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Level-rank duality of untwisted and twisted D-branes of the $\widehat{\mathrm{so}}(N)_K$ WZW model ☆

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Abstract

We analyze the level-rank duality of untwisted and ε-twisted D-branes of the $\widehat{\mathrm{so}}(N)_K$ WZW model. Untwisted D-branes of $\widehat{\mathrm{so}}(N)_K$ are characterized by integrable tensor and spinor representations of $\widehat{\mathrm{so}}(N)_K$. Level-rank duality maps untwisted $\widehat{\mathrm{so}}(N)_K$ D-branes corresponding to (equivalence classes of) tensor representations onto those of $\widehat{\mathrm{so}}(K)_N$. The ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ are characterized by (a subset of) integrable tensor and spinor representations of $\widehat{\mathrm{so}}(2n-1)_{2k+1}$. Level-rank duality maps spinor ε-twisted $\widehat{\mathrm{so}}(2n)_{2k}$ D-branes onto those of $\widehat{\mathrm{so}}(2k)_{2n}$. For both untwisted and ε-twisted D-branes, we prove that the spectrum of an open string ending on these D-branes is isomorphic to the spectrum of an open string ending on the level-rank-dual D-branes.

1. Introduction

It has long been known that the modular transformation matrix and fusion algebra of the Wess–Zumino–Witten (WZW) model with affine Lie algebra $\widehat{\mathrm{su}}(N)_K$ are closely related to those of the WZW model with affine Lie algebra $\widehat{\mathrm{su}}(K)_N$ (level-rank duality) [1–3]. Similar dualities have been shown for WZW models with affine Lie algebras related by $\widehat{\mathrm{sp}}(n)_k \leftrightarrow \widehat{\mathrm{sp}}(k)_n$ and $\widehat{\mathrm{so}}(N)_K \leftrightarrow \widehat{\mathrm{so}}(K)_N$ [2,3], and also for $\hat{\mathrm{u}}(N)_{K,N(K+N)} \leftrightarrow \hat{\mathrm{u}}(K)_{N,K(K+N)}$ [4].

More recently, it has been shown [5–7] that the untwisted and twisted D-branes in the boundary $\widehat{\mathrm{su}}(N)_K$ WZW model [8–25] respect level-rank duality; that is, there exists a one-to-one map

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between the (equivalence classes of) D-branes of $\widehat{\mathrm{su}}(N)_K$ and those of $\widehat{\mathrm{su}}(K)_N$. The open-string spectra associated with level-rank-dual D-branes are isomorphic, and the charges of level-rank-dual untwisted D-branes are equal (modulo sign), with a slightly more complicated relationship holding between the charges of twisted D-branes. Level-rank duality also holds for the D-branes of $\widehat{\mathrm{sp}}(n)_k$ [6].

In this paper, we continue the story by establishing the level-rank duality of the untwisted D-branes of $\widehat{\mathrm{so}}(N)_K$ and of the twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$. In this case, level-rank duality is partial and holds only for a subset of the D-branes of the theory. Moreover, in this case we find no simple relation between the charges of level-rank-dual D-branes.

We begin by summarizing our results. Untwisted D-branes of $\widehat{\mathrm{so}}(N)_K$ correspond to untwisted Cardy states $|\alpha\rangle\rangle_C$ (boundary states of the bulk WZW model), which are labelled by integrable highest-weight representations α (both tensors and spinors) of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(N)_K$. Only untwisted *tensor* D-branes exhibit level-rank duality,[1] and the duality is one-to-one between equivalence classes $[\alpha]$ of integrable tensor representations generated by the $_2$-automorphisms σ and (when N is even) ε of the $\widehat{\mathrm{so}}(N)_K$ algebra. (We denote by σ the simple current symmetry of $\widehat{\mathrm{so}}(N)_K$ that acts on the Dynkin indices of a representation by[2] $a_0 \leftrightarrow a_1$. We denote by ε the "chirality-flip" symmetry of $\widehat{\mathrm{so}}(2n)_K$ that acts on the Dynkin indices of a representation by $a_n \leftrightarrow a_{n-1}$. For $\widehat{\mathrm{so}}(2n+1)_K$, we define ε to be the identity.) The boundary state corresponding to the equivalence class $[\alpha]$ may be written as

$$\big|[\alpha]\big\rangle\!\big\rangle = \frac{1}{\sqrt{2}^{\,t(\alpha)-s(\alpha)+3}}\big[|\alpha\rangle\rangle_C + \big|\sigma(\alpha)\big\rangle\!\big\rangle_C + \big|\varepsilon(\alpha)\big\rangle\!\big\rangle_C + \big|\varepsilon\big(\sigma(\alpha)\big)\big\rangle\!\big\rangle_C\big], \tag{1.1}$$

where

$$s(\alpha) = \begin{cases} 1 & \text{if } \alpha \neq \varepsilon(\alpha), \\ 0 & \text{if } \alpha = \varepsilon(\alpha), \end{cases} \qquad t(\alpha) = \begin{cases} 1 & \text{if } \alpha = \sigma(\alpha), \\ 0 & \text{if } \alpha \neq \sigma(\alpha). \end{cases} \tag{1.2}$$

Equivalence classes $[\alpha]$ of integrable tensor representations of $\widehat{\mathrm{so}}(N)_K$ are characterized by Young tableaux with $N/2$ rows and $K/2$ columns. Level-rank duality acts by transposing these tableaux, inducing a one-to-one correspondence $[\alpha]$ $[\tilde{\alpha}]$ between equivalence classes of $\widehat{\mathrm{so}}(N)_K$ and $\widehat{\mathrm{so}}(K)_N$, and therefore between the untwisted D-branes that correspond to the boundary states (1.1). We show that the spectrum of representations carried by an open string stretched between untwisted $\widehat{\mathrm{so}}(N)_K$ D-branes corresponding to $[\alpha]$ and $[\beta]$ is isomorphic to that carried by an open string stretched between untwisted $\widehat{\mathrm{so}}(K)_N$ D-branes corresponding to $[\tilde{\alpha}]$ and $[\tilde{\beta}]$.

The $\widehat{\mathrm{so}}(2n)_K$ WZW model contains, in addition to untwisted D-branes, a class of D-branes twisted by the symmetry ε. These ε-twisted D-branes can be characterized [20] by (a subset of) the integrable highest-weight representations (both tensors and spinors) of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n-1)_{K+1}$. Only *spinor* ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ exhibit level-rank duality, which involves a one-to-one map α $\hat{\alpha}$ between the spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ and the spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2k)_{2n}$. We show that the spectrum of representations carried by an open string stretched between ε-twisted $\widehat{\mathrm{so}}(2n)_{2k}$ D-branes corresponding to α and β is isomorphic to that carried by an open string stretched between ε-twisted $\widehat{\mathrm{so}}(2k)_{2n}$ D-branes corresponding to $\hat{\alpha}$ and $\hat{\beta}$.

[1] Except for $\widehat{\mathrm{so}}(2n+1)_{2k+1}$, where a level-rank map can also be defined between equivalence classes of untwisted spinor D-branes.

