conjugate, while $\hat{\mu}$ necessarily is.) The proof of Eq. (6.2) is very similar to one given in section 6 of Ref. [5]. A consequence of Eq. (6.2) is that the $\widehat{\mathrm{su}}(2n)_{2k}$ modular transformation matrix S is related to the $\widehat{\mathrm{su}}(2k)_{2n}$ modular transformation matrix $\tilde{S}$ by

$$S^*_{\lambda\mu} = \sqrt{\frac{k}{n}}\,\tilde{S}_{\tilde{\lambda}\hat{\mu}} \tag{6.3}$$

[8] If μ has $\ell_1 = 2k$, $\tilde{\mu}$ is obtained by stripping off leading columns of length $2k$ from the transpose of μ.

which follows from [2,3]

$$S^*_{\lambda\mu} = \sqrt{\frac{k}{n}} \mathrm{e}^{2\pi i r(\lambda) r(\mu)/(4nk)} \tilde{S}_{\tilde{\lambda}\tilde{\mu}},$$
$$\tilde{S}_{\tilde{\lambda}\tilde{\mu}} = \mathrm{e}^{-2\pi i r(\lambda) r(\mu)/(4nk)} \tilde{S}_{\tilde{\lambda}\hat{\mu}}. \tag{6.4}$$

Having defined level-rank maps for the $_c$-twisted Cardy and Ishibashi states of $\widehat{\mathrm{su}}(2n)_{2k}$, we now turn to the duality of the open-string spectrum between $_c$-twisted D-branes. The coefficients of the partition function of an open string stretched between $_c$-twisted D-branes α and β are real numbers so we may write (4.6) as

$$s_{\beta\lambda}{}^{\alpha} = \left(\frac{1}{2}\right)^{\frac{1}{2}[t(\alpha)+t(\beta)]-2} \sum_{\mu'=\text{tensors I}} \frac{S'_{\alpha'\mu'} S^*_{\lambda\mu} S'_{\beta'\mu'}}{S^*_{0\mu}}. \tag{6.5}$$

In Ref. [3], the spinor–tensor components $S'_{\alpha'\mu'}$ of the modular transformation matrix of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ were shown to be related to the spinor-tensor components $\tilde{S}'_{\hat{\alpha}'\hat{\mu}'}$ of $\widehat{\mathrm{so}}(2k+1)_{2n+1}$ by

$$S'_{\alpha'\mu'} = 2^{t(\alpha)-\frac{1}{2}} (-1)^{r(\mu')} \tilde{S}'_{\hat{\alpha}'\hat{\mu}'} = 2^{\frac{1}{2}[t(\alpha)-\tilde{t}(\hat{\alpha})]} (-1)^{r(\mu')} \tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tag{6.6}$$

where we have used Eq. (6.1). Using Eqs. (6.3) and (6.6), we find

$$s_{\beta\lambda}{}^{\alpha} = \left(\frac{1}{2}\right)^{\frac{1}{2}[\tilde{t}(\hat{\alpha})+\tilde{t}(\hat{\beta})]-2} \sum_{\hat{\mu}'=\text{tensors I}} \frac{\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tilde{S}_{\tilde{\lambda}\hat{\mu}} \tilde{S}'_{\hat{\beta}'\hat{\mu}'}}{\tilde{S}_{0\hat{\mu}}} = \tilde{s}_{\hat{\beta}\tilde{\lambda}}{}^{\hat{\alpha}}. \tag{6.7}$$

Thus the (linear combination of) coefficients (4.6) of the open-string partition function of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_{2k}$ are equal to those of $\widehat{\mathrm{su}}(2k)_{2n}$ under the level-rank duality map acting on $_c$-twisted D-branes.

Duality of twisted states of $\widehat{\mathrm{su}}(2n+1)_{2k+1} \leftarrow \widehat{\mathrm{su}}(2k+1)_{2n+1}$

As we saw in Section 5, the $_c$-twisted Cardy states (and therefore the $_c$-twisted D-branes α) of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ map one-to-one to type I spinor integrable representations α' of $\widehat{\mathrm{so}}(2n+1)_{2k+3}$, and also to integrable representations α'' of $\widehat{\mathrm{sp}}(n)_k$. We define the level-rank duality map α $\hat{\alpha}$ for $_c$-twisted Cardy states by transposition of the associated $\widehat{\mathrm{sp}}(n)_k$ tableaux: that is, $\hat{\alpha}'' = (\tilde{\alpha}'')$. (In Ref. [5], we therefore denoted this map simply by α $\tilde{\alpha}$.) Exactly similar statements hold for the $_c$-twisted Ishibashi states μ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$.

The equality of the Cardy/Ishibashi coefficients of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{su}}(2k+1)_{2n+1}$

$$\psi_{\alpha\mu} = \tilde{\psi}_{\hat{\alpha}\hat{\mu}} \tag{6.8}$$

follows immediately from Eq. (5.5) together with level-rank duality of the $\widehat{\mathrm{sp}}(n)_k$ WZW model [3]

$$S''_{\alpha''\mu''} = \tilde{S}''_{\hat{\alpha}''\hat{\mu}''} \tag{6.9}$$

where S'' and $\tilde{S}''$ are the modular transformation matrices of $\widehat{\mathrm{sp}}(n)_k$ and $\widehat{\mathrm{sp}}(k)_n$ respectively. Moreover, by Eq. (5.3), we have

$$S'_{\alpha'\mu'} = \tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tag{6.10}$$

where S' and $\tilde{S}'$ are the modular transformation matrices of $\widehat{\mathrm{so}}(2n+1)_{2k+3}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+3}$ respectively, and the map $\alpha' \quad \hat{\alpha}'$ from $\widehat{\mathrm{so}}(2n+1)_{2k+3}$ to $\widehat{\mathrm{so}}(2k+1)_{2n+3}$ (induced from the transposition map $\alpha'' \quad \hat{\alpha}''$) is:

- reduce each of the row lengths of α' by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- add $\frac{1}{2}$ to each of the row lengths

and equivalently for $\mu' \quad \hat{\mu}'$. (Note that this map differs from spinor map defined in the last subsection by the omission of the complement map.) Note that Eq. (6.10) differs from the standard level-rank duality of WZW models [3], which relates $\widehat{\mathrm{so}}(N)_K$ to $\widehat{\mathrm{so}}(K)_N$.

