

Bowdoin College

Bowdoin Digital Commons

Physics Faculty Publications

Faculty Scholarship and Creative Work

6-1-2007

Level-rank duality of the $U(N)$ WZW model, Chern-Simons theory, and 2d qYM theory

Stephen G. Naculich
Bowdoin College

Howard J. Schnitzer
Brandeis University

Follow this and additional works at: <https://digitalcommons.bowdoin.edu/physics-faculty-publications>

Recommended Citation

Naculich, Stephen G. and Schnitzer, Howard J., "Level-rank duality of the $U(N)$ WZW model, Chern-Simons theory, and 2d qYM theory" (2007). *Physics Faculty Publications*. 129.
<https://digitalcommons.bowdoin.edu/physics-faculty-publications/129>

This Article is brought to you for free and open access by the Faculty Scholarship and Creative Work at Bowdoin Digital Commons. It has been accepted for inclusion in Physics Faculty Publications by an authorized administrator of Bowdoin Digital Commons. For more information, please contact mdoyle@bowdoin.edu, a.sauer@bowdoin.edu.

Level-rank duality of the $U(N)$ WZW model, Chern-Simons theory, and 2d qYM theory

To cite this article: Stephen G. Naculich and Howard J. Schnitzer JHEP06(2007)023

View the [article online](#) for updates and enhancements.

You may also like

- [Maxima of log-correlated fields: some recent developments](#)
E C Bailey and J P Keating
- [The Near-Infrared Photometric Properties of Bright Giants in the Central Regions of the Galactic Bulge](#)
T. J. Davidge
- [Classical setting and effective dynamics for spinfoam cosmology](#)
Etera R Livine and Mercedes Martín-Benito

Level-rank duality of the $U(N)$ WZW model, Chern-Simons theory, and 2d qYM theory

Stephen G. Naculich*

*Department of Physics, Bowdoin College,
Brunswick, ME 04011, U.S.A.
E-mail: naculich@bowdoin.edu*

Howard J. Schnitzer†

*Theoretical Physics Group, Martin Fisher School of Physics, Brandeis University,
Waltham, MA 02454, U.S.A.
E-mail: schnitzr@brandeis.edu*

ABSTRACT: We study the WZW, Chern-Simons, and 2d qYM theories with gauge group $U(N)$. The $U(N)$ WZW model is only well-defined for odd level K , and this model is shown to exhibit level-rank duality in a much simpler form than that for $SU(N)$. The $U(N)$ Chern-Simons theory on Seifert manifolds exhibits a similar duality, distinct from the level-rank duality of $SU(N)$ Chern-Simons theory on S^3 . When $q = e^{2\pi i/(N+K)}$, the observables of the 2d $U(N)$ qYM theory can be expressed as a sum over a *finite* subset of $U(N)$ representations. When N and K are odd, the qYM theory exhibits $N \leftrightarrow K$ duality, provided $q = e^{2\pi i/(N+K)}$ and $\theta = 0 \pmod{2\pi/(N+K)}$.

KEYWORDS: Field Theories in Lower Dimensions, Topological Field Theories, Chern-Simons Theories, Conformal and W Symmetry.

*Research supported in part by the NSF under grant PHY-0456944.

†Research supported in part by the DOE under grant DE-FG02-92ER40706.

Contents

1. Introduction	1
2. The $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model	2
2.1 Representations of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$	3
2.2 Affine characters and modular transformation matrices of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$	4
3. Level-rank duality of the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model	6
3.1 Level-rank duality of the $\widehat{\mathfrak{sp}}(n)_k$ WZW model	7
4. Chern-Simons theory and level-rank duality	8
4.1 Chern-Simons theory on Seifert manifolds	8
4.2 Level-rank duality of $SU(N)$ Chern-Simons theory on S^3	9
4.3 Level-rank duality of $U(N)$ and $Sp(n)$ CS theory on Seifert manifolds	10
5. 2d $U(N)$ qYM theory and $N \leftrightarrow K$ duality	10
5.1 $N \leftrightarrow K$ duality of the 2d $U(N)$ qYM theory	13
6. Conclusions	13
A. Representations of finite-dimensional and affine Lie algebras	14
A.1 Representations of $\mathfrak{u}(N)$	14
A.2 Representations of $\widehat{\mathfrak{su}}(N)_K$ and $\widehat{\mathfrak{u}}(1)_{K'}$	15

1. Introduction

While two-dimensional (Yang-Mills and Wess-Zumino-Witten) and three-dimensional (Chern-Simons) gauge theories have typically been analyzed using simple gauge groups, for some purposes it is more natural to consider these theories with gauge group $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$.

The proposed relation [1] between BPS black hole microstates and topological string theory amplitudes was illustrated in refs. [2, 3] by computing bound states of D-branes on a manifold that is locally a fibration over a Riemann surface Σ_g . The D4-brane worldvolume gauge theory reduces in this case to a two-dimensional q -deformed Yang-Mills (qYM) theory on Σ_g with gauge group $U(N)$, which was studied in refs. [4]–[10]. As we will demonstrate in this paper, the $U(N)$ qYM theory has the interesting feature — not shared by the qYM theory with gauge group $SU(N)$ — that, when $q = e^{2\pi i/(N+K)}$, its partition function is given by a sum restricted to a *finite* subset of $U(N)$ representations, namely, those with

Young tableaux with no more than K columns. Consequently, the 2d $U(N)$ qYM theory exhibits an $N \leftrightarrow K$ duality akin to level-rank duality of WZW models [11]–[14].

$U(N)$ also plays a more natural role than $SU(N)$ in large- N dualities of Chern-Simons theories [15], as pointed out in ref. [16]. The $U(N)$ Chern-Simons theory also has a natural realization in terms of free fermions [17]. More recently, Chern-Simons theory on Seifert manifolds (circle bundles over Σ_g) has received attention [18, 7, 9]. In this paper, we will show that level- K $U(N)$ Chern-Simons theory on certain Seifert manifolds exhibits an $N \leftrightarrow K$ duality (for odd N and K) that is distinct from the $N \leftrightarrow K$ duality that holds for $SU(N)$ Chern-Simons theory on S^3 [13, 19]–[22].

Both of the above-mentioned $N \leftrightarrow K$ dualities flow from the level-rank duality of the Wess-Zumino-Witten model with gauge group $U(N)$, which is explored in detail in this paper. The $U(N)$ WZW model has received much less attention than WZW models based on simple gauge groups, although its fusion rule algebra was explored in ref. [23], and also in ref. [24] in connection with the quantum cohomology of the Grassmannian. In several respects, level-rank duality is much simpler for $U(N)$ than for $SU(N)$, and closely resembles that for $Sp(n)$. For example, level-rank duality of the $SU(N)$ WZW model involves a 1-1 map between cominimal equivalence classes (simple-current orbits) of integrable representations, whereas for $U(N)$ the map is between the representations themselves.

The affine Lie algebra of the $U(N)$ WZW model is the quotient of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{K'}$ by \mathbf{Z}_N . To realize the $U(N)$ symmetry, the levels of the factor groups must be related by $K' = N(K + N)$, and moreover, the level K must be odd [23]. As we will show in section 2, the primary fields of the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model (for K odd) are in one-to-one correspondence with Young tableaux with at most N rows and K columns,¹ and the affine characters of these fields are infinite sums of characters of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{K'}$. By considering modular transformations of these affine characters, we derive the form of the S and T matrices for $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$.

In section 3, we demonstrate level-rank duality between the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ and $\widehat{\mathfrak{u}}(K)_{N,K(K+N)}$ WZW models (with N and K odd so that both theories are defined) and show the similarity to the duality between $\widehat{\mathfrak{sp}}(n)_k$ and $\widehat{\mathfrak{sp}}(k)_n$. In section 4, we use formulas derived from surgery on knots to demonstrate level-rank duality of observables of $U(N)$ (and also $Sp(n)$) Chern-Simons theory on a certain class of Seifert manifolds. In section 5, we show that the partition function of 2d q -deformed $U(N)$ Yang-Mills theory can be expressed, when $q = e^{2\pi i/(N+K)}$, as a sum over a truncated set of $U(N)$ representations. When K is odd, the observables (partition function, Wilson line expectation values) can be further expressed in terms of modular transformation matrices of the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model. Consequently, when N and K are odd, the $U(N)$ qYM theory exhibits $N \leftrightarrow K$ duality, provided $q = e^{2\pi i/(N+K)}$ and $\theta = 0 \pmod{2\pi/(N+K)}$.

