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ABSTRACT

We derive constraints on the color-ordered amplitudes of the L-loop four-point function in SU(N) gauge theories that arise solely from the structure of the gauge group. These constraints generalize well-known group theory relations, such as U(1) decoupling identities, to all loop orders.

1. Introduction

An exciting recent development in the study of perturbative amplitudes is the discovery of color-kinematic duality of gauge theory amplitudes at both tree and loop level [1,2]. This duality implies the existence of constraints on tree-level color-ordered amplitudes, which were proven in Refs. [3–6]. The BCJ conjecture was also veri ed through three loops for the $\mathcal{N}=4$ supersymmetric Yang–Mills four- [2,7] and ve-point [8,9] amplitudes. (See reviews in Refs. [7,10], which also contain references to related work on the subject.)

The BCJ constraints on tree-level color-ordered amplitudes hold in addition to various well-known SU(N) group theory relations, such as the U(1) decoupling or dual Ward identity [11,12] and the Kleiss–Kuijf relations [13]. Group-theory relations also hold for one-loop [14,15] and two-loop [16] color-ordered amplitudes. They can be elegantly derived by using an alternative color decomposition of the amplitude [17,18].

The purpose of this Letter is extend the SU(N) group theory relations for four-point amplitudes to all loops. We develop a recursive procedure to derive constraints satis ed by any L-loop diagram (containing only adjoint elds) obtained by attaching a rung between two external legs of an $L-1$)-loop diagram. We assume that the most general L-loop color factor can be obtained from this subset using Jacobi relations, an assumption that has been proven through $L=4$. Using this method, we nd four independent group-theory constraints for color-ordered four-point amplitudes at each loop level (except for $L=0$ and $L=1$, where there are one and three constraints respectively).

The color-ordered amplitudes of a gauge theory are the coefficients of the full amplitude in a basis using traces of generators in the fundamental representation of the gauge group. Color-ordered amplitudes have the advantage of being individually gauge-invariant. Four-point amplitudes of SU(N) gauge theories can be expressed in terms of single and double traces [14]

$$\begin{aligned} T_1 &= \mathrm{Tr}\big(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\big) + \mathrm{Tr}\big(T^{a_1}T^{a_4}T^{a_3}T^{a_2}\big), & T_4 &= \mathrm{Tr}\big(T^{a_1}T^{a_3}\big)\,\mathrm{Tr}\big(T^{a_2}T^{a_4}\big), \\ T_2 &= \mathrm{Tr}\big(T^{a_1}T^{a_2}T^{a_4}T^{a_3}\big) + \mathrm{Tr}\big(T^{a_1}T^{a_3}T^{a_4}T^{a_2}\big), & T_5 &= \mathrm{Tr}\big(T^{a_1}T^{a_4}\big)\,\mathrm{Tr}\big(T^{a_2}T^{a_3}\big), \\ T_3 &= \mathrm{Tr}\big(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\big) + \mathrm{Tr}\big(T^{a_1}T^{a_3}T^{a_2}T^{a_4}\big), & T_6 &= \mathrm{Tr}\big(T^{a_1}T^{a_2}\big)\,\mathrm{Tr}\big(T^{a_3}T^{a_4}\big). \end{aligned} \tag{1.1}$$

All other possible trace terms vanish in SU(N) since Tr $T^a) = 0$. The color-ordered amplitudes can be further decomposed [19] in powers of N as

$$L) = \sum_{\lambda=1}^{3}\left(\sum_{k=0}^{\lfloor \frac{L}{2} \rfloor} N^{L-2k} A_{\lambda}^{L,2k)}\right. T_\lambda + \sum_{\lambda=4}^{6}\left(\sum_{k=0}^{\lfloor \frac{L-1}{2} \rfloor} N^{L-2k-1} A_{\lambda}^{L,2k+1)}\right. T_\lambda \tag{1.2}$$

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where $A_\lambda^{L,0)}$ are leading-order-in-N (planar) amplitudes, and $A_\lambda^{L,k)}$, $k = 1, \dots, L$, are subleading-order, yielding $3L + 3)$ color-ordered amplitudes at L loops.

Alternatively, amplitudes may be decomposed into a basis of color factors [17,18]. It is in such a basis that color-kinematic duality is manifest [1,2]. The number of linearly-independent L-loop color factors, however, is less than the number of elements of the L-loop trace basis, implying the existence of constraints among $A_\lambda^{L,k)}$. In this Letter we show that, for even L, the color-ordered amplitudes must satisfy

$$6\sum_{\lambda=1}^{3} A_\lambda^{L,L-2)} - \sum_{\lambda=4}^{6} A_\lambda^{L,L-1)} = 0, \tag{1.3}$$

$$A_{\lambda+3}^{L,L-1)} + A_\lambda^{L,L)} = \text{independent of } \lambda, \tag{1.4}$$

$$\sum_{\lambda=1}^{3} A_\lambda^{L,L)} = 0, \tag{1.5}$$

while for odd L, the relations are

$$6\sum_{\lambda=1}^{3} A_\lambda^{L,L-3)} - \sum_{\lambda=4}^{6} A_\lambda^{L,L-2)} + 2\sum_{\lambda=1}^{3} A_\lambda^{L,L-1)} = 0, \tag{1.6}$$

$$6\sum_{\lambda=1}^{3} A_\lambda^{L,L-1)} - \sum_{\lambda=4}^{6} A_\lambda^{L,L)} = 0, \tag{1.7}$$

$$A_\lambda^{L,L)} = \text{independent of } \lambda. \tag{1.8}$$

These constraints generalize known group theory relations at tree-level [11,12], one loop [14], and two loops [16] to all loop orders. In particular, we note that Eqs. (1.3), (1.5), (1.7), and (1.8) can alternatively be derived by expanding the amplitude in a U(N) trace basis and requiring that any amplitude containing one or more gauge bosons in the U(1) subgroup vanish. Such U(1) decoupling arguments, however, cannot be used to obtain Eqs. (1.4) and (1.6).