[2] Except for $\widehat{\mathrm{so}}(4)_K$, in which case σ acts by $a_0 \leftrightarrow \min(a_1, a_2)$.

This paper is organized as follows. Section 2 briefly reviews the Ishibashi and Cardy states of the WZW model, and in Section 3, we characterize the integrable highest-weight representations of $\widehat{\mathrm{so}}(N)_K$. Section 4 describes the level-rank duality of the (equivalence classes of) untwisted D-branes corresponding to tensor representations of $\widehat{\mathrm{so}}(N)_K$ and to spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$. The ε-twisted Ishibashi and Cardy states of $\widehat{\mathrm{so}}(2n)_K$ are reviewed in Section 5, and in Section 6 we describe the level-rank duality of spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$.

2. Untwisted and twisted D-branes of WZW models

In this section, we briefly review some general aspects of untwisted and twisted D-branes of the WZW model and their relation to the Cardy and Ishibashi states of the closed-string sector, drawing on Refs. [9,10,18,20].

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is an untwisted affine Lie algebra $\hat{g}_K$ at level K. We only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}^{\mathrm{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \bar{V}_{\lambda^*}, \tag{2.1}$$

where V and $\bar{V}$ represent left- and right-moving states respectively, λ^* denotes the representation conjugate to λ, and P_+^K is the set of integrable highest-weight representations of $\hat{g}_K$.

D-branes of the WZW model may be described algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We only consider boundary conditions on the currents of the affine Lie algebra of the form

$$\left[J^a(z) - \quad \bar{J}^a(\bar{z})\right]\Big|_{z=\bar{z}} = 0 \tag{2.2}$$

where   is an automorphism of the Lie algebra g. These boundary conditions leave unbroken the $\hat{g}_K$ symmetry, as well as the conformal symmetry, of the theory.

Twisted Ishibashi states

Open-closed string duality allows one to correlate the boundary conditions (2.2) of the boundary WZW model with coherent states $|B\rangle\rangle \; \in \mathcal{H}^{\mathrm{closed}}$ of the bulk WZW model satisfying

$$\left[J_m^a + \quad \bar{J}_{-m}^a\right]|B\rangle\rangle \; = 0, \qquad m \in \quad , \tag{2.3}$$

where J_m^a are the modes of the affine Lie algebra generators. Solutions of Eq. (2.3) that belong to a single sector $V_\mu \otimes \bar{V}_{\;(\mu)^*}$ of the bulk WZW model are known as   -twisted Ishibashi states $|\mu\rangle\rangle_I$ [26]. As we are considering the diagonal closed-string theory (2.1),   -twisted Ishibashi states only exist when $\mu = \;(\mu)$, and so are labelled by $\mu \in \mathcal{E}$, where $\mathcal{E}$ is the subset of integrable representations of $\hat{g}_K$ that satisfy $\;(\mu) = \mu$. Equivalently, μ corresponds to an integrable representation of $\breve{g}$, the orbit Lie algebra associated with $\hat{g}_K$ [27].

Twisted Cardy states

A coherent state $|B\rangle\rangle$ that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions [28]. Solutions of Eq. (2.3) that satisfy the Cardy conditions are denoted -twisted Cardy states $|\alpha\rangle\rangle_C$, where the labels α take values in , the set of integrable representations of the -twisted affine Lie algebra $\hat{g}_K$ [10]. The -twisted Cardy states may be expressed as linear combinations of -twisted Ishibashi states

$$|\alpha\rangle\rangle_C = \sum_{\mu \in} \frac{\psi_{\alpha\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\rangle_I, \tag{2.4}$$

where $S_{\lambda\mu}$ is the modular transformation matrix of $\hat{g}_K$, 0 denotes the identity representation, and the coefficients $\psi_{\alpha\mu}$ may be identified with the modular transformation matrices of the -twisted affine Lie algebra $\hat{g}_K$ [10].

The -twisted D-branes of $\hat{g}_K$ correspond to the -twisted Cardy states $|\alpha\rangle\rangle_C$ and are therefore also labelled by $\alpha \in$. The spectrum of an open string stretched between -twisted D-branes labelled by α and β is encoded in the open-string partition function

$$Z^{\text{open}}_{\alpha\beta}(\tau) = \sum_{\lambda \in P^K_+} n_{\beta\lambda}{}^{\alpha} \chi_\lambda(\tau), \tag{2.5}$$

where $\chi_\lambda(\tau)$ is the affine character of the integrable highest-weight representation λ of $\hat{g}_K$. The multiplicity $n_{\beta\lambda}{}^{\alpha}$ of the representation λ carried by the open string may be expressed as [20]

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu \in} \frac{\psi^*_{\alpha\mu} S_{\lambda\mu} \psi_{\beta\mu}}{S_{0\mu}}. \tag{2.6}$$

Untwisted Ishibashi and Cardy states

Untwisted Cardy states $|\alpha\rangle\rangle_C$ and untwisted Ishibashi states $|\mu\rangle\rangle_I$ are solutions of Eq. (2.3) with $= 1$, and both are labelled by integrable representations of $\hat{g}_K$. The matrix $\psi_{\alpha\mu}$ in Eq. (2.4) relating the untwisted Cardy states to the untwisted Ishibashi states is given by the modular transformation matrix $S_{\alpha\mu}$ of $\hat{g}_K$ [28]. Consequently, by virtue of Eq. (2.6) and the Verlinde formula for the fusion coefficients [29]

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu \in P^K_+} \frac{S_{\beta\mu} S_{\lambda\mu} S^*_{\alpha\mu}}{S_{0\mu}} = N_{\beta\lambda}{}^{\alpha}, \tag{2.7}$$

the multiplicities $n_{\beta\lambda}{}^{\alpha}$ of the representations carried by an open string stretched between two untwisted D-branes α and β are given by the fusion coefficients $N_{\beta\lambda}{}^{\alpha}$ of the WZW model.

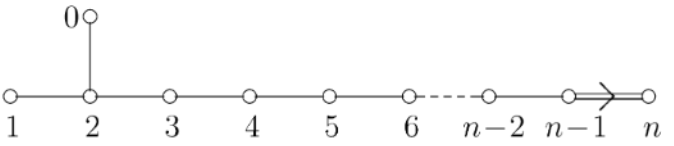
3. Integrable representations of so$(N)_K$

In this section, we review some details about the integrable representations of $\widehat{\text{so}}(N)_K$ used throughout this paper.[3] Integrable representations of an affine Lie algebra $\hat{g}_K$ have non-negative Dynkin indices $(a_0, a_1, \ldots, a_r)$ that satisfy $\sum_{i=0}^r m_i a_i = K$, where m_i are the dual Coxeter labels of the Dynkin diagram for $\hat{g}_K$, and $r+1$ is the rank of $\hat{g}_K$.

[3] Throughout this paper, $N \geqslant 3$ is understood.