2. The $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model

This section is devoted to an analysis of the representations of the $U(N)$ WZW model,

¹Integrable representations of $\widehat{\mathfrak{su}}(N)_K$ have Young tableaux with at most $N - 1$ rows and K columns.

their affine characters and modular transformation matrices. The $U(N)$ WZW model was previously studied in the context of its fusion rule algebra in refs. [23, 24].

The chiral algebra of the $U(N)$ WZW model is the quotient of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{K'}$ by \mathbf{Z}_N . To realize the $U(N)$ symmetry, the levels of the $\widehat{\mathfrak{su}}(N)_K$ and $\widehat{\mathfrak{u}}(1)_{K'}$ subalgebras must be related; the relation between K and K' may be determined by requiring that the conformal weight of the representation (R, Q) of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{K'}$ be proportional to the $\mathfrak{u}(N)$ Casimir (A.5)

$$h_{(R,Q)} = h_R + h'_Q = \frac{\frac{1}{2}C_2(R)}{K+N} + \frac{Q^2}{2K'} = \frac{\frac{1}{2}C_2(R, Q)}{K+N} \tag{2.1}$$

which implies that [23, 24]

$$K' = N(K+N). \tag{2.2}$$

(See the appendix for results and conventions for finite-dimensional and affine Lie algebras.) We will therefore denote the algebra of the $U(N)$ WZW model as

$$\widehat{\mathfrak{u}}(N)_{K,N(K+N)} \equiv [\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{N(K+N)}] / \mathbf{Z}_N. \tag{2.3}$$

2.1 Representations of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$

As in the case of the finite-dimensional $\mathfrak{u}(N)$ algebra (see the appendix), representations (R, Q) of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ must satisfy

$$Q = r \text{ mod } N \tag{2.4}$$

where r is the number of boxes of the Young tableau associated with R , but in addition we must impose [25, 23, 24] the equivalence relation

$$(R, Q) \simeq (\sigma(R), Q + N + K) \tag{2.5}$$

where σ is the simple current of $\widehat{\mathfrak{su}}(N)_K$, defined in the appendix. The identification (2.5) is necessary in order that the condition (2.4) be preserved under the modular transformation $\tau \rightarrow -1/\tau$. For the affine characters to be well-defined under $\tau \rightarrow \tau + 1$, moreover, the conformal weights of equivalent representations must differ by integers. Using eq. (A.14), one finds that

$$h_{(\sigma(R), Q+N+K)} - h_{(R,Q)} = \frac{2Q - 2r + NK + N}{2N} = s + \frac{1}{2}(K+1) \tag{2.6}$$

which is integer-valued if and only if K is odd [23]. Hence, the WZW model with chiral algebra $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ is only well-defined for odd values of K , which we henceforth assume.²

As a result of the equivalence relation (2.5), the number of primary fields of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ becomes finite, so the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model is a rational conformal field theory. To determine the number of primaries, we iterate eq. (2.5) N times and use $\sigma^N = 1$ to obtain the equivalence

$$(R, Q) \simeq (R, Q + N(N+K)). \tag{2.7}$$

²As another example of such a restriction, the WZW model with chiral algebra $SO(3)$ is only well-defined when the level is a multiple of four [26, 25, 27].

Hence Q may be restricted to the range $0 \leq Q < N(N + K)$. From this, we determine the number of primary fields of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ to be

$$\binom{N + K - 1}{K} \frac{N(N + K)}{N^2} = \binom{N + K}{K} \tag{2.8}$$

where $\binom{N+K-1}{K}$ is the number of integrable highest-weight $\widehat{\mathfrak{su}}(N)_K$ representations, $N(N + K)$ is the range of Q , and the constraint (2.4) and identification (2.5) each reduce the number of distinct fields by a factor of N . (There are no fixed points or short orbits of the equivalence relation.) The result (2.8) was also obtained in the context of Chern-Simons theory in ref. [17]. We emphasize, however, that this argument is only valid for odd values of K , where it is possible to impose the equivalence relation (2.5).

As explained in the appendix, $\mathfrak{u}(N)$ representations (R, Q) may be characterized by extended Young tableaux \mathcal{R} with row lengths $\bar{\ell}_i \in \mathbf{Z}$ ($i = 1, \dots, N$). It is straightforward to prove that, within each equivalence class of representations under (2.5), there is *exactly one* whose extended tableau \mathcal{R} satisfies $0 \leq \bar{\ell}_N \leq \dots \leq \bar{\ell}_1 \leq K$. Hence, the primary fields of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ (where K is odd) are in one-to-one correspondence with Young tableaux \mathcal{R} with no more than N rows and K columns.³ The number of such tableaux is $\binom{N+K}{N}$, in agreement with eq. (2.8).

2.2 Affine characters and modular transformation matrices of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$

The affine character for the representation (R, Q) of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{K'}$ is given by the product of eqs. (A.11) and (A.18),

$$\chi_{(R,Q)}(\tau) = \chi_R(\tau) \chi'_Q(\tau). \tag{2.9}$$

The primary fields of $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$, however, are characterized as equivalence classes of representations (R, Q) under the identification (2.5), so the affine character⁴ of the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ representation \mathcal{R} is given by the following sum of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{N(K+N)}$ characters

$$\begin{aligned} \chi_{\mathcal{R}}(\tau) &= \sum_{n \in \mathbf{Z}} \chi_{(\sigma^n(R), Q+n(N+K))}(\tau) \\ &= \sum_{n=0}^{N-1} \sum_{t \in \mathbf{Z}} \chi_{\sigma^n(R)}(\tau) \chi'_{Q+(N+K)Nt+(N+K)n}(\tau) \\ &= \frac{1}{\eta(\tau)} \sum_{n=0}^{N-1} \chi_{\sigma^n(R)}(\tau) \Theta_{Q+(N+K)n, N(N+K)/2}(\tau) \end{aligned} \tag{2.10}$$

³This is also the conclusion of ref. [10], although the restriction to odd K is not mentioned.

⁴Whereas we are considering the basic specialization of the affine characters, Frenkel [28] considered the *principally-specialized* characters of $\mathfrak{u}(N)$, but without imposing the identification (2.5). He demonstrated the invariance of these characters under transposition of tableaux and $N \leftrightarrow K$. Historically, this was one of the first manifestations of level-rank duality.

where $N(N + K)/2$ is integer-valued since K must be odd, and

$$\Theta_{Q,L}(\tau) \equiv \sum_{t \in \mathbb{Z}} e^{2\pi i \tau L(t+Q/2L)^2}. \quad (2.11)$$

Using the modular transformation property of the theta function

$$\frac{\Theta_{Q,L}}{\eta} \left(-\frac{1}{\tau} \right) = \sum_{Q'=0}^{2L-1} \frac{e^{-\pi i Q Q' / L}}{\sqrt{2L}} \frac{\Theta_{Q',L}}{\eta}(\tau) \quad (2.12)$$

together with eqs. (A.12) and (A.15), one finds

$$\begin{aligned} \chi_{\mathcal{R}}(-1/\tau) &= \frac{1}{\eta(\tau)} \sum_{R'} \sum_{Q'=0}^{N(N+K)-1} \left[\sum_{n=0}^{N-1} e^{2\pi i n(r'-Q')/N} \right] \\ &\quad \times S_{RR'} \frac{e^{-2\pi i Q Q' / N(N+K)}}{\sqrt{N(N+K)}} \chi_{R'}(\tau) \Theta_{Q',N(N+K)/2}(\tau). \end{aligned} \quad (2.13)$$