Finally, we turn to the duality of the open-string spectrum between $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{su}}(2k+1)_{2n+1}$. In Ref. [5], $\widehat{\mathrm{sp}}(n)_k$ level-rank duality (6.9) was used to show the level-rank duality of the coefficients of the open string partition function. We can equivalently use Eqs. (5.4) and (6.10) to show the same result

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'}S^*_{\lambda\mu}S'_{\beta'\mu'}}{S^*_{0\mu}} = \sum_{\hat{\mu}'=\text{spinors I}} \frac{4\tilde{S}'_{\hat{\alpha}'\hat{\mu}'}\tilde{S}_{\tilde{\lambda}\hat{\mu}}\tilde{S}'_{\hat{\beta}'\hat{\mu}'}}{\tilde{S}_{0\hat{\mu}}} = \tilde{n}_{\hat{\beta}\tilde{\lambda}}{}^{\hat{\alpha}} \tag{6.11}$$

since $\mu' \quad \hat{\mu}'$ maps type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+3}$ to type I spinors of $\widehat{\mathrm{so}}(2k+1)_{2n+3}$, and we have also used

$$S^*_{\lambda\mu} = \sqrt{\frac{2k+1}{2n+1}}\,\tilde{S}_{\tilde{\lambda}\hat{\mu}} \tag{6.12}$$

which was proved in Ref. [5].

Duality of twisted states of $\widehat{\mathrm{su}}(2n+1)_{2k} \leftarrow \quad \widehat{\mathrm{su}}(2k)_{2n+1}$

Recall that the $_c$-twisted D-branes α of $\widehat{\mathrm{su}}(2n+1)_{2k}$ correspond to type I spinor representations α' of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$, and the equivalence classes of $_c$-twisted D-branes $\hat{\alpha}$ of $\widehat{\mathrm{su}}(2k)_{2n+1}$ correspond to type I spinor representations $\hat{\alpha}'$ of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$. The number of such spinor representations is equal, and we define the one-to-one level-rank map $\alpha' \quad \hat{\alpha}'$ from $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ to $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ (for $k > 1$) as follows:

- reduce each of the row lengths of α' by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- take the complement with respect to a $k \times n$ rectangle,
- add $\frac{1}{2}$ to each of the row lengths.

(By comparison, the definition of $\alpha' \quad \hat{\alpha}'$ from $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ to $\widehat{\mathrm{so}}(2k+1)_{2n+1}$ is the same, but in that case type I spinors are mapped to type II spinors and vice versa.) The map $\alpha' \quad \hat{\alpha}'$ lifts to a one-to-one map $\alpha \quad \hat{\alpha}$ between $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k}$ and *equivalence classes of* $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2k)_{2n+1}$. (The map $\alpha'' \quad \hat{\alpha}''$ between the corresponding representations of $\widehat{\mathrm{sp}}(n)_{2k+n}$ and $\widehat{\mathrm{sp}}(k)_{2n+k}$ is given by the middle two steps above.)

Next, we turn to the level-rank map between $_c$-twisted Ishibashi states. The $_c$-twisted Ishibashi states μ of $\widehat{\mathrm{su}}(2n+1)_{2k}$ correspond to type I spinor representations μ' of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$. The $_c$-twisted Ishibashi states $\hat{\mu}$ of $\widehat{\mathrm{su}}(2k)_{2n+1}$ correspond to type I tensor and type I spinor representations $\hat{\mu}'$ of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$. The number of such representations on each side is not equal, and the level-rank map $\mu' \quad \hat{\mu}'$ takes type I spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ to *only* the type I tensor representations of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$. (Just as for $\widehat{\mathrm{su}}(2k)_{2n}$, there is no level-rank correspondence for the spinor Ishibashi states of $\widehat{\mathrm{su}}(2k)_{2n+1}$.) The map $\mu' \quad \hat{\mu}'$ from $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ to $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ is defined as follows:

- reduce each of the row lengths of μ' by $\frac{1}{2}$, so that they all become integers, and
- transpose the resulting tableau.

The map $\mu' \quad \hat{\mu}'$ then lifts to a map $\mu \quad \hat{\mu}$ between $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n+1)_{2k}$ and a subset of $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2k)_{2n+1}$. One may show that

$$\hat{\mu} = \sigma^{-r(\mu)/(2n+1)}(\tilde{\mu}) \tag{6.13}$$

where $\tilde{\mu}$ is the transpose[9] of the Young tableau of the self-conjugate representation μ of $\widehat{\mathrm{su}}(2n+1)_{2k}$, and $r(\mu)$ is the number of boxes of this $\widehat{\mathrm{su}}(2n+1)_{2k}$ tableau. The proof of Eq. (6.13) is very similar to one given in Section 6 of Ref. [5]. Consequently, the modular transformation matrices S of $\widehat{\mathrm{su}}(2n+1)_{2k}$ and $\tilde{S}$ of $\widehat{\mathrm{su}}(2k)_{2n+1}$ are related by

$$S^*_{\lambda\mu} = \sqrt{\frac{2k}{2n+1}}\,\tilde{S}_{\tilde{\lambda}\hat{\mu}} \tag{6.14}$$

which follows from [2,3]

$$\begin{aligned} S^*_{\lambda\mu} &= \sqrt{\frac{2k}{2n+1}}\,\mathrm{e}^{2\pi i r(\lambda) r(\mu)/(2n+1)(2k)}\,\tilde{S}_{\tilde{\lambda}\tilde{\mu}}, \\ \tilde{S}_{\tilde{\lambda}\tilde{\mu}} &= \mathrm{e}^{-2\pi i r(\lambda) r(\mu)/(2n+1)(2k)}\,\tilde{S}_{\tilde{\lambda}\hat{\mu}}. \end{aligned} \tag{6.15}$$

Finally, in Appendix A of this paper, we show that

$$S'_{\alpha'\mu'} = (-1)^{r(\hat{\mu}')+k}\,\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tag{6.16}$$

where $S'_{\alpha'\mu'}$ and $\tilde{S}'_{\hat{\alpha}'\hat{\mu}'}$ are modular transformation matrices of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ respectively. As before, we observe that Eq. (6.16) is *not* the standard $\widehat{\mathrm{so}}(N)_K \leftrightarrow \widehat{\mathrm{so}}(K)_N$ duality of WZW models.