Integrable representations of $\widehat{\mathrm{so}}(2n+1)_K$

The Dynkin diagram for $\widehat{\mathrm{so}}(2n+1)_K$ is



and the dual Coxeter labels are $(m_0, m_1, m_2, \ldots, m_{n-1}, m_n) = (1, 1, 2, \ldots, 2, 1)$, where the labelling of nodes is indicated on the diagram. Integrable representations of $\widehat{\mathrm{so}}(2n+1)_K$ thus have Dynkin indices that satisfy[4]

$$a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n = K. \tag{3.1}$$

An even or odd value of a_n corresponds, respectively, to a tensor or spinor representation of $\mathrm{so}(2n+1)$. With each irreducible tensor representation of $\mathrm{so}(2n+1)$ may be associated a Young tableau whose row lengths ℓ_i are given by

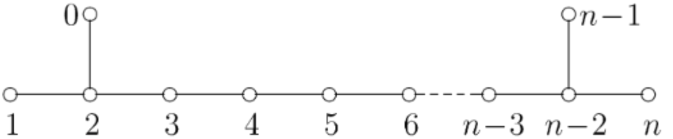
$$\ell_i = \begin{cases} \frac{1}{2}a_n + \sum_{j=i}^{n-1} a_j, & \text{for } 1 \le i \le n-1, \\ \frac{1}{2}a_n, & \text{for } i = n. \end{cases} \tag{3.2}$$

The integrability condition (3.1) is equivalent to the constraint $\ell_1 + \ell_2 \le K$ on the row lengths of the tableau.

We may also formally use Eq. (3.2) to define row lengths for a spinor representation. These row lengths are all half-integers, and correspond to a "Young tableau" containing a column of "half-boxes".

Integrable representations of $\widehat{\mathrm{so}}(2n)_K$

The Dynkin diagram for $\widehat{\mathrm{so}}(2n)_K$ is



and the dual Coxeter labels are $(m_0, m_1, m_2, \ldots, m_{n-2}, m_{n-1}, m_n) = (1, 1, 2, \ldots, 2, 1, 1)$, where the labelling of nodes is indicated on the diagram. Integrable representations of $\widehat{\mathrm{so}}(2n)_K$ thus have Dynkin indices that satisfy[5]

$$a_0 + a_1 + 2(a_2 + \cdots + a_{n-2}) + a_{n-1} + a_n = K. \tag{3.3}$$

An even or odd value of $a_n - a_{n-1}$ corresponds, respectively, to a tensor or spinor representation of $\mathrm{so}(2n)$.

[4] Throughout this paper, by $\widehat{\mathrm{so}}(3)_K$ we mean the affine Lie algebra $\widehat{\mathrm{su}}(2)_{2K}$. Its integrable representations have so(3) Young tableaux that obey $\ell_1 \le K$. Since $\ell_1 = \frac{1}{2}a_1$, this means that Eq. (3.1) is replaced by $a_0 + a_1 = 2K$ when $n = 1$.

[5] For $\widehat{\mathrm{so}}(4)_K$, we take the integrability condition to be $a_0 + \max(a_1, a_2) = K$, which is equivalent to $\ell_1 + |\ell_2| \le K$.

The Dynkin diagram of so$(2n)$ (and also $\widehat{\mathrm{so}}(2n)_K$) is invariant under the exchange of the $(n-1)$th and nth nodes. This gives rise to a $\mathbb{Z}_2$-automorphism ε of the so$(2n)$ Lie algebra, which exchanges representations with Dynkin indices $(\ldots, a_{n-1}, a_n)$ and $(\ldots, a_n, a_{n-1})$. This automorphism may be dubbed [20] "chirality flip" as it exchanges the two fundamental spinor representations of opposite chirality.

For each representation of so$(2n)$ we may define

$$\ell_i = \begin{cases} \frac{1}{2}(a_n + a_{n-1}) + \sum_{j=i}^{n-2} a_j, & \text{for } 1 \leq i \leq n-2, \\ \frac{1}{2}(a_n + a_{n-1}), & \text{for } i = n-1, \\ \frac{1}{2}(a_n - a_{n-1}), & \text{for } i = n, \end{cases} \tag{3.4}$$

in terms of which the integrability condition (3.3) becomes $\ell_1 + \ell_2 \leq K$. The *absolute values* of ℓ_i represent the row lengths of a Young tableau A with up to n rows. (For spinor representations, these row lengths are all half-integer, and correspond to a Young tableau containing a column of half-boxes.) When $a_n = a_{n-1}$, the Young tableau has $n-1$ or fewer rows, and corresponds to a unique irreducible so$(2n)$ representation a, one which is invariant under ε. When $a_n \neq a_{n-1}$, the Young tableau has precisely n rows and corresponds to two distinct representations, a and $\varepsilon(a)$. Hence we may consider the Young tableau A as labelling either an irreducible (a) or a reducible $(a \oplus \varepsilon(a))$ representation of so$(2n)$, depending respectively on whether the representation a is or is not invariant under ε. We thus write

$$A = 2^{s(a)-1}\left[a \oplus \varepsilon(a)\right], \tag{3.5}$$

where

$$s(a) = \begin{cases} 1, & \text{if } \ell_n \neq 0, \text{ that is, } a \neq \varepsilon(a), \\ 0, & \text{if } \ell_n = 0, \text{ that is, } a = \varepsilon(a). \end{cases} \tag{3.6}$$

(For *all* representations of so$(2n+1)$, we define $\varepsilon(a) = a$ and $s(a) = 0$.)

Let S_{ab} denote the (symmetric) modular transformation matrix of $\widehat{\mathrm{so}}(N)_K$, an explicit formula for which may be found, for example, in Ref. [3]. We define

$$\begin{aligned} S_{Ab} &= 2^{s(a)-1}[S_{ab} + S_{\varepsilon(a)b}], \\ S_{AB} &= 2^{s(b)-1}[S_{Ab} + S_{A\varepsilon(b)}]. \end{aligned} \tag{3.7}$$

Since the modular transformation matrix obeys

$$S_{\varepsilon(a)b} = S_{a\varepsilon(b)}, \tag{3.8}$$

it follows that

$$\begin{aligned} S_{Ab} &= S_{A\varepsilon(b)}, \\ S_{AB} &= 2^{s(b)} S_{Ab}. \end{aligned} \tag{3.9}$$

Simple current orbits of $\widehat{\mathrm{so}}(N)_K$

Both $\widehat{\mathrm{so}}(2n+1)_K$ and $\widehat{\mathrm{so}}(2n)_K$ Dynkin diagrams have a $\mathbb{Z}_2$-symmetry that exchanges the 0th and 1st nodes. This symmetry induces a simple-current symmetry (denoted by σ) of the $\widehat{\mathrm{so}}(N)_K$ WZW model that pairs integrable representations related by $a_0 \leftrightarrow a_1$, with the other Dynkin indices unchanged.[6] Their respective Young tableaux are related by $\ell_1 \quad K-\ell_1$. Under σ, tensor representations are mapped to tensors, and spinor representations to spinors.