The sum in brackets vanishes unless $Q' = r' \pmod{N}$, so we may restrict the sum over Q' to obey this constraint

$$\begin{aligned} \chi_{\mathcal{R}}(-1/\tau) &= \frac{1}{\eta(\tau)} \sum_{R'} \sum_{\substack{0 \leq Q' < N(N+K) \\ Q' = r' \pmod{N}}} N S_{RR'} \frac{e^{-2\pi i Q Q' / N(N+K)}}{\sqrt{N(N+K)}} \chi_{R'}(\tau) \Theta_{Q',N(N+K)/2}(\tau) \\ &= \frac{1}{\eta(\tau)} \sum_{R'} \sum_{\substack{0 \leq Q' < N(N+K) \\ Q' = r' \pmod{N}}} S_{RR'} \frac{e^{-2\pi i Q Q' / N(N+K)}}{\sqrt{N(N+K)}} \\ &\quad \times \sum_{n'=0}^{N-1} \chi_{\sigma^{n'}(R')}(\tau) \Theta_{Q'+(N+K)n',N(N+K)/2}(\tau) \\ &= \sum_{\mathcal{R}'} S_{RR'} \frac{N e^{-2\pi i Q Q' / N(N+K)}}{\sqrt{N(N+K)}} \chi_{R'}(\tau) \end{aligned} \quad (2.14)$$

where in the second line, we have used eq. (A.15) together with $Q = r \pmod{N}$. The factor of N in the last line arises because each primary field \mathcal{R}' corresponds to N distinct values of (R', Q') satisfying $0 \leq Q' < N(N + K)$ and $Q' = r' \pmod{N}$. From eq. (2.14), we read off the modular transformation matrix for the $\hat{u}(N)_{K,N(K+N)}$ characters

$$S_{\mathcal{R}\mathcal{R}'} = \sqrt{\frac{N}{N+K}} S_{RR'} e^{-2\pi i Q Q' / N(N+K)}. \quad (2.15)$$

(In order not to unduly clutter the notation, we let the subscripts R or \mathcal{R} indicate whether the modular transformation matrix refers to $\hat{su}(N)_K$ or $\hat{u}(N)_{K,N(K+N)}$ respectively.) In ref. [23], the quantity $S_{\mathcal{R}\mathcal{R}'}$ was obtained in a different way by specializing the $u(N)$ Weyl character. This alternative approach allows one to formally define $S_{\mathcal{R}\mathcal{R}'}$ when K is not odd, or even when K is not a real number [6], but in that case $S_{\mathcal{R}\mathcal{R}'}$ does not represent the transformation matrix of affine characters under $\tau \rightarrow -1/\tau$. The phase factor in eq. (2.15)

was obtained in ref. [16] in the context of $U(N)$ Chern-Simons theory, where $S_{\mathcal{R}\mathcal{R}'}$ represents the expectation value of the Wilson line of the Hopf link on S^3 with canonical framing.

The modular transformation matrix of $\widehat{u}(N)_{K,N(K+N)}$ characters (2.10) under $\tau \rightarrow \tau + 1$ is

$$T_{\mathcal{R}\mathcal{R}'} = \exp \left[2\pi i \left(h_{\mathcal{R}} - \frac{c}{24} \right) \right] \delta_{\mathcal{R}\mathcal{R}'} \quad (2.16)$$

where from (2.1) we have

$$h_{\mathcal{R}} = \frac{\frac{1}{2}C_2(\mathcal{R})}{K+N} \quad (2.17)$$

and

$$c = \frac{K(N^2 - 1)}{K+N} + 1 = \frac{N(NK + 1)}{K+N}. \quad (2.18)$$

Note that $T_{\mathcal{R}\mathcal{R}'}$ is only well-defined when K is odd because, when K is even, the conformal weight $h_{\mathcal{R}}$ of the character (2.10) is not well-defined modulo \mathbf{Z} .

3. Level-rank duality of the $\widehat{u}(N)_{K,N(K+N)}$ WZW model

In the previous section, we showed that the WZW model with affine Lie algebra $\widehat{u}(N)_{K,N(K+N)}$ is only well-defined for odd values of K , for which the equivalence relation (2.5) may be consistently imposed. Throughout this section, we restrict ourselves to odd values of N and K , so that both $\widehat{u}(N)_{K,N(K+N)}$ and $\widehat{u}(K)_{N,K(K+N)}$ are well-defined.

Primary fields of $\widehat{u}(N)_{K,N(K+N)}$, which are characterized (for K odd) by Young tableaux \mathcal{R} with no more than N columns and K rows, are in one-to-one correspondence⁵ with the primary fields of $\widehat{u}(K)_{N,K(K+N)}$ under transposition of \mathcal{R} . (The number $\binom{N+K}{K}$ of such tableaux is manifestly invariant under $N \leftrightarrow K$.) In terms of representations of $\widehat{su}(N)_K \times \widehat{u}(1)_{N(K+N)}$ and $\widehat{su}(K)_N \times \widehat{u}(1)_{K(K+N)}$, the correspondence is

$$\mathcal{R} = (R, Q) \quad \rightarrow \quad \widetilde{\mathcal{R}} = (\sigma^s(\widetilde{R}), Q), \quad s = \frac{Q-r}{N}. \quad (3.1)$$

The conformal weights of level-rank-dual primary fields satisfy

$$h_{\mathcal{R}} + \widetilde{h}_{\widetilde{\mathcal{R}}} = \frac{1}{2}Q. \quad (3.2)$$

This follows from eqs. (2.17) and (A.7) and the fact that $T(\mathcal{R}) = -T(\widetilde{\mathcal{R}})$, which becomes evident when we rewrite eq. (A.8) as

$$T(\mathcal{R}) = \sum \bar{\ell}_i^2 - \sum \bar{k}_j^2 \quad (3.3)$$

where $\bar{\ell}_i$ and \bar{k}_j are the row and column lengths, respectively, of \mathcal{R} .

From eq. (2.18), we have that the central charges of the level-rank-dual theories are related by

$$c + \widetilde{c} = NK + 1. \quad (3.4)$$

⁵This is simpler than level-rank duality between the $\widehat{su}(N)_K$ and $\widehat{su}(K)_N$ WZW models [11]–[14], in which case the correspondence is one-to-one between simple-current orbits of primary fields.

Together with eq. (3.2) this implies that

$$\tilde{T}_{\tilde{\mathcal{R}}\tilde{\mathcal{R}}} = (-)^Q e^{-\pi i(KN+1)/12} T_{\mathcal{R}\mathcal{R}}^* \tag{3.5}$$

where T and \tilde{T} are the modular transformation matrices (2.16) for $\hat{u}(N)_{K,N(K+N)}$ and $\hat{u}(K)_{N,K(K+N)}$ respectively.

The S modular transformation matrix of $\hat{u}(K)_{N,K(K+N)}$ may be obtained using eqs. (2.15) and (3.1),

$$\tilde{S}_{\tilde{\mathcal{R}}\tilde{\mathcal{R}'}} = \sqrt{\frac{K}{K+N}} \tilde{S}_{\sigma^s(\tilde{R})\sigma^{s'}(\tilde{R}')} e^{-2\pi i Q Q' / K(N+K)}. \tag{3.6}$$

Equation (A.15) implies

$$\tilde{S}_{\sigma^s(\tilde{R})\sigma^{s'}(\tilde{R}')} = e^{2\pi i s Q' / K} \tilde{S}_{\tilde{R}\sigma^{s'}(\tilde{R}')}, \quad \tilde{S}_{\tilde{R}\sigma^{s'}(\tilde{R}')} = e^{2\pi i s' r / K} \tilde{S}_{\tilde{R}\tilde{\mathcal{R}'}}. \tag{3.7}$$

Finally, under level-rank duality, the $\hat{u}(N)_K$ and $\hat{u}(K)_N$ modular transformation matrices⁶ S and \tilde{S} are related by [12, 13]

$$\tilde{S}_{\tilde{R}\tilde{R}'} = \sqrt{\frac{N}{K}} e^{2\pi i r r' / NK} S_{RR'}^* \tag{3.8}$$

which, together with the previous equations, yields the much simpler level-rank relation

$$\tilde{S}_{\tilde{\mathcal{R}}\tilde{\mathcal{R}'}} = S_{\mathcal{R}\mathcal{R}'}^* \tag{3.9}$$

between the $\hat{u}(N)_{K,N(K+N)}$ and $\hat{u}(K)_{N,K(K+N)}$ modular transformation matrices S and \tilde{S} .

Equation (3.9) has the immediate consequence, via the Verlinde formula [29], that the fusion coefficients of $\hat{u}(N)_{K,N(K+N)}$ are level-rank dual:

$$\tilde{N}_{\tilde{\mathcal{R}}\tilde{\mathcal{R}''}} = N_{\mathcal{R}\mathcal{R}''}. \tag{3.10}$$

Duality of the fusion coefficients also follows, as shown in ref. [24], from the symmetry of the Grassmannian $G(N, N+K) \simeq G(K, N+K)$.