Eqs. (6.14) and (6.16) may be used to establish the level-rank duality of the coefficients of the partition functions (5.4) and (4.6) of an open string stretched between $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k}$ and $\widehat{\mathrm{su}}(2k)_{2n+1}$

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'}S^*_{\lambda\mu}S'_{\beta'\mu'}}{S^*_{0\mu}} = \sum_{\hat{\mu}'=\text{tensors I}} \frac{4\tilde{S}'_{\hat{\alpha}'\hat{\mu}'}\tilde{S}_{\tilde{\lambda}\hat{\mu}}\tilde{S}'_{\hat{\beta}'\hat{\mu}'}}{\tilde{S}_{0\hat{\mu}}} = \tilde{s}_{\hat{\beta}\tilde{\lambda}}{}^{\hat{\alpha}} \tag{6.17}$$

where the last equality follows because $\hat{\alpha}$ and $\hat{\beta}$ are both type I spinor representations of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$, so that $\tilde{t}(\hat{\alpha}) = \tilde{t}(\hat{\beta}) = 0$.

---

9 If μ has $\ell_1 = 2k$, $\tilde{\mu}$ is obtained by stripping off leading columns of length $2k$ from the transpose of μ.

7. Level-rank duality of twisted D-brane charges

In this section, we ascertain the relationship between the charges of level-rank-dual $_c$-twisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$. Recall from Ref. [22] that the D0-brane charge of the $_c$-twisted D-brane of $\widehat{\mathrm{su}}(N)_K$ labelled by α is given by

$$Q_{\alpha}{}^{c} = (\dim \alpha'')_{\mathrm{sp}(n)} \bmod x_{N,K} \quad \text{for } \widehat{\mathrm{su}}(N)_K \tag{7.1}$$

where α'' is the sp(n) representation corresponding to the $_c$-twisted Cardy state α of $\widehat{\mathrm{su}}(2n)_K$ or $\widehat{\mathrm{su}}(2n+1)_K$, as described in Sections 4 and 5.

Since the charges of $\widehat{\mathrm{su}}(N)_K$ D-branes (both untwisted and twisted) are defined only modulo $x_{N,K}$, and those of $\widehat{\mathrm{su}}(K)_N$ D-branes modulo $x_{K,N}$, comparison of charges of level-rank-dual D-branes is only possible modulo $x \equiv \gcd\{x_{N,K}, x_{K,N}\} = \min\{x_{N,K}, x_{K,N}\}$. In Refs. [4,5], the charges of untwisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ model and those of the level-rank-dual $\widehat{\mathrm{su}}(K)_N$ model were shown to be equal modulo x, up to a (known) sign (1.5), (1.6). In Ref. [5], the charges of $_c$-twisted D-branes of the $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ model and those of the level-rank-dual $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ model were also shown to be equal, modulo x. As we will see below, the relationship between charges of level-rank-dual $_c$-twisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$ is more complicated when N and K are not both odd.

Charges of cominimally-equivalent twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$

Since level-rank duality is a correspondence between $\mathbb{Z}_2$-cominimal equivalence classes of $_c$-twisted D-branes when either N or K is even, we must first demonstrate that cominimally-equivalent $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ have the same charge (modulo sign and modulo $x_{2n,K}$). The sp(n) representation α'' is related to the so$(2n+1)$ representation α' by reducing each row length of the tableau for the latter by one-half. As demonstrated in Appendix A of Ref. [22] (see also Ref. [42]), the respective dimensions of these representation are related by the "miraculous dimension formula"

$$(\dim \alpha')_{\mathrm{so}(2n+1)} = 2^n (\dim \alpha'')_{\mathrm{sp}(n)}. \tag{7.2}$$

Next, in Appendix B of Ref. [22], it is shown that

$$\big(\dim \sigma(\lambda)\big)_{\mathrm{so}(2n+1)} = -(\dim \lambda)_{\mathrm{so}(2n+1)} \bmod x_{2n,K} \tag{7.3}$$

where $\sigma(\lambda)$ is the $\widehat{\mathrm{so}}(2n+1)_{K+1}$ representation cominimally-equivalent to λ. Using conjecture $\mathrm{B}^{\mathrm{spin}}$ of Ref. [22], and the facts that the dimensions of all spinor representations of so$(2n+1)$ are multiples of 2^n and that $(\dim \sigma(0))_{\mathrm{so}(2n+1)} = -1 \bmod x_{2n,K}$ [22], Eq. (7.3) may be strengthened to

$$\big(\dim \sigma(\alpha')\big)_{\mathrm{so}(2n+1)} = -(\dim \alpha')_{\mathrm{so}(2n+1)} \bmod 2^n x_{2n,K} \tag{7.4}$$

for α' a spinor representation of $\widehat{\mathrm{so}}(2n+1)_{K+1}$. Together with Eq. (7.2), this implies that the charges of cominimally-equivalent $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$ are related by

$$Q_{\sigma(\alpha)}{}^{c} = -Q_{\alpha}{}^{c} \bmod x_{2n,K} \tag{7.5}$$

analogous to Eq. (1.4) for untwisted D-branes.