We will refer to representations of $\widehat{\mathrm{so}}(N)_K$ with $\ell_1<\frac{1}{2}K$, $\ell_1=\frac{1}{2}K$, and $\ell_1>\frac{1}{2}K$ as being of types I, II, and III, respectively. Type II representations are invariant under σ, and are tensors (resp. spinors) when K is even (resp. odd). Each simple-current orbit of $\widehat{\mathrm{so}}(N)_K$ contains either a type I and type III representation, or a single type II representation. We define

$$t(a)=\begin{cases}1, & \text{if } \ell_1=\frac{1}{2}K \text{ (type II), that is, } a=\sigma(a),\\ 0, & \text{if } \ell_1\neq\frac{1}{2}K \text{ (type I or III), that is, } a\neq\sigma(a).\end{cases} \tag{3.10}$$

Finally, the modular transformation matrix of $\widehat{\mathrm{so}}(N)_K$ obeys

$$S_{\sigma(a)b}=\pm S_{\varepsilon(a)b}, \quad \text{for } b \text{ a } \begin{Bmatrix}\text{tensor}\\ \text{spinor}\end{Bmatrix} \text{ representation.} \tag{3.11}$$

4. Level-rank duality of untwisted D-branes of $\mathrm{so}(N)_K$

Having reviewed the characterization of integrable representations of $\widehat{\mathrm{so}}(N)_K$ in the previous section, we now turn to the untwisted D-branes of the $\widehat{\mathrm{so}}(N)_K$ WZW model, which are labelled by those representations. In this section, we will demonstrate a level-rank duality between the untwisted D-branes of $\widehat{\mathrm{so}}(N)_K$ and those of $\widehat{\mathrm{so}}(K)_N$.

Since the multiplicities of the representations carried by an open string stretched between two untwisted D-branes are given by the fusion coefficients of the WZW model (2.7), level-rank duality of the untwisted D-branes of the $\widehat{\mathrm{so}}(N)_K$ model is closely related to level-rank duality of the fusion coefficients of this model, which was described in Ref. [3]. We recall two salient aspects of this duality:

- The level-rank map is partial: it only relates the *tensor* representations[7] of $\widehat{\mathrm{so}}(N)_K$ to those of $\widehat{\mathrm{so}}(K)_N$.
- The level-rank map is not one-to-one between integrable tensor representations a, but rather between *equivalence classes* of representations,[8] denoted by $[a]$. These equivalence classes are characterized by tensor Young tableaux with $N/2$ rows and $K/2$ columns (termed "reduced and cominimally-reduced" in Ref. [3]). Level-rank duality acts by transposing these tableaux, and maps the set of tensor Young tableaux with $N/2$ rows and $K/2$ columns one-to-one onto the set of tensor Young tableaux with $K/2$ rows and $N/2$ columns.

The equivalence classes of integrable tensor representations fall into several categories, which we now describe, using the notation of the previous section.

(1) $s(a)=0$ and $t(a)=1$: the equivalence class labelled by a tensor Young tableau with $\ell_1=\frac{1}{2}K$ columns (only possible when K is even) and with fewer than $\frac{1}{2}N$ rows corresponds to a

[6] Except for $\widehat{\mathrm{so}}(4)_K$, in which case σ acts by $a_0\leftrightarrow\min(a_1,a_2)$. Thus, if a has Dynkin indices (a_0,a_1,a_2) then $\sigma(a)$ has Dynkin indices $(\min(a_1,a_2), K-a_2, K-a_1)$.

[7] For $\widehat{\mathrm{so}}(2n+1)_{2k+1}$, a level-rank map between the spinor representations also exists [3], and thus a level-rank map can be defined for all the untwisted D-branes of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$.

[8] This is also the case for level-rank duality of $\widehat{\mathrm{su}}(N)_K$.

single (type II) irreducible representation a, whose Dynkin indices satisfy $a_0 = a_1$ and (for $N = 2n$) $a_n = a_{n-1}$. This representation is invariant under both σ and ε.

(2) $s(a) = 0$ and $t(a) = 0$: the equivalence class labelled by a tensor Young tableau with $\ell_1 < \frac{1}{2}K$ columns and with fewer than $\frac{1}{2}N$ rows corresponds to a *pair* of irreducible representations a and $\sigma(a)$ (of type I and type III) whose Dynkin indices are related by $a_0 \leftrightarrow a_1$. When $N = 2n$, the Dynkin indices of these representations satisfy $a_n = a_{n-1}$, i.e., these representations are invariant under ε.

(3) $s(a) = 1$ and $t(a) = 1$: the equivalence class labelled by a tensor Young tableau with $\ell_1 = \frac{1}{2}K$ columns (only possible when K is even) and with exactly $\frac{1}{2}N$ rows (only possible when N is even) corresponds to a *pair* of (type II) irreducible representations a and $\varepsilon(a)$, whose Dynkin indices are related by $a_n \leftrightarrow a_{n-1}$ where $N = 2n$, and obey[9] $a_0 = a_1$, i.e., these representations are invariant under σ.

(4) $s(a) = 1$ and $t(a) = 0$: the equivalence class labelled by a tensor Young tableau with $\ell_1 < \frac{1}{2}K$ columns and with exactly $\frac{1}{2}N$ rows (only possible when N is even) corresponds to *four* irreducible representations: a, $\sigma(a)$, $\varepsilon(a)$, and $\varepsilon(\sigma(a))$ (two of type I and two of type III).

Let $[\tilde{a}]$ denote the transpose of the Young tableau characterizing the equivalence class $[a]$. Then

$$t(a) = \tilde{s}(\tilde{a}) \quad \text{and} \quad s(a) = \tilde{t}(\tilde{a}), \tag{4.1}$$

where $\tilde{s}$ and $\tilde{t}$ are the quantities (3.6) and (3.10) defined for $\widehat{\mathrm{so}}(K)_N$. Under level-rank duality, equivalence classes $[a]$ in categories (1), (2), (3), and (4) map into equivalence classes $[\tilde{a}]$ in categories (4), (2), (3), and (1), respectively.

We now elucidate the implications of level-rank duality for the untwisted tensor D-branes of the $\widehat{\mathrm{so}}(N)_K$ WZW model. Consider the linear combination of untwisted Cardy states

$$\big|[\alpha]\big\rangle\!\!\big\rangle = \frac{1}{\sqrt{2}^{\,t(\alpha)-s(\alpha)+3}}\Big[|\alpha\rangle\!\rangle_C + \big|\sigma(\alpha)\big\rangle\!\!\big\rangle_C + \big|\varepsilon(\alpha)\big\rangle\!\!\big\rangle_C + \big|\varepsilon\big(\sigma(\alpha)\big)\big\rangle\!\!\big\rangle_C\Big], \tag{4.2}$$

which corresponds to an equivalence class $[\alpha]$ of integrable tensor representations. Using Eqs. (2.7) and (3.11), we find that the multiplicity $n_{[\beta][\lambda]}{}^{[\alpha]}$ of the equivalence class of representations $[\lambda]$ carried by an open string stretched between untwisted D-branes corresponding to the states $|[\alpha]\rangle\!\rangle$ and $[\beta]\rangle\!\rangle$ is given by

$$\begin{aligned} n_{[\beta][\lambda]}{}^{[\alpha]} &= \frac{1}{\sqrt{2}^{\,t(\alpha)+t(\beta)+t(\lambda)-s(\alpha)-s(\beta)-s(\lambda)+3}} \\ &\quad \times \sum_{\substack{\mu=\text{tensor} \\ \text{representations}}} \frac{(S_{\beta\mu} + S_{\varepsilon(\beta)\mu})(S_{\lambda\mu} + S_{\varepsilon(\lambda)\mu})(S^*_{\alpha\mu} + S^*_{\varepsilon(\alpha)\mu})}{S_{0\mu}}, \end{aligned} \tag{4.3}$$

where only integrable *tensor* representations μ remain in the sum as a consequence of Eq. (3.11). Using Eqs. (3.7) and (3.9), we express $n_{[\beta][\lambda]}{}^{[\alpha]}$ in terms of Young tableaux A, B, Λ, M, related to α, β, λ, and μ by Eq. (3.5),