3.1 Level-rank duality of the $\hat{sp}(n)_k$ WZW model

We note that the level-rank duality of the $\hat{u}(N)_{K,N(K+N)}$ WZW model is very similar to level-rank duality of the $\hat{sp}(n)_k$ WZW model [13], where $n = \text{rank } \mathfrak{sp}(n)$. In the case of $\hat{sp}(n)_k$, integrable highest-weight representations are characterized by Young tableaux R with no more than n rows and k columns, with the level-rank dual representation given by the transpose of R . The relations [13] between the central charges and conformal weights of $\hat{sp}(n)_k$ and $\hat{sp}(k)_n$ representations

$$c + \tilde{c} = 2nk, \quad h_R + \tilde{h}_{\tilde{R}} = \frac{1}{2}r \tag{3.11}$$

and between the modular transformation matrices

$$\tilde{S}_{\tilde{R}\tilde{R}'} = S_{RR'}, \quad \tilde{T}_{\tilde{R}\tilde{R}} = (-)^r e^{-\pi i nk / 6} T_{RR}^* \tag{3.12}$$

are quite similar to those derived above for $\hat{u}(N)_{K,N(K+N)}$.

⁶The modular matrix S here differs from that given in ref. [13] by complex conjugation.

4. Chern-Simons theory and level-rank duality

Because of the close connection between WZW models in two dimensions and Chern-Simons gauge theory in three dimensions [30] (see ref. [31] for a review), level-rank duality of WZW models has implications for Chern-Simons theory as well [13, 19]–[22].

Since Chern-Simons theory is a topological field theory, its observables are topological invariants of the 3-manifold \mathcal{M} on which the theory is defined. The partition function depends only on \mathcal{M} , the gauge group G , the Chern-Simons coupling K , and a choice of framing of the manifold [30]. Other gauge-invariant observables of this theory are expectation values of path-ordered integrals $\text{Tr}_R P \exp(\oint_{\mathcal{K}} A)$ around a closed path (knot) \mathcal{K} in \mathcal{M} , with the trace taken in some irreducible representation R of G . More generally, one may consider Wilson lines on a link comprising several, possibly interconnected, knots, each associated with its own representation. Expectation values of these Wilson lines (together with a choice of framing) are topological invariants of the knots and links in \mathcal{M} , and are related to well-known knot polynomials.

Observables of the Chern-Simons theory may be expressed in terms of quantities of the related $\widehat{\mathfrak{g}}_K$ WZW model. For example, consider Chern-Simons theory on the manifold $\mathcal{M} = \Sigma_g \times S^1$, where Σ_g is a genus g Riemann surface. The expectation value of n vertical lines (i.e. wrapped around S^1) in $\Sigma_g \times S^1$, carrying representations R_i , is [30, 32, 33, 9]

$$W_{R_1 \dots R_n}[\Sigma_g \times S^1, G, K] = \sum_R S_{0R}^{2-n-2g} \prod_{i=1}^n S_{RR_i} \quad (4.1)$$

where $S_{RR'}$ is the modular transformation matrix of the $\widehat{\mathfrak{g}}_K$ WZW model, the sum is over integrable representations of $\widehat{\mathfrak{g}}_K$, and 0 denotes the identity representation. Equation (4.1) will serve as a building block for observables of Chern-Simons theory on manifolds other than $\Sigma_g \times S^1$.

4.1 Chern-Simons theory on Seifert manifolds

More generally, one may consider [18, 7, 9] Chern-Simons theory on a Seifert manifold $\mathcal{M}_{(g,p)}$, a circle bundle over Σ_g with first Chern class p . (The special case $\mathcal{M}_{(0,p)}$ is the lens space S^3/\mathbf{Z}_p , a circle bundle over S^2 .) The Seifert manifold $\mathcal{M}_{(g,p)}$ may be obtained from $\Sigma_g \times S^1$ by surgery, though different choices of surgery yield different framings of $\mathcal{M}_{(g,p)}$. Wilson line expectation values on $\mathcal{M}_{(g,p)}$ may be computed by the methods of ref. [30]. The link consisting of $n - 1$ circles carrying representations R_2, \dots, R_n all linked to a single circle carrying representation R_1 has expectation value [9]

$$\begin{aligned} W_{R_1 \dots R_n}[\mathcal{M}_{(g,p)}, G, K] &= \sum_R K_{R_1 R}^{(p)} W_{RR_2 \dots R_n}[\Sigma_g \times S^1, G, K] \\ &= \sum_R (K^{(p)} S)_{R_1 R} S_{0R}^{2-n-2g} \prod_{i=2}^n S_{RR_i} \end{aligned} \quad (4.2)$$

where the line carrying R_1 is in the tubular neighborhood where surgery occurs, but the other lines are not, and where $K^{(p)}$ is a matrix that depends on the framing of the manifold. For $p = 0$ (the trivial circle bundle), $K^{(0)} = 1$, and eq. (4.2) reduces to eq. (4.1).

When $g = 0$ and $p = 1$, the Seifert manifold is just S^3 . For the canonical framing of S^3 , the matrix $K^{(1)}$ is equal to the modular transformation matrix S [30], so that eq. (4.2) with $n = 2$ yields the Hopf link on S^3

$$W_{R_1 R_2}[S^3, G, K] = S_{\bar{R}_1 R_2} = S_{R_1 \bar{R}_2} = S_{R_1 R_2}^* \quad (\text{canonical framing}) \quad (4.3)$$

where \bar{R} is the representation conjugate to R , and we have used $S^2 = C$ (where C is the charge conjugation matrix), $C^2 = 1$, and $S^{-1} = S^\dagger = S^*$.

A different choice of framing for $\mathcal{M}_{(g,p)}$, called Seifert framing, was considered in refs. [18, 7, 9]. For this framing, one has $K^{(p)} = (TST)^p$ [9]. Using the relation $(ST)^3 = S^2$, one can rewrite $K^{(p)} = (ST^{-1}S^{-1})^p = ST^{-p}S^{-1}$. In Seifert framing, the expectation value (4.2) on $\mathcal{M}_{(g,p)}$ becomes

$$W_{R_1 \dots R_n}[\mathcal{M}_{(g,p)}, G, K] = \sum_R T_{RR}^{-p} S_{0R}^{2-n-2g} \prod_{i=1}^n S_{RR_i} \quad (\text{Seifert framing}). \quad (4.4)$$

Note that, in Seifert framing, the Hopf link expectation value on the three-sphere

$$W_{R_1 R_2}[S^3, G, K] = (ST^{-1}S)_{R_1 R_2} = (TSTC)_{R_1 R_2} = T_{R_1 R_1} S_{R_1 \bar{R}_2} T_{\bar{R}_2 R_2} \quad (\text{Seifert framing}) \quad (4.5)$$

differs from the result in canonical framing (4.3) by the representation-dependent phase factors in T .

4.2 Level-rank duality of $SU(N)$ Chern-Simons theory on S^3

We now review level-rank duality of observables of the $SU(N)$ Chern-Simons theory on S^3 [13, 19]–[22]. These results can be obtained by using the expression (4.3) for the Hopf link expectation value on S^3 (with canonical framing) together with the relation (3.8) between the modular transformation matrices of $\widehat{\text{su}}(N)_K$ and $\widehat{\text{su}}(K)_N$. For example, the partition functions of level- K $SU(N)$ Chern-Simons theory and level- N $SU(K)$ Chern-Simons theory on S^3 (with canonical framing) are related by [19]

$$Z_{\text{CS}}[S^3, SU(N), K] = \sqrt{\frac{K}{N}} Z_{\text{CS}}[S^3, SU(K), N]. \quad (4.6)$$

Also, the expectation values of unknots on S^3 (normalized by the partition function) of level-rank-dual representations are equal

$$\frac{W_R[S^3, SU(N), K]}{Z_{\text{CS}}[S^3, SU(N), K]} = \frac{W_{\bar{R}}[S^3, SU(K), N]}{Z_{\text{CS}}[S^3, SU(K), N]} \quad (4.7)$$

and more generally, one can show by using skein relations that the normalized expectation value of a knot taken in representation R of $SU(N)$ is equal to the expectation value of the mirror-image knot taken in representation \tilde{R} of $SU(K)$ [19, 20]. Finally, normalized expectation values of Hopf links of level-rank-dual theories are related by [13]

$$\frac{W_{R_1 R_2}[S^3, SU(N), K]}{Z_{\text{CS}}[S^3, SU(N), K]} = e^{-2\pi i r_1 r_2 / NK} \frac{W_{\tilde{R}_1 \tilde{R}_2}^*[S^3, SU(K), N]}{Z_{\text{CS}}[S^3, SU(K), N]} \quad (4.8)$$

where complex conjugation corresponds to the mirror-image Hopf link. Note that, for $SU(N)$, duality only holds because cominimally-equivalent unknots have identical expectation values on S^3 .