Finally, we turn to the relationship between the charges of level-rank-dual $_c$-twisted D-branes.

Duality of twisted D-brane charges under $\widehat{\mathrm{su}}(2n+1)_{2k+1} \leftarrow\ \ \widehat{\mathrm{su}}(2k+1)_{2n+1}$

Let $x = \gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\}$. In Ref. [5], it was shown that

$$(\dim \alpha'')_{\mathrm{sp}(n)} = (\dim \hat{\alpha}'')_{\mathrm{sp}(k)} \bmod x \tag{7.6}$$

where $\hat{\alpha}''$ is obtained from α'' by tableau transposition. Since $\hat{\alpha}''$ is the sp(k) representation corresponding to the level-rank-dual $_c$-twisted D-brane $\hat{\alpha}$ of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$, it immediately follows from Eq. (7.1) that the charges of level-rank-dual $_c$-twisted D-branes are equal

$$Q_{\alpha}{}^{c} = \tilde{Q}_{\hat{\alpha}}{}^{c} \bmod x. \tag{7.7}$$

This was previously presented in Ref. [5] and is included here for completeness.

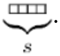
Duality of twisted D-brane charges under $\widehat{\mathrm{su}}(2n+1)_{2k} \leftarrow\ \ \widehat{\mathrm{su}}(2k)_{2n+1}$

Let $x = \gcd\{x_{2n+1,2k}, x_{2k,2n+1}\}$. We begin with the relationship

$$(\dim \Lambda_s)_{\mathrm{sp}(n)} = (\dim \Lambda_s)_{\mathrm{su}(2n+1)} - (\dim \Lambda_{s-1})_{\mathrm{su}(2n+1)} \tag{7.8}$$

where Λ_s is the completely antisymmetric representation with Young tableau . Next, as shown in Ref. [4],

$$(\dim \Lambda_s)_{\mathrm{su}(2n+1)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{su}(2k)} \bmod x \tag{7.9}$$

where $\tilde{\Lambda}_s$ is the completely symmetric representation with Young tableau . Finally,

$$(\dim \tilde{\Lambda}_s)_{\mathrm{so}(2k+1)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(2k)} + (\dim \tilde{\Lambda}_{s-1})_{\mathrm{su}(2k)}. \tag{7.10}$$

Combining these three equations, we obtain

$$(\dim \Lambda_s)_{\mathrm{sp}(n)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{so}(2k+1)} \bmod x. \tag{7.11}$$

This result can be used in the determinantal formulas (A.44) and (A.60) of Ref. [43], following the approach of Ref. [5], to establish a relationship between arbitrary representations of sp(n) and so$(2k+1)$,

$$(\dim \alpha'')_{\mathrm{sp}(n)} = (-1)^{r(\alpha'')} (\dim \widetilde{\alpha''})_{\mathrm{so}(2k+1)} \bmod x \tag{7.12}$$

where $\widetilde{\alpha''}$ is the transpose of the tableau of α''.

Now, from the level-rank map of Section 6, the representation $\widetilde{\alpha''}$ is related to the representation $\hat{\alpha}'$ that corresponds to the level-rank dual $_c$-twisted D-brane by taking the complement of the tableau with respect to a $k \times (n+\frac{1}{2})$ rectangle. This maps a type I tensor representation of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ to a type I spinor representation. We conjecture a relationship[10]

$$(\dim \widetilde{\alpha''})_{\mathrm{so}(2k+1)} = (-1)^{k(k+1)/2} (\dim \hat{\alpha}')_{\mathrm{so}(2k+1)} \bmod x_{2k,2n+1} \quad \text{(conjecture)} \tag{7.13}$$

[10] After v1 of this paper appeared, we learned that an equivalent version of this relationship has been independently conjectured by Stefan Fredenhagen and collaborators [44].

between the dimensions of $\widetilde{\alpha''}$ and $\hat{\alpha}'$. To justify this, consider the expression for the dimension of the so$(2k+1)$ representation $\widetilde{\alpha''}$:

$$(\dim\widetilde{\alpha''})_{\mathrm{so}(2k+1)} = \frac{\prod_{i=1}^{k}(2\phi_i)\prod_{i<j}(\phi_i-\phi_j)(\phi_i+\phi_j)}{\prod_{i=1}^{k}(2k+1-i)\prod_{i<j}(j-i)(2k+1-i-j)} \tag{7.14}$$

where $\phi_i = \ell_i(\widetilde{\alpha''}) - \frac{1}{2} + k - i$. All the factors in parentheses are integers. The row lengths of $\hat{\alpha}'$ are related to those of $\widetilde{\alpha''}$ by $\ell_i(\hat{\alpha}') = n + \frac{1}{2} - \ell_{k+1-i}(\widetilde{\alpha''})$. Hence

$$(\dim\hat{\alpha}')_{\mathrm{so}(2k+1)} = \frac{\prod_{i=1}^{k}(X-2\phi_{k+1-i})\prod_{i<j}(\phi_{k+1-j}-\phi_{k+1-i})(X-\phi_{k+1-i}-\phi_{k+1-j})}{\prod_{i=1}^{k}(2k+1-i)\prod_{i<j}(j-i)(2k+1-i-j)} \tag{7.15}$$

where $X \equiv 2n+2k+1$. Then

$$(\dim\widetilde{\alpha''})_{\mathrm{so}(2k+1)} - (-1)^{k(k+1)/2}(\dim\hat{\alpha}')_{\mathrm{so}(2k+1)} = XR \tag{7.16}$$

where R is a rational number with denominator $\prod_{i=1}^{k}(2k+1-i)\prod_{i<j}(j-i)(2k+1-i-j)$. If X is prime, then none of the factors in the denominator of R (which are all less than $2k+1$) divide X, and since the left-hand side is an integer, R must also be an integer, in which case the left-hand side is a multiple of X. This establishes Eq. (7.13) when X is prime, since $x_{2k,2n+1} = X$ in that case. When X is not prime, some of the factors in the denominator of R may divide X, but we believe (proved for $k=2$, and based on strong numerical evidence for $k=3$, 4, and 5, with arbitrary n) that the right-hand side of Eq. (7.16) is always a multiple of $x_{2k,2n+1}$, and therefore that the conjecture (7.13) holds.