[9] For $\widehat{\mathrm{so}}(4)_K$, they obey $a_0 = \min(a_1, a_2)$.

$$
n_{[\beta][\lambda]}{}^{[\alpha]} = \frac{1}{\sqrt{2}^{\,t(\alpha)+t(\beta)+t(\lambda)+s(\alpha)+s(\beta)+s(\lambda)-3}} \times \sum_{\substack{M=\text{tensor}\\ \text{tableaux I, II, III}}} \frac{1}{2^{s(\mu)}} \frac{S_{BM} S_{\Lambda M} S^*_{AM}}{S_{0M}}. \tag{4.4}
$$

Finally, since $S_{A\sigma(M)} = S_{AM}$, the sum may be restricted to tableaux of types I and II ("cominimally-reduced" tableaux)

$$
n_{[\beta][\lambda]}{}^{[\alpha]} = \frac{1}{\sqrt{2}^{\,t(\alpha)+t(\beta)+t(\lambda)+s(\alpha)+s(\beta)+s(\lambda)-3}} \times \sum_{\substack{M=\text{tensor}\\ \text{tableaux I, II}}} \frac{1}{2^{s(\mu)+t(\mu)-1}} \frac{S_{BM} S_{\Lambda M} S^*_{AM}}{S_{0M}}. \tag{4.5}
$$

The multiplicities $n_{[\beta][\lambda]}{}^{[\alpha]}$ are closely related to $\Sigma_{B\Lambda}{}^{A}$ defined in Eq. (3.16) of Ref. [3].

Level-rank duality maps the state $|[\alpha]\rangle\rangle$ of $\widehat{\text{so}}(N)_K$ to the state $|[\tilde{\alpha}]\rangle\rangle$ of $\widehat{\text{so}}(K)_N$. Let $\tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}{}^{[\tilde{\alpha}]}$ denote the quantity (4.3) defined for $\widehat{\text{so}}(K)_N$. The form of Eq. (4.5) makes manifest the equality of the multiplicities

$$
n_{[\beta][\lambda]}{}^{[\alpha]} = \tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}{}^{[\tilde{\alpha}]} \tag{4.6}
$$

as a consequence of three facts: (1) the set of cominimally-reduced tableaux M of $\widehat{\text{so}}(N)_K$ are in one-to-one correspondence with those of $\widehat{\text{so}}(K)_N$, (2) Eq. (4.1) holds for all tensor representations, and (3) the quantities S_{AB}, defined by Eq. (3.7), are level-rank dual ($S_{AB} = \tilde{S}_{\tilde{A}\tilde{B}}$) as was proved in the appendix of Ref. [3]. Hence, the spectrum of representations carried by open strings stretched between untwisted tensor D-branes of $\widehat{\text{so}}(N)_K$ is level-rank dual.

We end this section by describing the level-rank duality of untwisted spinor D-branes of $\widehat{\text{so}}(2n+1)_{2k+1}$. The equivalence classes $[\alpha]$ of spinor representations of $\widehat{\text{so}}(2n+1)_{2k+1}$ are characterized by type I and type II spinor tableaux, where a type I tableau represents a pair of spinor representations α and $\sigma(\alpha)$, and a type II tableau represents a single irreducible spinor representation α that obeys $\sigma(\alpha) = \alpha$. The level-rank map $[\alpha]$ $[\hat{\alpha}]$ between equivalence classes of spinor representations of $\widehat{\text{so}}(2n+1)_{2k+1}$ and $\widehat{\text{so}}(2k+1)_{2n+1}$ was presented in Ref. [3]:

- reduce each of the row lengths by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- take the complement with respect to a $k \times n$ rectangle and rotate 180 degrees,
- add $\frac{1}{2}$ to each of the row lengths.

This takes type I spinor tableaux of $\widehat{\text{so}}(2n+1)_{2k+1}$ to type II spinor tableaux of $\widehat{\text{so}}(2k+1)_{2n+1}$ and vice versa: $t(\alpha) = 1 - \tilde{t}(\hat{\alpha})$. This procedure thus defines a map between an untwisted spinor D-brane of $\widehat{\text{so}}(2n+1)_{2k+1}$ corresponding to the boundary state

$$
\big|[\alpha]\big\rangle\!\big\rangle = \frac{1}{\sqrt{2}^{\,t(\alpha)+1}} \big[|\alpha\rangle\rangle_C + \big|\sigma(\alpha)\big\rangle\!\big\rangle_C \big] \tag{4.7}
$$

and an untwisted spinor D-brane $|[\hat{\alpha}]\rangle\rangle$ of $\widehat{\text{so}}(2k+1)_{2n+1}$. The multiplicity of the (equivalence class of) representations $[\lambda]$ carried by an open string stretched between untwisted spinor

D-branes $[\alpha]$ and $[\beta]$ of $\widehat{\text{so}}(2n+1)_{2k+1}$ obeys

$$n_{[\beta][\lambda]}{}^{[\alpha]} = \frac{1}{\sqrt{2}^{\,t(\alpha)+t(\beta)-5}} \sum_{\substack{\mu=\text{tensor}\\ \text{tableau I}}} \frac{S_{\beta\mu} S_{\lambda\mu} S^*_{\alpha\mu}}{S_{0\mu}} = \tilde{n}_{[\hat{\beta}][\tilde{\lambda}]}{}^{[\hat{\alpha}]}, \tag{4.8}$$

using Eq. (3.25) of Ref. [3]. Hence, the spectrum of representations carried by open strings stretched between untwisted spinor D-branes of $\widehat{\text{so}}(2n+1)_{2k+1}$ is also level-rank dual.

5. The -twisted D-branes of the $\text{so}(2n)_K$ model

In the previous section, we proved the level-rank duality of untwisted D-branes of $\widehat{\text{so}}(N)_K$. In this section, we will describe a class of twisted D-branes of $\widehat{\text{so}}(2n)_K$, and in the next section, we will prove the level-rank duality of a subset of these twisted D-branes.

Recall from in Section 3 that the finite Lie algebra $\text{so}(2n)$ possesses (when $n \geqslant 2$) a ${}_2$-automorphism ε (chirality flip), under which the Dynkin indices a_{n-1} and a_n of an irreducible representation are exchanged. This automorphism lifts to an automorphism of the affine Lie algebra $\widehat{\text{so}}(2n)_K$, and gives rise to a set of ε-twisted Ishibashi states and ε-twisted Cardy states of the bulk $\widehat{\text{so}}(2n)_K$ WZW model, and a corresponding class of ε-twisted D-branes of the boundary model. In this section we characterize these twisted states, relying heavily on Ref. [20].