4.3 Level-rank duality of $U(N)$ and $Sp(n)$ CS theory on Seifert manifolds

Level-rank duality of $SU(N)$ Chern-Simons observables does *not* generally hold on manifolds other than S^3 , because the level-rank-dual map for $\widehat{\mathfrak{su}}(N)_K$ is between cominimal equivalence classes. As the simplest example, consider the unknot expectation value on $\mathcal{M}_{(0,0)} = S^2 \times S^1$:

$$W_R[S^2 \times S^1, SU(N), K] = C_{0R}. \quad (4.9)$$

This vanishes unless R is the identity representation, but if R corresponds to the cominimal representation $\sigma(0)$, the level-rank-dual unknot does not vanish because \widetilde{R} is equivalent to the identity representation.

Unlike $SU(N)$ Chern-Simons theory, $U(N)$ Chern-Simons theory on $\mathcal{M}_{(g,p)}$ *does* exhibit level-rank duality (for even p), as we will now see. This is because, as shown in section 3, level-rank duality of $U(N)$ WZW models involves a map between primary fields, not cominimal equivalence classes thereof. (As before, however, we must restrict ourselves to odd values of N and K .) By combining the link expectation value (4.4) on a Seifert manifold (in Seifert framing) with the level-rank-dual relationships (3.5) and (3.9) for modular transformation matrices of $\widehat{\mathfrak{u}}(N)_{K, N(K+N)}$, we find

$$\begin{aligned} W_{\mathcal{R}_1 \dots \mathcal{R}_n}[\mathcal{M}_{(g,p)}, U(N), K] &= \sum_{\mathcal{R}} T_{\mathcal{R}\mathcal{R}}^{-p} S_{0\mathcal{R}}^{2-n-2g} \prod_{i=1}^n S_{\mathcal{R}\mathcal{R}_i} \\ &= e^{\pi i p(KN+1)/12} \sum_{\widetilde{\mathcal{R}}} (-1)^{pQ} \widetilde{T}_{\widetilde{\mathcal{R}}\widetilde{\mathcal{R}}}^{*-p} \widetilde{S}_{0\widetilde{\mathcal{R}}}^{*2-n-2g} \prod_{i=1}^n \widetilde{S}_{\widetilde{\mathcal{R}}\widetilde{\mathcal{R}}_i}^*. \end{aligned} \quad (4.10)$$

When p is odd, the factor of $(-1)^Q$ in the relation (3.5) introduces an unwanted relative sign between the terms in the sum. For even p , however, the sign is absent, and we obtain the level-rank-dual relation

$$W_{\mathcal{R}_1 \dots \mathcal{R}_n}[\mathcal{M}_{(g,p)}, U(N), K] = e^{\pi i p(KN+1)/12} W_{\widetilde{\mathcal{R}}_1 \dots \widetilde{\mathcal{R}}_n}^*[\mathcal{M}_{(g,p)}, U(K), N] \quad (p \text{ even}) \quad (4.11)$$

between these link expectation values in $U(N)$ and $U(K)$ Chern-Simons theories on a Seifert manifold (in Seifert framing). The cases $n = 0$, $n = 1$, and $n = 2$ of eq. (4.11) yield level-rank-dual relationships between partition functions, unknot expectation values, and Hopf link expectation values on a Seifert manifold respectively.

A similar relation

$$W_{R_1 \dots R_m}[\mathcal{M}_{(g,p)}, Sp(n), k] = e^{\pi i p n k / 6} W_{\widetilde{R}_1 \dots \widetilde{R}_m}^*[\mathcal{M}_{(g,p)}, Sp(k), n] \quad (p \text{ even}) \quad (4.12)$$

holds between expectation values of $Sp(n)$ and $Sp(k)$ Chern-Simons theories on $\mathcal{M}_{(g,p)}$ (for p even), because level-rank duality of $\widehat{\mathfrak{sp}}(n)_k$ WZW models [13] is also a map between representations.

5. 2d $U(N)$ qYM theory and $N \leftrightarrow K$ duality

The two-dimensional q -deformed $U(N)$ Yang-Mills theory that arises [2, 3] from computations of bound states of D-branes on a fibration over a Riemann surface Σ_g has a close

connection to both $U(N)$ Chern-Simons theory on a Seifert manifold [3, 34, 5, 18, 7, 9] and to the $\widehat{u}(N)_{K,N(K+N)}$ WZW model. In particular, as we will show, when $q = e^{2\pi i/(N+K)}$, the $U(N)$ qYM partition function can be written as a sum over a finite subset of $U(N)$ representations. Consequently, the $U(N)$ qYM theory exhibits a level-rank-type duality under $N \leftrightarrow K$.

The partition function of the two-dimensional q -deformed $U(N)$ Yang-Mills theory on a genus g Riemann surface Σ_g is given (up to an overall normalization factor) by [3]

$$Z_{\text{qYM}}[U(N), q, p, \theta] \sim \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{2-2g} q^{-\frac{1}{2}pC_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})} \quad (5.1)$$

where the sum is over all representations $\mathcal{R} = (R, Q)$ of $U(N)$, whose Young tableaux have row lengths $\bar{\ell}_i$ obeying $-\infty < \bar{\ell}_N \leq \bar{\ell}_{N-1} \leq \dots \leq \bar{\ell}_1 < \infty$. In eq. (5.1), p is related to the coupling in the YM action,⁷ $C_2(\mathcal{R})$ is the quadratic Casimir (A.7), $C_1(\mathcal{R}) = Q = \sum_{i=1}^N \bar{\ell}_i$, and $\dim_q \mathcal{R}$ is the “quantum dimension” of the representation \mathcal{R}

$$\dim_q \mathcal{R} = \prod_{1 \leq i < j \leq N} \frac{[\bar{\ell}_i - \bar{\ell}_j + j - i]_q}{[j - i]_q} \quad \text{where} \quad [x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}. \quad (5.2)$$

Letting $m_i = \bar{\ell}_i - i \in \mathbb{Z}$, we may rewrite eq. (5.1) as

$$Z_{\text{qYM}}[U(N), q, p, \theta] \sim \sum_{m_N < \dots < m_1} \left(\prod_{1 \leq i < j \leq N} \frac{[m_i - m_j]_q}{[j - i]_q} \right)^{2-2g} \times \quad (5.3)$$

$$q^{-\frac{1}{2}p \sum_i [m_i + \frac{1}{2}(N+1)]^2 + \frac{1}{24}pN(N^2-1)} e^{i\theta \sum_i [m_i + \frac{1}{2}(N+1)]}.$$

The invariance of eq. (5.3) under permutations of the m_i allows us to extend the sum to all m_i (modulo a factor of $N!$), omitting the locus where two or more of the m_i are equal. The resulting expression is actually the result of a path integral derivation of the 2d qYM partition function [3], in which terms with $m_i = m_j$ correspond to singular points that give no contribution to the path integral [33].

Next, we show that when $q = e^{2\pi i/(N+K)}$, with K and p both integer-valued, the infinite sum over representations in (5.1) either vanishes or is proportional to a sum over a *finite* subset of $U(N)$ representations, namely those whose tableaux have no more than K columns.⁸ First, when $q = e^{2\pi i/(N+K)}$, one has $[x]_q = \sin(\frac{\pi x}{N+K})/\sin(\frac{\pi}{N+K})$, so that terms in which two m_i 's differ by a multiple of $N + K$ are singular, and should be omitted from the sum (5.3) [33]. The resulting partition function is

$$Z_{\text{qYM}}[U(N), e^{2\pi i/(N+K)}, p, \theta] \sim \sum'_{m_i \in \mathbb{Z}} \left(\prod_{1 \leq i < j \leq N} \frac{[m_i - m_j]_q}{[j - i]_q} \right)^{2-2g} \times \quad (5.4)$$

$$\exp \left\{ -\frac{\pi i p}{N+K} \sum_i \left[m_i + \frac{1}{2}(N+1) \right]^2 + \frac{\pi i p N(N^2-1)}{12(N+K)} + i\theta \sum_i \left[m_i + \frac{1}{2}(N+1) \right] \right\}$$

⁷Our p is opposite in sign from that in refs. [3, 7] and the same as that in ref. [9].