Finally, from Eq. (7.2), we have

$$(\dim\hat{\alpha}')_{\mathrm{so}(2k+1)} = 2^k(\dim\hat{\alpha}'')_{\mathrm{sp}(k)}. \tag{7.17}$$

Putting together Eqs. (7.12), (7.13), and (7.17), we obtain the relationship between the charge of the $_c$-twisted D-brane α of $\widehat{\mathrm{su}}(2n+1)_{2k}$ and the level-rank-dual $_c$-twisted D-brane $\hat{\alpha}$ of $\widehat{\mathrm{su}}(2k)_{2n+1}$

$$Q_{\alpha}{}^{c} = 2^k(-1)^{r(\alpha'')+k(k+1)/2}\tilde{Q}_{\hat{\alpha}}{}^{c} \bmod x \tag{7.18}$$

whose validity is subject only to the conjectured relation (7.13).

Duality of twisted D-brane charges under $\widehat{\mathrm{su}}(2n)_{2k} \leftarrow \quad \widehat{\mathrm{su}}(2k)_{2n}$

Let $x = \gcd\{x_{2n,2k}, x_{2k,2n}\}$. As shown in Ref. [5], if $n=k$, then $x=4$ if $n=2^m$, otherwise $x=1$. If $n \neq k$, then $x=2$ if $n+k=2^m$, otherwise $x=1$.

We saw above that the charges of level-rank dual $_c$-twisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$ are equal (modulo x) when both N and K are odd. This equality (modulo x) no longer holds if either N or K is even. When both N and K are even, the charges are again not equal (even modulo x and modulo sign), as may be checked in a specific case (e.g., $\widehat{\mathrm{su}}(4)_4$, with $\alpha'' = \yng(1,1)$ and $\hat{\alpha}'' = \yng(2)$, since $5 \neq \pm 10 \bmod 4$). On the basis of Eq. (7.18), one might expect a relationship such as

$$2^n Q_{\alpha}{}^{c} = \pm 2^k \tilde{Q}_{\hat{\alpha}}{}^{c} \bmod x. \tag{7.19}$$

However, any such relationship is trivially satisfied, since $_c$-twisted branes exist only when n, $k > 1$, and x is either 1, 2, or 4.

8. Conclusions

In this paper, we have considered D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model twisted by the charge-conjugation symmetry $_c$. Such D-branes exist for all $N > 2$, and possess integer D0-brane charge, defined modulo $x_{N,K}$.

For $\widehat{\mathrm{su}}(2n)_K$ and $\widehat{\mathrm{su}}(2n+1)_K$, the $_c$-twisted D-branes are labelled by a subset of the integrable representations of $\widehat{\mathrm{so}}(2n+1)_{K+1}$ and $\widehat{\mathrm{so}}(2n+1)_{K+2}$ respectively. In the former case, the D-branes belong to cominimal equivalence classes generated by the $\mathbb{Z}_2$ simple current symmetry of $\widehat{\mathrm{so}}(2n+1)_{K+1}$. We showed that the D0-brane charges of cominimally equivalent D-branes are equal and opposite modulo $x_{2n,K}$.

We then showed that level-rank-duality of $\widehat{\mathrm{su}}(N)_K$ WZW models extends to the $_c$-twisted D-branes of the theory when both N and K are greater than two. In particular, we demonstrated a one-to-one mapping $\alpha \quad \hat{\alpha}$ between the $_c$-twisted D-branes for N odd (or cominimal equivalence classes of D-branes for N even) of $\widehat{\mathrm{su}}(N)_K$ and the $_c$-twisted D-branes for K odd (or cominimal equivalence classes of D-branes for K even) of $\widehat{\mathrm{su}}(K)_N$.

We then showed that the spectrum of an open string stretched between $_c$-twisted D-branes is invariant under level-rank duality. More precisely, we showed that the coefficients ${n_{\beta\lambda}}^{\alpha}$ of the open-string partition function (or ${s_{\beta\lambda}}^{\alpha}$, the appropriate linear combination (1.8) of those coefficients corresponding to cominimal equivalence classes of $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n)_K$) are invariant under $\alpha \quad \hat{\alpha}$, $\beta \quad \hat{\beta}$, and $\lambda \quad \tilde{\lambda}$. The proof of this required the existence of a *partial* level-rank mapping between the $_c$-twisted Ishibashi states of each theory. (That is, the map only involved a subset of the $_c$-twisted Ishibashi states of $\widehat{\mathrm{su}}(2n)_K$.)

Finally, we analyzed the relation between the D0-brane charges of level-rank-dual $_c$-twisted D-branes (or cominimal equivalence classes thereof), modulo $x = \gcd\{x_{N,K}, x_{K,N}\}$. When N and K are both odd, the charges are equal mod x (as previously demonstrated in Ref. [5]), but in other cases this simple relationship does not hold. For $N = 2n+1$ and $K = 2k$, the relation between the charges of level-rank-dual $_c$-twisted D-branes is

$$Q_{\alpha}{}^{c} = 2^k (-1)^{r(\alpha'')+k(k+1)/2} \tilde{Q}_{\hat{\alpha}}{}^{c} \bmod x \tag{8.1}$$

subject to the validity of a certain conjecture (7.13) stated in Section 7.

It would interesting to know whether level-rank duality extends to any of the other twisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model [23].

Acknowledgements

The authors wish to express their gratitude to M. Gaberdiel for several extremely helpful correspondences. We also thank P. Bouwknegt and D. Ridout for useful comments.