ε-twisted Ishibashi states of $\widehat{\text{so}}(2n)_K$

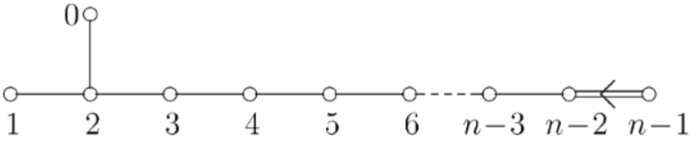
The ε-twisted Ishibashi states $|\mu\rangle\rangle^{\varepsilon}_{I}$ of the $\widehat{\text{so}}(2n)_K$ WZW model are labelled by integrable representations $\mu \in \mathcal{E}^{\varepsilon}$ of $\widehat{\text{so}}(2n)_K = (D_n^{(1)})_K$ that obey $\varepsilon(\mu) = \mu$ (i.e., integrable representations characterized by $\text{so}(2n)$ tensor Young tableaux with no more than $n-1$ rows). These representations have Dynkin indices

$$(\mu_0, \mu_1, \ldots, \mu_{n-2}, \mu_{n-1}, \mu_{n-1}), \tag{5.1}$$

that satisfy[10]

$$\mu_0 + \mu_1 + 2(\mu_2 + \cdots + \mu_{n-1}) = K. \tag{5.2}$$

Equivalently, the ε-twisted Ishibashi states of $\widehat{\text{so}}(2n)_K$ may be characterized by the integrable representations of the associated orbit Lie algebra $\breve{g}^{\varepsilon} = (A^{(2)}_{2n-3})_K$ [20] whose Dynkin diagram is



and whose dual Coxeter numbers are $(m_0, m_1, m_2, \ldots, m_{n-1}) = (1, 1, 2, \ldots, 2)$, where the labelling of nodes is indicated on the diagram. The $(D_n^{(1)})_K$ representation with Dynkin indices (5.1) corresponds to the $(A^{(2)}_{2n-3})_K$ representation with Dynkin indices $(\mu_0, \mu_1, \ldots, \mu_{n-2}, \mu_{n-1})$, whose integrability condition is precisely (5.2).

10 For $\widehat{\text{so}}(4)_K$, the Dynkin indices (μ_0, μ_1, μ_1) satisfy $\mu_0 + \mu_1 = K$.

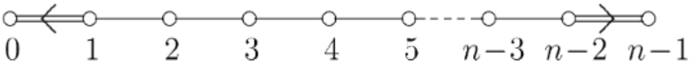
It was shown in Ref. [20] that each ε-twisted Ishibashi state μ of $\widehat{\mathrm{so}}(2n)_K$ may be mapped to a *spinor* representation μ' of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n-1)_{K+1}$ with Dynkin indices[11]

$$\mu'_i = \mu_i \quad (0 \quad i \quad n-2) \quad \text{and} \quad \mu'_{n-1} = 2\mu_{n-1} + 1. \tag{5.3}$$

The constraint (5.2) on μ is precisely equivalent to the integrability constraint (3.1) on the representation μ' of $\widehat{\mathrm{so}}(2n-1)_{K+1}$. Hence, *$\varepsilon$-twisted Ishibashi states of $\widehat{\mathrm{so}}(2n)_K$ are in one-to-one correspondence with the set of integrable spinor representations of $\widehat{\mathrm{so}}(2n-1)_{K+1}$ of type I, type II (when K is even), and type III.*

ε-twisted Cardy states of $\widehat{\mathrm{so}}(2n)_K$

The ε-twisted Cardy states $|\alpha\rangle\rangle_C^\varepsilon$ (and therefore the ε-twisted D-branes) of the $\widehat{\mathrm{so}}(2n)_K$ WZW model[12] are labelled by the integrable representations $\alpha \in {}^{\varepsilon}$ of the ε-twisted affine Lie algebra $\hat{g}_K^\varepsilon = (D_n^{(2)})_K$ [20], whose Dynkin diagram is



and whose dual Coxeter numbers are $(m_0, m_1, \ldots, m_{n-2}, m_{n-1}) = (1, 2, \ldots, 2, 1)$, where the labelling of nodes is indicated on the diagram. The Dynkin indices $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1})$ of α thus satisfy[13]

$$\alpha_0 + 2(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1} = K. \tag{5.4}$$

It was shown in Ref. [20] that each ε-twisted Cardy state α of $\widehat{\mathrm{so}}(2n)_K$ may be mapped to a representation α' of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n-1)_{K+1}$ with Dynkin indices[14]

$$\alpha'_0 = \alpha_0 + \alpha_1 + 1 \quad \text{and} \quad \alpha'_i = \alpha_i \quad (1 \quad i \quad n-1). \tag{5.5}$$

The constraint (5.4) on α implies that α' is an integrable representation of $\widehat{\mathrm{so}}(2n-1)_{K+1}$, and the constraint $\alpha_0 \geqslant 0$ further implies that α' is a type I representation of $\widehat{\mathrm{so}}(2n-1)_{K+1}$ (i.e., corresponds to a Young tableau whose first row length obeys $\ell'_1 \quad \frac{1}{2}K$). Therefore, *$\varepsilon$-twisted D-branes of $\widehat{\mathrm{so}}(2n)_K$ are in one-to-one correspondence with the set of integrable type I representations of $\widehat{\mathrm{so}}(2n-1)_{K+1}$, of both tensor and spinor types.*

Although the ε-twisted Ishibashi states and the ε-twisted Cardy states of $\widehat{\mathrm{so}}(2n)_K$ are characterized differently in terms of integrable representations of $\widehat{\mathrm{so}}(2n-1)_{K+1}$, they are equal in number. (For K even, the number of each is $\binom{n-1+K/2}{n-1} + \binom{n-2+K/2}{n-1}$, while for K odd, the number of each is $2\binom{n-1+(K-1)/2}{n-1}$.) Thus, the ε-twisted Cardy states α may be written as linear combinations (2.4) of ε-twisted Ishibashi states μ, with the transformation coefficients $\psi_{\alpha\mu}$ given by the modular transformation matrix of $(D_n^{(2)})_K$. In Ref. [20], it was shown that these coefficients are proportional to the (real) matrix elements $S'_{\alpha'\mu'}$ of the modular transformation matrix of the

11 Except for $\widehat{\mathrm{so}}(4)_K$, in which case $\mu'_0 = 2\mu_0 + 1$ and $\mu'_1 = 2\mu_1 + 1$. Also, recall footnote 4.

12 $n \geqslant 2$ is understood.

13 For $\widehat{\mathrm{so}}(4)_K$, the condition is $\alpha_0 + \alpha_1 = K$.

14 Except for $\widehat{\mathrm{so}}(4)_K$, in which case $\alpha'_0 = 2\alpha_0 + \alpha_1 + 2$ and $\alpha'_1 = \alpha_1$. Also, recall footnote 4.

untwisted affine Lie algebra $\widehat{\mathrm{so}}(2n-1)_{K+1}$:

$$\psi_{\alpha\mu} = \sqrt{2} S'_{\alpha'\mu'} = \sqrt{2} S'^{*}_{\alpha'\mu'} \tag{5.6}$$

where α' and μ' are the $\widehat{\mathrm{so}}(2n-1)_{K+1}$ representations related to α and μ by Eqs. (5.5) and (5.3), respectively.