⁸When K is not an integer, there is no truncation of representations [3].

where the prime on the sum denotes the omission of the locus $m_i = m_j \bmod N + K$ for all $i \neq j$.

Next, observe that under $m_i \rightarrow m_i + N + K$ (with the other m_j fixed), the summand is multiplied by $e^{-i\pi p(K+1) + i(N+K)\theta}$, provided $p \in \mathbb{Z}$. If $\theta \neq \frac{\pi p(K+1)}{N+K} \bmod \frac{2\pi}{N+K}$, the resulting infinite sum over phases causes the partition function to vanish. On the other hand, if $\theta = \frac{\pi p(K+1)}{N+K} \bmod \frac{2\pi}{N+K}$, the summand is invariant under $m_i \rightarrow m_i + N + K$, and so the sum may be restricted (modulo an infinite factor) to the hypercube $-N \leq m_i < K - 1$, still omitting terms in which any two m_i 's are equal. Finally, due to the invariance under permutations of m_i , the sum may be restricted to $-N \leq m_N < \dots < m_1 \leq K - 1$, which corresponds to $0 \leq \bar{\ell}_N \leq \bar{\ell}_{N-1} \leq \dots \leq \bar{\ell}_1 \leq K$. Hence

$$Z_{\text{qYM}} \left[\text{U}(N), e^{2\pi i/(N+K)}, p, \theta \right] = \sum_{0 \leq \bar{\ell}_N \leq \dots \leq \bar{\ell}_1 \leq K} (\dim_q \mathcal{R})^{2-2g} q^{-\frac{1}{2}pC_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})} \quad (5.5)$$

provided $p \in \mathbb{Z}$ and $\theta = \frac{\pi p(K+1)}{N+K} \bmod \frac{2\pi}{N+K}$. To summarize, when these conditions hold, the 2d $\text{U}(N)$ qYM partition function (5.1) is proportional to eq. (5.5), which has the same form but is restricted to Young tableaux with no more than K columns. This result holds for both even and odd K ; the allowed set of Young tableaux corresponds in the latter case to primary fields of $\widehat{\text{u}}(N)_{K, N(K+N)}$.

We note that the analogous truncation does *not* occur in the case of the $\text{SU}(N)$ qYM partition function even for $q = e^{2\pi i/(N+K)}$; the extra term $-r^2/N$ in the $\text{SU}(N)$ Casimir (A.2) ruins the invariance of the summand under $m_i \rightarrow m_i + N + K$ in that case.

As we saw in section 2, when K is odd, the set of Young tableaux with up to N rows and K columns precisely corresponds to the set of primary fields of $\widehat{\text{u}}(N)_{K, N(K+N)}$. Moreover, the quantities in eq. (5.5) may be expressed in terms of modular transformation matrices (2.15) and (2.16) of $\widehat{\text{u}}(N)_{K, N(K+N)}$, namely $q^{\frac{1}{2}C_2(\mathcal{R})} = T_{\mathcal{R}\mathcal{R}}/T_{00}$ and

$$\begin{aligned} \dim_q \mathcal{R} &= \prod_{1 \leq i < j \leq N} \frac{[\bar{\ell}_i - \bar{\ell}_j + j - i]_q}{[j - i]_q} = \prod_{1 \leq i < j \leq N} \frac{[\ell_i - \ell_j + j - i]_q}{[j - i]_q} = \dim_q R = \left(\frac{S_{0R}}{S_{00}} \right)_{\widehat{\text{u}}(N)_K} \\ &= \left(\frac{S_{0\mathcal{R}}}{S_{00}} \right)_{\widehat{\text{u}}(N)_{K, N(K+N)}} \end{aligned} \quad (5.6)$$

where $\ell_i = \bar{\ell}_i - \bar{\ell}_N$ are the row lengths of the $\text{SU}(N)$ representation R , and we have used eq. (2.15). Hence, when K is odd and $\theta = 2\pi t/(N + K)$ with $t \in \mathbb{Z}$, the $\text{U}(N)$ qYM partition function (5.5) may be expressed as

$$Z_{\text{qYM}} \left[\text{U}(N), e^{2\pi i/(N+K)}, p, \frac{2\pi t}{N+K} \right] = T_{00}^p S_{00}^{2g-2} \sum_{\mathcal{R} \text{ integrable}} S_{0\mathcal{R}}^{2-2g} T_{\mathcal{R}\mathcal{R}}^{-p} e^{2\pi i t Q/(N+K)} \quad (K \text{ odd}). \quad (5.7)$$

Finally, as observed in refs. [3, 5, 18, 7, 9], the $\text{U}(N)$ qYM partition function (for $q = e^{2\pi i/(N+K)}$ and $\theta = 0$) may be expressed in terms of the level- K $\text{U}(N)$ Chern-Simons partition function on $\mathcal{M}_{(g,p)}$ with Seifert framing (4.4)

$$Z_{\text{qYM}}[\text{U}(N), e^{2\pi i/(N+K)}, p, 0] = T_{00}^p S_{00}^{2g-2} Z_{\text{CS}}[\mathcal{M}_{(g,p)}, \text{U}(N), K] \quad (K \text{ odd}). \quad (5.8)$$

5.1 $N \leftrightarrow K$ duality of the 2d $U(N)$ qYM theory

With the 2d $U(N)$ qYM partition function in the form (5.7), duality under $N \leftrightarrow K$ becomes manifest by using eqs. (3.5) and (3.9):

$$Z_{\text{qYM}} \left[U(N), e^{2\pi i/(N+K)}, p, \frac{2\pi t}{N+K} \right] = Z_{\text{qYM}}^* \left[U(K), e^{2\pi i/(N+K)}, p, \frac{2\pi t}{N+K} + \pi p \right] \quad (N, K \text{ odd}). \quad (5.9)$$

That is, the 2d $U(N)$ qYM partition function is the complex conjugate of the 2d $U(K)$ qYM partition function, provided $q = e^{2\pi i/(N+K)}$ and $\theta = 0 \pmod{2\pi/(N+K)}$, with N and K both odd. When p is odd, θ must be shifted in order to compensate for the representation-dependent minus sign in (3.5), but when p is even, no shift is necessary.

It has been shown [7] that a certain class of Wilson line observables in qYM theory (for $q = e^{2\pi i/(N+K)}$) are proportional to the expectation values of links of Chern-Simons theory on $\mathcal{M}_{(g,p)}$. For K odd, these Wilson line observables in $U(N)$ qYM theory may be expressed in terms of modular transformation matrices of $\widehat{u}(N)_{K,N(K+N)}$:

$$W_{\mathcal{R}_1 \dots \mathcal{R}_n} \left[U(N), e^{2\pi i/(N+K)}, p, \frac{2\pi t}{N+K} \right] \sim \sum_{\mathcal{R} \text{ integrable}} S_{0\mathcal{R}}^{2-2g-n} T_{\mathcal{R}\mathcal{R}}^{-p} e^{2\pi i t Q/(N+K)} \prod_{i=1}^n S_{\mathcal{R}\mathcal{R}_i} \quad (K \text{ odd}). \quad (5.10)$$

(The sum over all $u(N)$ representations restricts to a sum over a finite subset of representations under the same conditions as before.) Once again, eqs. (3.5) and (3.9) can be used to show that these observables, like the partition function, exhibit $N \leftrightarrow K$ duality (5.9).

6. Conclusions

In this paper we have examined three different exactly-soluble gauge theories in two and three dimensions with gauge group $U(N)$, and the $N \leftrightarrow K$ dualities that these theories exhibit.

First, we examined the $U(N)$ WZW model, whose affine Lie algebra is $\widehat{u}(N)_{K,N(K+N)} \equiv [\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{N(K+N)}] / \mathbf{Z}_N$. The primary fields of this theory are given by equivalence classes of representations of $\widehat{\mathfrak{su}}(N)_K \times \widehat{\mathfrak{u}}(1)_{N(K+N)}$. These equivalence classes are only defined when K is odd, and are in one-to-one correspondence with Young tableaux with up to N rows and K columns. By explicitly considering the affine characters of the primary fields, we derived the modular transformation matrices of the $\widehat{u}(N)_{K,N(K+N)}$ WZW model. We then showed that the $\widehat{u}(N)_{K,N(K+N)}$ and $\widehat{u}(K)_{N,K(K+N)}$ WZW models are level-rank dual (when N and K are odd), with a one-to-one correspondence between the primary fields of these theories. This is simpler than the level-rank duality between $\widehat{\mathfrak{su}}(N)_K$ and $\widehat{\mathfrak{su}}(K)_N$ WZW models, where the correspondence is between the simple-current orbits of primary fields.