Appendix A

In this appendix, we establish the relationship between certain matrix elements of the modular transformation matrices of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ through the use of Jacobi's theorem, following the approach of Ref. [3]. Note that this is *not* the usual level-rank duality between $\widehat{\mathrm{so}}(N)_K$ and $\widehat{\mathrm{so}}(K)_N$.

Let α' (μ') be an integrable type I spinor representation of $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ corresponding to a $_c$-twisted Cardy (Ishibashi) state of $\widehat{\mathrm{su}}(2n+1)_{2k}$. The $\widehat{\mathrm{so}}(2n+1)_{2k+2}$ modular transformation matrix has the matrix element [45]

$$S'_{\alpha'\mu'} = (-1)^{n(n-1)/2} 2^{n-1} (2k+2n+1)^{-n/2} \det \mathbf{M} \tag{A.1}$$

where $\mathbf{M}$ is an $n \times n$ matrix with matrix elements

$$M_{ij} = \sin\left(\frac{\pi \phi_i(\alpha')\phi_j(\mu')}{k+n+\frac{1}{2}}\right), \quad \phi_i(\alpha') = \ell_i(\alpha') + n + \frac{1}{2} - i, \ i = 1, \ldots, n. \tag{A.2}$$

Let $\hat\alpha'$ ($\hat\mu'$) be the integrable type I spinor (tensor) representation of $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ related to α' (μ') by the level-rank duality map described in Section 6. The $\widehat{\mathrm{so}}(2k+1)_{2n+2}$ modular transformation matrix has matrix element

$$\tilde S'_{\hat\alpha'\hat\mu'} = (-1)^{k(k-1)/2} 2^{k-1} (2k+2n+1)^{-k/2} \det \tilde{\mathbf{M}} \tag{A.3}$$

where $\tilde{\mathbf{M}}$ is a $k \times k$ matrix with matrix elements

$$\tilde M_{ij} = \sin\left(\frac{\pi \tilde\phi_i(\hat\alpha')\tilde\phi_j(\hat\mu')}{k+n+\frac{1}{2}}\right), \quad \tilde\phi_i(\hat\alpha') = \ell_i(\hat\alpha') + k + \frac{1}{2} - i, \ i = 1, \ldots, k. \tag{A.4}$$

Next, define the index sets for the $_c$-twisted Cardy states

$$I = \left\{\phi_i(\alpha'), \quad i = 1, \ldots, n\right\},$$
$$\bar I = \left\{\tilde\phi_i(\hat\alpha'), \quad i = 1, \ldots, k\right\}. \tag{A.5}$$

Using the level-rank duality map α' $\hat\alpha'$ given in Section 6, one may establish that I and $\bar I$ are complementary sets of integers:

$$I \cup \bar I = \{1, 2, \ldots, n+k\}, \qquad I \cap \bar I = 0. \tag{A.6}$$

Also, define the index sets for the $_c$-twisted Ishibashi states

$$J = \left\{\phi_j(\mu'), \quad j = 1, \ldots, n\right\},$$
$$\bar J = \left\{n + k + \frac{1}{2} - \tilde\phi_j(\hat\mu'), \quad j = 1, \ldots, k\right\}. \tag{A.7}$$

Using the level-rank duality map μ' $\hat\mu'$ given in Section 6, one may also establish that J and $\bar J$ are complementary sets of integers:

$$J \cup \bar J = \{1, 2, \ldots, n+k\}, \quad J \cap \bar J = 0. \tag{A.8}$$

Now, define the $L \times L$ matrix $\mathbf{\Omega}$ with matrix elements

$$\Omega_{ij} = \sin\left(\frac{\pi i j}{L+\frac{1}{2}}\right), \quad i, j = 1, \ldots, L \tag{A.9}$$

where $L = n + k$. This matrix has determinant

$$\det \mathbf{\Omega} = (-1)^{L(L-1)/2} \left(\frac{2L+1}{4}\right)^{L/2} \tag{A.10}$$

and obeys

$$\mathbf{\Omega}^{-1} = \left(\frac{4}{2L+1}\right) \mathbf{\Omega}. \tag{A.11}$$

Define $(\mathbf{\Omega})_{IJ}$ to be the $n \times n$ submatrix obtained from the larger $\mathbf{\Omega}$ by considering only those rows indexed by the elements of I and those columns indexed by the elements of J. Jacobi's theorem [46] states that

$$\det\left[\left(\mathbf{\Omega}^{-1}\right)^T\right]_{IJ} = (-)^{\Sigma_I+\Sigma_J}(\det\mathbf{\Omega})^{-1}\det(\mathbf{\Omega})_{\bar{I}\bar{J}}, \tag{A.12}$$

where

$$\Sigma_I = \sum_{i\in I} i \quad \text{and} \quad \Sigma_J = \sum_{j\in J} j. \tag{A.13}$$

One may observe that

$$\det\mathbf{M} = \det(\mathbf{\Omega})_{IJ}, \qquad \det\tilde{\mathbf{M}} = (-1)^{k+\Sigma_{\bar{I}}+k(k-1)/2}\det(\mathbf{\Omega})_{\bar{I}\bar{J}} \tag{A.14}$$

where the last contribution to the sign results from reversing the ordering of the rows of $\tilde{\mathbf{M}}$. Assembling Eqs. (A.1), (A.3), (A.10)–(A.12), and (A.14), and using

$$(-1)^{\Sigma_I+\Sigma_J+\Sigma_{\bar{I}}} = (-1)^{\Sigma_{\bar{J}}} = (-1)^{nk+k(k-1)/2+r(\hat{\mu}')} \tag{A.15}$$

one concludes that

$$S'_{\alpha'\mu'} = (-1)^{r(\hat{\mu}')+k}\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tag{A.16}$$

which is used in proving the level-rank duality of the open string spectrum in the last subsection of Section 6.