Twisted open string partition function of $\widehat{\mathrm{so}}(2n)_K$

Combining Eqs. (2.6) and (5.6), we may write the multiplicities of the representations carried by an open string stretched between ε-twisted D-branes α and β of $\widehat{\mathrm{so}}(2n)_K$ as

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I,II,III}} \frac{2 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}}, \tag{5.7}$$

where $S_{\lambda\mu}$ and $S'_{\alpha'\mu'}$ are modular transformation matrix elements of $\widehat{\mathrm{so}}(2n)_K$ and $\widehat{\mathrm{so}}(2n-1)_{K+1}$ respectively, and the sum is over all ε-twisted Ishibashi states μ of $\widehat{\mathrm{so}}(2n)_K$, or equivalently, over all integrable spinor representations μ' of $\widehat{\mathrm{so}}(2n-1)_{K+1}$. (Type II spinors of $\widehat{\mathrm{so}}(2n-1)_{K+1}$ are present only when K is even.)

Although the ε-twisted D-branes correspond to both tensor and spinor representations of $\widehat{\mathrm{so}}(2n-1)_{K+1}$, for the remainder of this section we will restrict α and β to correspond to spinor representations α' and β' of $\widehat{\mathrm{so}}(2n-1)_{K+1}$, which allows us to simplify Eq. (5.7) considerably. Recall from Eq. (3.11) that the modular transformation matrix elements of $\widehat{\mathrm{so}}(2n-1)_{K+1}$ obey

$$S'_{\alpha'\sigma(\mu')} = -S'_{\alpha'\mu'}, \quad \text{for } \alpha' = \text{spinor}. \tag{5.8}$$

As a consequence, type I and III representations μ', which are related by σ, may be combined in Eq. (5.7)

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \left[\frac{2 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} + \frac{2 S'_{\alpha'\mu'} S_{\lambda\sigma(\mu)} S'_{\beta'\mu'}}{S_{0\sigma(\mu)}} \right], \quad \text{for } \alpha', \beta' \text{ both spinors} \tag{5.9}$$

and type II representations, which obey $\sigma(\mu') = \mu'$, drop out of the sum since $S'_{\alpha'\mu'} = 0$. (We have also used the fact that the $\widehat{\mathrm{so}}(2n-1)_{K+1}$ representation $\sigma(\mu')$, related to μ' by ℓ'_1 $K+1-\ell'_1$, corresponds via the map (5.3) to the ε-twisted Ishibashi state $\sigma(\mu)$ of $\widehat{\mathrm{so}}(2n)_K$, related to μ by ℓ_1 $K-\ell_1$.) Finally, recalling that the modular transformation matrix elements of $\widehat{\mathrm{so}}(2n)_K$ obey

$$S_{\lambda\sigma(\mu)} = \pm S_{\lambda\varepsilon(\mu)}, \quad \text{for } \lambda \text{ a } \begin{Bmatrix} \text{tensor} \\ \text{spinor} \end{Bmatrix} \text{ representation} \tag{5.10}$$

and that ε-twisted Ishibashi states obey $\varepsilon(\mu) = \mu$, we finally obtain

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}}, \quad \text{for } \alpha', \beta' \text{ both spinors and } \lambda = \text{tensor}, \tag{5.11}$$

and

$$n_{\beta\lambda}{}^{\alpha} = 0, \quad \text{for } \alpha', \beta' \text{ both spinors and } \lambda = \text{spinor}. \tag{5.12}$$

This result will allow us to demonstrate in the next section the level-rank duality of the spectrum of an open string stretched between spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$.

6. Level-rank duality of -twisted D-branes of $\mathrm{so}(2n)_{2k}$

As we saw in the previous section, the $\widehat{\mathrm{so}}(2n)_K$ WZW model possesses twisted D-branes corresponding to the chirality-flip symmetry ε of the $\mathrm{so}(2n)$ Dynkin diagram, and these ε-twisted D-branes are characterized by integrable type I tensor and spinor representations of $\widehat{\mathrm{so}}(2n-1)_{K+1}$. We will refer to these as tensor and spinor ε-twisted D-branes, respectively.

In this section, we will exhibit a level-rank duality[15] between the ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ and those of $\widehat{\mathrm{so}}(2k)_{2n}$. This duality is partial, and only holds between *spinor* ε-twisted D-branes (just as the level-rank duality of untwisted D-branes only holds between tensor D-branes). The restriction to spinor ε-twisted D-branes can be anticipated by observing that the number of tensor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ is $\binom{n+k-1}{n-1}$ and the number of spinor ε-twisted D-branes is $\binom{n+k-2}{n-1}$, and only the latter is invariant under $n \leftrightarrow k$.

First we define an explicit one-to-one map α $\hat{\alpha}$ between the spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ and $\widehat{\mathrm{so}}(2k)_{2n}$. The map α $\hat{\alpha}$ is defined by specifying its action[16] on the corresponding $\widehat{\mathrm{so}}(2n-1)_{2k+1}$ and $\widehat{\mathrm{so}}(2k-1)_{2n+1}$ representations α' and $\hat{\alpha}'$:

- reduce each of the row lengths of α' by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- add $\frac{1}{2}$ to each of the row lengths.

The same procedure defines a one-to-one map μ' $\hat{\mu}'$ between type I spinor representations of $\widehat{\mathrm{so}}(2n-1)_{2k+1}$ and $\widehat{\mathrm{so}}(2k-1)_{2n+1}$ corresponding to ε-twisted Ishibashi states. By virtue of Eq. (5.3), this map lifts to a map μ $\tilde{\mu}$ between (a subset of) the ε-twisted Ishibashi states. As suggested by the notation, this map is simply transposition of the type I $\widehat{\mathrm{so}}(2n)_{2k}$ Young tableau corresponding to μ.

Next, we turn to the level-rank duality of the spectrum of an open string stretched between ε-twisted D-branes. In the previous section, it was shown that the multiplicity of the representation λ carried by an open string stretched between spinor ε-twisted D-branes α and β of $\widehat{\mathrm{so}}(2n)_{2k}$ is given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}}, \qquad \text{for } \lambda = \text{tensor}, \tag{6.1}$$

with $n_{\beta\lambda}{}^{\alpha}$ vanishing for $\lambda = $ spinor. As in Section 4, however, we consider the multiplicity corresponding to the equivalence class of tensor representations $[\lambda]$:

$$n_{\beta[\lambda]}{}^{\alpha} = \frac{1}{\sqrt{2}^{\,t(\lambda)-s(\lambda)+3}} \left[n_{\beta\lambda}{}^{\alpha} + n_{\beta\varepsilon(\lambda)}{}^{\alpha} + n_{\beta\sigma(\lambda)}{}^{\alpha} + n_{\beta\varepsilon(\sigma(\lambda))}{}^{\alpha} \right]. \tag{6.2}$$

Using Eqs. (6.1), (3.11), and (3.7), we find

$$n_{\beta[\lambda]}{}^{\alpha} = \frac{1}{\sqrt{2}^{\,t(\lambda)-s(\lambda)+1}} \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'} (S_{\lambda\mu} + S_{\varepsilon(\lambda)\mu}) S'_{\beta'\mu'}}{S_{0\mu}}$$

---

[15] Clearly the ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k+1}$ have no level-rank duals, since $\widehat{\mathrm{so}}(2k+1)_{2n}$ has no ε-twisted D-branes.