Next we considered $U(N)$ Chern-Simons theories on Seifert manifolds: circle bundles over Σ_g with first Chern class p . The partition function and knot and link expectation values of this theory were expressed (using surgery) in terms of the $\widehat{u}(N)_{K,N(K+N)}$ modular transformation matrices. The level-rank duality of the latter was used to show the $N \leftrightarrow K$

duality of the Chern-Simons observables, for N and K odd, and p even. A similar result was shown for $\text{Sp}(n)$ Chern-Simons theory on Seifert manifolds.

Finally, we considered two-dimensional q -deformed $U(N)$ Yang-Mills theory. We showed that when $q = e^{2\pi i/(N+K)}$, the partition function may be expressed in terms of a finite sum over $\mathfrak{u}(N)$ representations. This result, together with the level-rank duality of the $\widehat{\mathfrak{u}}(N)_{K,N(K+N)}$ WZW model, was then used to show that the 2d $U(N)$ qYM theory exhibits a level-rank-type duality under $N \leftrightarrow K$ (for N and K both odd), provided $q = e^{2\pi i/(N+K)}$ and $\theta = 0 \pmod{2\pi/(N+K)}$.

The $N \leftrightarrow K$ duality of q -deformed YM theory discussed in this paper is not immediately relevant to the counting of bound states of BPS black holes in string theory because in that case the deformation parameter q is real, not a pure phase. On the other hand, level-rank of Chern-Simons observables could have important implications for large- N dualities of Chern-Simons theories since it relates the large- N limit to the large- K limit of finite N observables.

A. Representations of finite-dimensional and affine Lie algebras

In this appendix, we summarize various results for finite-dimensional and affine Lie algebras needed in the main part of the paper.

A.1 Representations of $\mathfrak{u}(N)$

The $\mathfrak{su}(N)$ generators T^a ($a = 1, \dots, N^2 - 1$) satisfy $[T^a, T^b] = if^{ab}_c T^c$, and the Killing form is defined as $g^{ab} = \text{Tr}(T^a_{\square} T^b_{\square})$, with the trace taken in the defining representation of $\mathfrak{su}(N)$. (With this normalization of g^{ab} , the roots of $\mathfrak{su}(N)$ have length two.) A representation R of $\mathfrak{su}(N)$ is characterized by a Young tableau with row lengths ℓ_i ($i = 1, \dots, N - 1$) satisfying $\ell_1 \geq \ell_2 \geq \dots \geq \ell_{N-1} \geq 0$, and with number of boxes $r = \sum_{i=1}^{N-1} \ell_i$. The quadratic Casimir $C_2(R)$ of the $\mathfrak{su}(N)$ representation R is given by

$$\sum_{a,b=1}^{N^2-1} g_{ab} T_R^a T_R^b = C_2(R) \mathbf{1}_{\dim R} \tag{A.1}$$

$$C_2(R) = Nr + T(R) - \frac{r^2}{N} \tag{A.2}$$

$$T(R) = \sum_{i=1}^{N-1} \ell_i(\ell_i - 2i + 1) \tag{A.3}$$

where g_{ab} is the inverse of g^{ab} .

Representations of $\mathfrak{u}(N) = [\mathfrak{su}(N) \times \mathfrak{u}(1)]/\mathbf{Z}_N$ are those representations (R, Q) of $\mathfrak{su}(N) \times \mathfrak{u}(1)$ that satisfy $Q = r \pmod{N}$. Here, Q is the eigenvalue of the $\mathfrak{u}(1)$ generator T^0 , which is normalized so that the defining representation has $Q = 1$, implying that

$g^{00} = N$. The quadratic Casimir of the $u(N)$ representation (R, Q) is then

$$\sum_{a,b=0}^{N^2-1} g_{ab} T_{(R,Q)}^a T_{(R,Q)}^b = C_2(R, Q) \mathbf{1}_{\dim R} \quad (\text{A.4})$$

$$C_2(R, Q) = C_2(R) + \frac{Q^2}{N} = Nr + T(R) + 2sr + Ns^2 \quad (\text{A.5})$$

$$Q = r + Ns. \quad (\text{A.6})$$

One can characterize the $u(N)$ representation (R, Q) by an extended Young tableau \mathcal{R} , obtained by prepending s columns of N boxes to the Young tableau for R , giving a tableau with row lengths $\bar{\ell}_i = \ell_i + s$ ($i = 1, \dots, N-1$) and $\bar{\ell}_N = s$, and with number of boxes $Q = \sum_{i=1}^N \bar{\ell}_i$. Since $s \in \mathbf{Z}$, the $\bar{\ell}_i$ may be negative, in which case one can represent the tableau using “anti-boxes”; cf. appendix D of ref. [6]. The quadratic Casimir (A.5) may be re-expressed as

$$C_2(\mathcal{R}) = C_2(R, Q) = NQ + T(\mathcal{R}) \quad (\text{A.7})$$

$$T(\mathcal{R}) = \sum_{i=1}^N \bar{\ell}_i(\bar{\ell}_i - 2i + 1). \quad (\text{A.8})$$

A.2 Representations of $\widehat{su}(N)_K$ and $\widehat{u}(1)_{K'}$

Next, we review the representations of the affine Lie algebras $\widehat{su}(N)_K$ and $\widehat{u}(1)_{K'}$.

The generators of $\widehat{su}(N)_K$ satisfy

$$[J_m^a, J_n^b] = if_c^{ab} J_{m+n}^c + Kg^{ab} m\delta_{m+n} \quad (\text{A.9})$$

where the level, K , is integer-valued when g^{ab} is normalized as above. Integrable highest-weight representations of $\widehat{su}(N)_K$ are characterized by $su(N)$ representations R whose Young tableaux have $\ell_1 \leq K$. This representation has conformal weight

$$h_R = \frac{\frac{1}{2}C_2(R)}{K + N} \quad (\text{A.10})$$

and affine character

$$\chi_R(\tau) = \text{Tr} \left(q^{L_0 - c/24} \right) = q^{h_R - c/24} (\dim R + \dots), \quad q = e^{2\pi i \tau} \quad (\text{A.11})$$

where the central charge is $c = K(N^2 - 1)/(K + N)$. The characters of integrable highest-weight representations transform into one another under the modular transformation $\tau \rightarrow -1/\tau$:

$$\chi_R(-1/\tau) = \sum_{R'} S_{RR'} \chi_{R'}(\tau) \quad (\text{A.12})$$

and into themselves, up to a phase, under $\tau \rightarrow \tau + 1$:

$$\chi_R(\tau + 1) = \sum_{R'} T_{RR'} \chi_{R'}(\tau), \quad T_{RR'} = e^{2\pi i(h_R - c/24)} \delta_{RR'}. \quad (\text{A.13})$$

Integrable highest-weight representations of $\widehat{\mathfrak{su}}(N)_K$ may be grouped into cominimal equivalence classes (or simple-current orbits) of order N (or some divisor of N). These classes are generated by the simple current σ , which rotates the Dynkin indices of the Dynkin diagram of $\widehat{\mathfrak{su}}(N)_K$. In terms of Young tableaux, $\sigma(R)$ is obtained by adding a row of K boxes to the top of the Young tableau of R , and stripping off any columns of length N that may result. The conformal weights of cominimally-equivalent representations are related by

$$h_{\sigma(R)} = h_R + \frac{NK - K - 2r}{2N} \tag{A.14}$$

and the modular transformation matrix⁹ obeys

$$S_{\sigma^n(R)R'} = e^{2\pi i nr'/N} S_{RR'} \tag{A.15}$$

where r' is the number of boxes of the tableau representing R' .