References

[1] S.G. Naculich, H.J. Schnitzer, Duality between SU$(N)_k$ and SU$(k)_N$ WZW models, Nucl. Phys. B 347 (1990) 687–742;
S.G. Naculich, H.J. Schnitzer, Duality relations between SU$(N)_k$ and SU$(k)_N$ WZW models and their braid matrices, Phys. Lett. B 244 (1990) 235–240;
S.G. Naculich, H.A. Riggs, H.J. Schnitzer, Group level duality in WZW models and Chern–Simons theory, Phys. Lett. B 246 (1990) 417–422;
J. Fuchs, P. van Driel, Some symmetries of quantum dimensions, J. Math. Phys. 31 (1990) 1770–1775.

[2] D. Altschuler, M. Bauer, C. Itzykson, The branching rules of conformal embeddings, Commun. Math. Phys. 132 (1990) 349–364;
See also M.A. Walton, Conformal branching rules and modular invariants, Nucl. Phys. B 322 (1989) 775;
H. Saleur, D. Altschuler, Level rank duality in quantum groups, Nucl. Phys. B 354 (1991) 579–613;
A. Kuniba, T. Nakanishi, Level rank duality in fusion RSOS models, in: Proceedings of the International Colloquium on Modern Quantum Field Theory, Bombay, India, January 1990, World Scientific, Singapore, 1991.

[3] E.J. Mlawer, S.G. Naculich, H.A. Riggs, H.J. Schnitzer, Group level duality of WZW fusion coefficients and Chern–Simons link observables, Nucl. Phys. B 352 (1991) 863–896.

[4] S.G. Naculich, H.J. Schnitzer, Level-rank duality of D-branes on the SU(N) group manifold, Nucl. Phys. B 740 (2006) 181–194, hep-th/0511083.

[5] S.G. Naculich, H.J. Schnitzer, Level-rank duality of untwisted and twisted D-branes, Nucl. Phys. B 742 (2006) 295–311, hep-th/0601175.

[6] C. Klimcik, P. Severa, Open strings and D-branes in WZNW models, Nucl. Phys. B 488 (1997) 653–676, hep-th/9609112;
M. Kato, T. Okada, D-branes on group manifolds, Nucl. Phys. B 499 (1997) 583–595, hep-th/9612148;
A.Y. Alekseev, V. Schomerus, D-branes in the WZW model, Phys. Rev. D 60 (1999) 061901, hep-th/9812193;
K. Gawedzki, Conformal field theory: A case study, hep-th/9904145.

[7] R.E. Behrend, P.A. Pearce, V.B. Petkova, J.-B. Zuber, On the classification of bulk and boundary conformal field theories, Phys. Lett. B 444 (1998) 163–166, hep-th/9809097;
R.E. Behrend, P.A. Pearce, V.B. Petkova, J.-B. Zuber, Boundary conditions in rational conformal field theories, Nucl. Phys. B 570 (2000) 525–589, hep-th/9908036.

[8] J. Fuchs, C. Schweigert, Symmetry breaking boundaries. I: General theory, Nucl. Phys. B 558 (1999) 419–483, hep-th/9902132.
[9] L. Birke, J. Fuchs, C. Schweigert, Symmetry breaking boundary conditions and WZW orbifolds, Adv. Theor. Math. Phys. 3 (1999) 671–726, hep-th/9905038.
[10] A.Y. Alekseev, A. Recknagel, V. Schomerus, Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres, JHEP 9909 (1999) 023, hep-th/9908040;
A.Y. Alekseev, A. Recknagel, V. Schomerus, Brane dynamics in background fluxes and non-commutative geometry, JHEP 0005 (2000) 010, hep-th/0003187.
[11] G. Felder, J. Frohlich, J. Fuchs, C. Schweigert, The geometry of WZW branes, J. Geom. Phys. 34 (2000) 162–190, hep-th/9909030.
[12] S. Stanciu, D-branes in group manifolds, JHEP 0001 (2000) 025, hep-th/9909163;
S. Stanciu, A note on D-branes in group manifolds: Flux quantization and D0-charge, JHEP 0010 (2000) 015, hep-th/0006145;
S. Stanciu, An illustrated guide to D-branes in SU(3), hep-th/0111221;
J.M. Figueroa-O'Farrill, S. Stanciu, D-brane charge, flux quantization and relative (co)homology, JHEP 0101 (2001) 006, hep-th/0008038.
[13] C. Bachas, M.R. Douglas, C. Schweigert, Flux stabilization of D-branes, JHEP 0005 (2000) 048, hep-th/0003037;
J. Pawelczyk, SU(2) WZW D-branes and their noncommutative geometry from DBI action, JHEP 0008 (2000) 006, hep-th/0003057;
W. Taylor, D2-branes in B fields, JHEP 0007 (2000) 039, hep-th/0004141.
[14] A. Alekseev, V. Schomerus, RR charges of D2-branes in the WZW model, hep-th/0007096;
A.Y. Alekseev, A. Recknagel, V. Schomerus, Open strings and non-commutative geometry of branes on group manifolds, Mod. Phys. Lett. A 16 (2001) 325–336, hep-th/0104054.
[15] S. Fredenhagen, V. Schomerus, Branes on group manifolds, gluon condensates, and twisted K-theory, JHEP 0104 (2001) 007, hep-th/0012164.
[16] J.M. Maldacena, G.W. Moore, N. Seiberg, Geometrical interpretation of D-branes in gauged WZW models, JHEP 0107 (2001) 046, hep-th/0105038.
[17] J.M. Maldacena, G.W. Moore, N. Seiberg, D-brane instantons and K-theory charges, JHEP 0111 (2001) 062, hep-th/0108100.
[18] K. Gawedzki, Boundary WZW, G/H, G/G and CS theories, Ann. Henri Poincaré 3 (2002) 847–881, hep-th/0108044;
K. Gawedzki, N. Reis, WZW branes and gerbes, Rev. Math. Phys. 14 (2002) 1281–1334, hep-th/0205233.
[19] H. Ishikawa, Boundary states in coset conformal field theories, Nucl. Phys. B 629 (2002) 209–232, hep-th/0111230.
[20] V.B. Petkova, J.B. Zuber, Boundary conditions in charge conjugate sl(N) WZW theories, hep-th/0201239.
[21] M.R. Gaberdiel, T. Gannon, Boundary states for WZW models, Nucl. Phys. B 639 (2002) 471–501, hep-th/0202067.
[22] M.R. Gaberdiel, T. Gannon, The charges of a twisted brane, JHEP 0401 (2004) 018, hep-th/0311242.
[23] M.R. Gaberdiel, T. Gannon, D. Roggenkamp, The D-branes of SU(n), JHEP 0407 (2004) 015, hep-th/0403271;
M.R. Gaberdiel, T. Gannon, D. Roggenkamp, The coset D-branes of SU(n), JHEP 0410 (2004) 047, hep-th/0404112.
[24] A.Y. Alekseev, S. Fredenhagen, T. Quella, V. Schomerus, Non-commutative gauge theory of twisted D-branes, Nucl. Phys. B 646 (2002) 127–157, hep-th/0205123;
T. Quella, Branching rules of semi-simple Lie algebras using affine extensions, J. Phys. A 35 (2002) 3743–3754, math-ph/0111020.
[25] T. Quella, V. Schomerus, Symmetry breaking boundary states and defect lines, JHEP 0206 (2002) 028, hep-th/0203161;
T. Quella, On the hierarchy of symmetry breaking D-branes in group manifolds, JHEP 0212 (2002) 009, hep-th/0209157.
[26] P. Bouwknegt, P. Dawson, D. Ridout, D-branes on group manifolds and fusion rings, JHEP 0212 (2002) 065, hep-th/0210302.
[27] P. Bouwknegt, D. Ridout, A note on the equality of algebraic and geometric D-brane charges in WZW models, JHEP 0405 (2004) 029, hep-th/0312259.
[28] H. Ishikawa, T. Tani, Novel construction of boundary states in coset conformal field theories, Nucl. Phys. B 649 (2003) 205–242, hep-th/0207177;
H. Ishikawa, A. Yamaguchi, Twisted boundary states in $c = 1$ coset conformal field theories, JHEP 0304 (2003) 026, hep-th/0301040;
H. Ishikawa, T. Tani, Twisted boundary states and representation of generalized fusion algebra, hep-th/0510242.
[29] H. Ishikawa, T. Tani, Twisted boundary states in Kazama–Suzuki models, Nucl. Phys. B 678 (2004) 363–397, hep-th/0306227;