[16] Note that the "hat" map defined here differs from that defined in Section 4 between spinor representations of $\widehat{\mathrm{so}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{so}}(2k+1)_{2n+1}$. The map defined here also characterizes the map between $_c$-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ and $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ [7].

$$= \frac{1}{\sqrt{2}^{\,t(\lambda)+s(\lambda)-1}} \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'} S_{\Lambda M} S'_{\beta'\mu'}}{S_{0M}}, \tag{6.3}$$

where $\Lambda = 2^{s(\lambda)-1}[\lambda \oplus \varepsilon(\lambda)]$ and $M = \mu$ since $\varepsilon(\mu) = \mu$. The form of Eq. (6.3) makes manifest the equality of the multiplicities

$$n_{\beta[\lambda]}{}^{\alpha} = \frac{1}{\sqrt{2}^{\,t(\lambda)+s(\lambda)-1}} \sum_{\mu'=\text{spinors I}} \frac{4S'_{\alpha'\mu'} S_{\Lambda M} S'_{\beta'\mu'}}{S_{0M}} \tag{6.4}$$

$$= \frac{1}{\sqrt{2}^{\,\tilde{s}(\tilde{\lambda})+\tilde{t}(\tilde{\lambda})-1}} \sum_{\hat{\mu}'=\text{spinors I}} \frac{4\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tilde{S}_{\tilde{\Lambda}\tilde{M}} \tilde{S}'_{\hat{\beta}'\hat{\mu}'}}{\tilde{S}_{0\tilde{M}}} \tag{6.5}$$

$$= \tilde{n}_{\hat{\beta}[\tilde{\lambda}]}{}^{\hat{\alpha}}, \tag{6.6}$$

where we have used Eq. (4.1) and the facts that:

(1) type I spinors μ' of $\widehat{\text{so}}(2n-1)_{2k+1}$ map one-to-one to type I spinors $\hat{\mu}'$ of $\widehat{\text{so}}(2k-1)_{2n+1}$,
(2) $S_{\Lambda M} = \tilde{S}_{\tilde{\Lambda}\tilde{M}}$ [3], where S and $\tilde{S}$ are the modular transformation matrices of $\widehat{\text{so}}(2n)_{2k}$ and $\widehat{\text{so}}(2k)_{2n}$ respectively, and
(3) $S'_{\alpha'\mu'} = \tilde{S}'_{\hat{\alpha}'\hat{\mu}'}$ for α' and μ' both type I spinor representations (Eq. (6.10) of Ref. [7]), where S' and $\tilde{S}'$ are the modular transformation matrices of $\widehat{\text{so}}(2n-1)_{2k+1}$ and $\widehat{\text{so}}(2k-1)_{2n+1}$ respectively. Since by Eq. (5.12) only tensor representations λ appear in the ε-twisted open-string partition function (2.5), we have established that the spectrum of representations carried by open strings stretched between ε-twisted D-branes of $\widehat{\text{so}}(2n)_{2k}$ is level-rank dual.

7. Conclusions

We have analyzed the level-rank duality of the untwisted D-branes of $\widehat{\text{so}}(N)_K$ and of the ε-twisted D-branes of $\widehat{\text{so}}(2n)_{2k}$. In each case, only a subset of the D-branes are mapped onto those of the level-rank-dual theory.

Untwisted D-branes of $\widehat{\text{so}}(N)_K$ are characterized by integrable tensor and spinor representations of $\widehat{\text{so}}(N)_K$. Only the untwisted *tensor* D-branes participate in level-rank duality.[17] The tensor representations α of $\widehat{\text{so}}(N)_K$ fall into equivalence classes $[\alpha]$ generated by the $_2$-isomorphisms σ and ε (the latter non-trivial only for N even), and characterized by Young tableaux with $N/2$ rows and $K/2$ columns. Level-rank duality acts by transposing these tableaux, and thus maps the equivalence classes $[\alpha]$ of untwisted tensor D-branes of $\widehat{\text{so}}(N)_K$ onto $[\tilde{\alpha}]$ of $\widehat{\text{so}}(K)_N$. We showed that the multiplicity $n_{[\beta][\lambda]}{}^{[\alpha]}$ of the (equivalence class of) representations $[\lambda]$ carried by an open string stretched between untwisted $\widehat{\text{so}}(N)_K$ D-branes corresponding to $[\alpha]$ and $[\beta]$ is equal to $\tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}{}^{[\tilde{\alpha}]}$, the multiplicity of the (equivalence class of) representations $[\tilde{\lambda}]$ carried by an open string stretched between untwisted $\widehat{\text{so}}(K)_N$ D-branes corresponding to $[\tilde{\alpha}]$ and $[\tilde{\beta}]$. A similar result was shown for untwisted spinor D-branes of $\widehat{\text{so}}(2n+1)_{2k+1}$.

The ε-twisted D-branes of $\widehat{\text{so}}(2n)_{2k}$, associated with the chirality-flip symmetry ε of the $\text{so}(2n)$ Dynkin diagram, are characterized by type I integrable tensor and spinor representations of $\widehat{\text{so}}(2n-1)_{2k+1}$. Only the *spinor* ε-twisted D-branes participate in level-rank duality.

[17] Except for $\widehat{\text{so}}(2n+1)_{2k+1}$, where the untwisted spinor D-branes also respect level-rank duality.

We defined a one-to-one map $\alpha \to \hat{\alpha}$ between the spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2n)_{2k}$ and the spinor ε-twisted D-branes of $\widehat{\mathrm{so}}(2k)_{2n}$. We then showed that the multiplicity ${n_{\beta[\lambda]}}^{\alpha}$ of the (equivalence class of) representations $[\lambda]$ carried by an open string stretched between ε-twisted $\widehat{\mathrm{so}}(2n)_{2k}$ D-branes corresponding to α and β is equal to ${\tilde{n}_{\hat{\beta}[\tilde{\lambda}]}}^{\hat{\alpha}}$, the multiplicity of the (equivalence class of) representations $[\tilde{\lambda}]$ carried by an open string stretched between ε-twisted $\widehat{\mathrm{so}}(2k)_{2n}$ D-branes corresponding to $\hat{\alpha}$ and $\hat{\beta}$.

Hence, for both untwisted and ε-twisted D-branes, we have established an isomorphism between the spectrum of an open string ending on these D-branes and the spectrum of an open string ending on the level-rank-dual D-branes.

In both the $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{sp}}(n)_k$ WZW theories, the charges of level-rank-dual untwisted D-branes are equal (modulo sign) [5,6], with a slightly more complicated relationship holding between the charges of twisted D-branes [7]. In the case of $\widehat{\mathrm{so}}(N)_K$, however, the charges of the D-branes do not exhibit any simple relationship under level-rank duality.

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