The generators of $\widehat{\mathfrak{u}}(1)_{K'}$ satisfy

$$[J_m^0, J_n^0] = K' m \delta_{m+n}. \tag{A.16}$$

Highest-weight representations of this algebra are labelled by their J_0^0 eigenvalue Q , and have conformal weights

$$h'_Q = \frac{\frac{1}{2}Q^2}{K'} \tag{A.17}$$

and affine characters

$$\chi'_Q(\tau) = \frac{q^{h'_Q}}{\eta(\tau)} = q^{h'_Q - \frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)}. \tag{A.18}$$

References

- [1] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev. D* **70** (2004) 106007 [[hep-th/0405146](#)].
- [2] C. Vafa, *Two dimensional Yang-Mills, black holes and topological strings*, [hep-th/0406058](#).
- [3] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, *Black holes, q-deformed 2d Yang-Mills and non-perturbative topological strings*, *Nucl. Phys. B* **715** (2005) 304 [[hep-th/0411280](#)].
- [4] E. Buffenoir and P. Roche, *Two-dimensional lattice gauge theory based on a quantum group*, *Commun. Math. Phys.* **170** (1995) 669 [[hep-th/9405126](#)];
C. Klimčík, *The formulae of Kontsevich and Verlinde from the perspective of the Drinfeld double*, *Commun. Math. Phys.* **217** (2001) 203 [[hep-th/9911239](#)].
- [5] S. de Haro, *Chern-Simons theory, 2d Yang-Mills and Lie algebra wanderers*, *Nucl. Phys. B* **730** (2005) 312 [[hep-th/0412110](#)];
S. de Haro and M. Tierz, *Discrete and oscillatory matrix models in Chern-Simons theory*, *Nucl. Phys. B* **731** (2005) 225 [[hep-th/0501123](#)].
- [6] M. Aganagic, A. Neitzke and C. Vafa, *BPS microstates and the open topological string wave function*, [hep-th/0504054](#).

⁹The modular matrix S here differs from that given in ref. [13] by complex conjugation.

- [7] S. de Haro, *A note on knot invariants and q -deformed 2d Yang-Mills*, *Phys. Lett. B* **634** (2006) 78 [[hep-th/0509167](#)].
- [8] X. Arsiwalla, R. Boels, M. Marino and A. Sinkovics, *Phase transitions in q -deformed 2d Yang-Mills theory and topological strings*, *Phys. Rev. D* **73** (2006) 026005 [[hep-th/0509002](#)];
 D. Jafferis and J. Marsano, *A DK phase transition in q -deformed Yang-Mills on S^2 and topological strings*, [hep-th/0509004](#);
 N. Caporaso et al., *Topological strings and large- N phase transitions. I: nonchiral expansion of q -deformed Yang-Mills theory*, *JHEP* **01** (2006) 035 [[hep-th/0509041](#)]; *Topological strings and large- N phase transitions. II: chiral expansion of q -deformed Yang-Mills theory*, *JHEP* **01** (2006) 036 [[hep-th/0511043](#)];
 M. Aganagic, D. Jafferis and N. Saulina, *Branes, black holes and topological strings on toric Calabi-Yau manifolds*, *JHEP* **12** (2006) 018 [[hep-th/0512245](#)];
 S. de Haro, S. Ramgoolam and A. Torrielli, *Large- N expansion of q -deformed two-dimensional Yang-Mills theory and Hecke algebras*, [hep-th/0603056](#);
 N. Caporaso, L. Griguolo, M. Marino, S. Pasquetti and D. Seminara, *Phase transitions, double-scaling limit and topological strings*, *Phys. Rev. D* **75** (2007) 046004 [[hep-th/0606120](#)];
 L. Griguolo, D. Seminara, R.J. Szabo and A. Tanzini, *Black holes, instanton counting on toric singularities and q -deformed two-dimensional Yang-Mills theory*, *Nucl. Phys. B* **772** (2007) 1 [[hep-th/0610155](#)].
- [9] M. Blau and G. Thompson, *Chern-Simons theory on S^1 -bundles: abelianisation and q -deformed Yang-Mills theory*, *JHEP* **05** (2006) 003 [[hep-th/0601068](#)].
- [10] N. Caporaso et al., *Topological strings, two-dimensional Yang-Mills theory and Chern-Simons theory on torus bundles*, [hep-th/0609129](#).
- [11] S.G. Naculich and H.J. Schnitzer, *Duality between $SU(N)_k$ and $SU(K)_N$ WZW models*, *Nucl. Phys. B* **347** (1990) 687; *Duality relations between $SU(N)_k$ and $SU(K)_N$ WZW models and their braid matrices*, *Phys. Lett. B* **244** (1990) 235;
 J. Fuchs and P. van Driel, *Some symmetries of quantum dimensions*, *J. Math. Phys.* **31** (1990) 1770.
- [12] D. Altschuler, M. Bauer and C. Itzykson, *The branching rules of conformal embeddings*, *Commun. Math. Phys.* **132** (1990) 349;
 H. Saleur and D. Altschuler, *Level rank duality in quantum groups*, *Nucl. Phys. B* **354** (1991) 579;
 A. Kuniba and T. Nakanishi, “Level rank duality in fusion RSOS models.” In *Proceedings of the International Colloquium on Modern Quantum Field Theory*, Bombay, January 1990, World Scientific, Singapore (1991);
 M.A. Walton, *Conformal branching rules and modular invariants*, *Nucl. Phys. B* **322** (1989) 775.
- [13] E.J. Mlawer, S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Group level duality of WZW fusion coefficients and Chern-Simons link observables*, *Nucl. Phys. B* **352** (1991) 863.
- [14] T. Nakanishi and A. Tsuchiya, *Level rank duality of WZW models in conformal field theory*, *Commun. Math. Phys.* **144** (1992) 351.
- [15] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [[hep-th/9811131](#)];

- H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys. B* **577** (2000) 419 [[hep-th/9912123](#)].
- [16] M. Marino and C. Vafa, *Framed knots at large- N* , [hep-th/0108064](#).
- [17] M.R. Douglas, *Chern-Simons-Witten theory as a topological Fermi liquid*, [hep-th/9403119](#).
- [18] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, *J. Diff. Geom.* **70** (2005) 183 [[hep-th/0503126](#)].
- [19] M. Camperi, F. Levstein and G. Zemba, *The large- N limit of Chern-Simons gauge theory*, *Phys. Lett. B* **247** (1990) 549.
- [20] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Group level duality in WZW models and Chern-Simons theory*, *Phys. Lett. B* **246** (1990) 417.
- [21] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Simple current symmetries, rank level duality and linear skein relations for Chern-Simons graphs*, *Nucl. Phys. B* **394** (1993) 445 [[hep-th/9205082](#)].
- [22] J.M.F. Labastida, M. Marino and C. Vafa, *Knots, links and branes at large- N* , *JHEP* **11** (2000) 007 [[hep-th/0010102](#)].
- [23] D. Gepner, *Foundations of rational quantum field theory. 1*, [hep-th/9211100](#).
- [24] E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, [hep-th/9312104](#).
- [25] G.W. Moore and N. Seiberg, *Taming the conformal zoo*, *Phys. Lett. B* **220** (1989) 422.
- [26] D. Gepner and E. Witten, *String theory on group manifolds*, *Nucl. Phys. B* **278** (1986) 493.
- [27] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, *Commun. Math. Phys.* **129** (1990) 393.
- [28] I. Frenkel, *Representations of affine Lie algebras, Hecke modular forms, and Korteweg-de Vries type equations*, in *Lie algebras and related topics*, Lecture Notes in Mathematics, no. 933, D. Winter, ed. Springer-Verlag, Berlin (1982).
- [29] E.P. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, *Nucl. Phys. B* **300** (1988) 360.
- [30] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.
- [31] M. Marino, *Chern-Simons theory and topological strings*, *Rev. Mod. Phys.* **77** (2005) 675 [[hep-th/0406005](#)].
- [32] M. Blau and G. Thompson, *Derivation of the Verlinde formula from Chern-Simons theory and the G/G model*, *Nucl. Phys. B* **408** (1993) 345 [[hep-th/9305010](#)].
- [33] M. Blau and G. Thompson, *Lectures on 2-d gauge theories: topological aspects and path integral techniques*, [hep-th/9310144](#).
- [34] S. de Haro and M. Tierz, *Brownian motion, Chern-Simons theory and 2d Yang-Mills*, *Phys. Lett. B* **601** (2004) 201 [[hep-th/0406093](#)];
S. de Haro, *Chern-Simons theory in lens spaces from 2d Yang-Mills on the cylinder*, *JHEP* **08** (2004) 041 [[hep-th/0407139](#)].