S. Schafer-Nameki, D-branes in $N = 2$ coset models and twisted equivariant K-theory, hep-th/0308058.
[30] V. Braun, S. Schafer-Nameki, Supersymmetric WZW models and twisted K-theory of SO(3), hep-th/0403287.
[31] M.R. Gaberdiel, T. Gannon, D-brane charges on non-simply connected groups, JHEP 0404 (2004) 030, hep-th/0403011;
S. Fredenhagen, D-brane charges on SO(3), JHEP 0411 (2004) 082, hep-th/0404017;
M. Vasudevan, Charges of exceptionally twisted branes, JHEP 0507 (2005) 035, hep-th/0504006;
S. Fredenhagen, M.R. Gaberdiel, T. Mettler, Charges of twisted branes: The exceptional cases, JHEP 0505 (2005) 058, hep-th/0504007;
S. Fredenhagen, T. Quella, Generalised permutation branes, JHEP 0511 (2005) 004, hep-th/0509153.
[32] V. Schomerus, Lectures on branes in curved backgrounds, Class. Quantum Grav. 19 (2002) 5781–5847, hep-th/0209241.
[33] J.L. Cardy, Boundary conditions, fusion rules and the Verlinde formula, Nucl. Phys. B 324 (1989) 581.
[34] P. Bouwknegt, V. Mathai, D-branes, B-fields and twisted K-theory, JHEP 0003 (2000) 007, hep-th/0002023.
[35] D.S. Freed, The Verlinde algebra is twisted equivariant K-theory, Turk. J. Math. 25 (2001) 159–167, math.RT/0101038;
D.S. Freed, Twisted K-theory and loop groups, math.AT/0206237;
D.S. Freed, M.J. Hopkins, C. Teleman, Twisted K-theory and loop group representations, math.AT/0312155.
[36] V. Braun, Twisted K-theory of Lie groups, JHEP 0403 (2004) 029, hep-th/0305178;
C.L. Douglas, On the twisted K-homology of simple Lie groups, math.AT/0404017.
[37] N. Ishibashi, The boundary and crosscap states in conformal field theories, Mod. Phys. Lett. A 4 (1989) 251.
[38] J. Fuchs, B. Schellekens, C. Schweigert, From Dynkin diagram symmetries to fixed point structures, Commun. Math. Phys. 180 (1996) 39–98, hep-th/9506135.
[39] V. Kac, Infinite-Dimensional Lie Algebras, third ed., Cambridge Univ. Press, Cambridge, 1990.
[40] P. Goddard, D.I. Olive, Kac–Moody and Virasoro algebras in relation to quantum physics, Int. J. Mod. Phys. A 1 (1986) 303.
[41] J. Fuchs, Affine Lie Algebras and Quantum Groups, Cambridge Univ. Press, Cambridge, 1992.
[42] P. Bouwknegt, D. Ridout, Presentations of Wess–Zumino–Witten fusion rings, Rev. Math. Phys. 18 (2006) 201–232, hep-th/0602057.
[43] W. Fulton, J. Harris, Representation Theory, Springer-Verlag, Berlin, 1991.
[44] S. Fredenhagen, private communication.
[45] V.G. Kac, D.H. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. Math. 53 (1984) 125–264;
V.G. Kac, M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, Adv. Math. 70 (1988) 156.
[46] H. Jeffreys, B. Jeffreys, Methods of Mathematical Physics, third ed., Cambridge Univ. Press, Cambridge, 1956.