Since the space of L-loop color factors is by construction at least $3L - 1)$-dimensional (for $L \geqslant 2$), Eqs. (1.3)–(1.8) are the maximal set of constraints on color-ordered amplitudes that follow from SU(N) group theory alone.[1] It is interesting that these constraints only involve the three or four most-subleading-in-$1/N$ color-ordered amplitudes at a given loop order; other amplitudes are not constrained at all by group theory. Of course, color-kinematic duality implies further relations among the amplitudes [1,2]. Other recent work on constraints among loop-level amplitudes includes Refs. [20–22].

In Section 2, we describe the relation between color and trace bases, and how to use this to derive constraints among color-ordered amplitudes. In Section 3, we apply this to four-point amplitudes through two loops, and then develop and solve all-loop-order recursion relations yielding constraints for four-point color-ordered amplitudes. In Appendix A, we provide details about the three- and four-loop cases.

2. Color and trace bases

In this section, we schematically outline the approach we use to obtain constraints among color-ordered amplitudes. This approach was used in Ref. [23] for tree-level and one-loop ve-point amplitudes.

The n-point amplitude in a gauge theory containing only elds in the adjoint representation of SU(N) (such as pure Yang–Mills or supersymmetric Yang–Mills theory) can be written in a loop expansion, with the L-loop contribution given by a sum of L-loop Feynman diagrams. Suppressing n and L, as well as all momentum and polarization dependence, we can express the L-loop amplitude in the "parent-graph" decomposition [24]

$$= \sum_i a_i c_i \tag{2.1}$$

where $\{c_i\}$ represents a complete set of color factors of L-loop n-point diagrams built from cubic vertices with a factor of the SU(N) structure constants $\tilde{f}^{abc}$ at each vertex. Contributions from Feynman diagrams containing quartic vertices with factors of $\tilde{f}^{abe}\tilde{f}^{cde}$, $\tilde{f}^{ace}\tilde{f}^{bde}$, and $\tilde{f}^{ade}\tilde{f}^{bce}$ can be parceled out among other diagrams containing only cubic vertices. The set of color factors may be overcomplete, in which case they satisfy relations of the form

$$\sum_i \ell_i c_i = 0. \tag{2.2}$$

In fact, it is often necessary to use an overcomplete basis to make color-kinematic duality manifest [1,8]. Although the amplitude (2.1) is gauge invariant, the individual terms in the sum may not be. Any gauge-dependent pieces of the form $a_i = \ell_i f$ (where f is independent of i) will cancel out due to Eq. (2.2).

[1] If our recursive procedure together with the Jacobi relations do not yield the entire space of L-loop color factors, then some of these constraints could be violated for $L > 4$, though we think this unlikely.

The L-loop amplitude may alternatively be expressed in terms of a trace basis $\{t_\lambda\}$ as

$$= \sum_\lambda A_\lambda t_\lambda \tag{2.3}$$

where A_λ are gauge-invariant color-ordered amplitudes. One can convert the amplitude (2.1) into the trace basis by writing

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \mathrm{Tr}\big(\big[T^a, T^b\big]T^c\big) \tag{2.4}$$

and using the SU(N) identities

$$\begin{aligned} &\mathrm{Tr}\big(PT^a\big)\,\mathrm{Tr}\big(QT^a\big) = \mathrm{Tr}\ \ PQ) - \frac{1}{N}\mathrm{Tr}\ \ P)\,\mathrm{Tr}\ \ Q), \\ &\mathrm{Tr}\big(PT^aQT^a\big) = \mathrm{Tr}\ \ P)\,\mathrm{Tr}\ \ Q) - \frac{1}{N}\mathrm{Tr}\ \ PQ) \end{aligned} \tag{2.5}$$

to express the color factor c_i as a linear combination of traces

$$c_i = \sum_\lambda M_{i\lambda} t_\lambda. \tag{2.6}$$

The color-ordered amplitudes are then given by

$$A_\lambda = \sum_i a_i M_{i\lambda}. \tag{2.7}$$

Any constraints (2.2) among the color factors correspond to left null eigenvectors of the transformation matrix

$$\sum_i \ell_i M_{i\lambda} = 0. \tag{2.8}$$

The transformation matrix will also have a set of right null eigenvectors

$$\sum_\lambda M_{i\lambda} r_\lambda = 0. \tag{2.9}$$

Each right null eigenvector implies a constraint

$$\sum_\lambda A_\lambda r_\lambda = 0 \tag{2.10}$$

on the color-ordered amplitudes.

3. Constraints on color-ordered four-point amplitudes

In Eq. (1.2), we decomposed the L-loop four-point amplitude in terms of the six-dimensional trace basis $\{T_\lambda\}$ de ned in Eq. (1.1). The $1/N$ expansion suggests enlarging the trace basis to the $3L+3$)-dimensional basis $\{t_\lambda^{L)}\}$:

$$\begin{aligned} t_{1+6k}^{L)} &= N^{L-2k} T_1, & t_{4+6k}^{L)} &= N^{L-2k-1} T_4, \\ t_{2+6k}^{L)} &= N^{L-2k} T_2, & t_{5+6k}^{L)} &= N^{L-2k-1} T_5, \\ t_{3+6k}^{L)} &= N^{L-2k} T_3, & t_{6+6k}^{L)} &= N^{L-2k-1} T_6, \end{aligned} \tag{3.1}$$

in terms of which Eq. (1.2) becomes

$$^{L)} = \sum_{\lambda=1}^{3L+3} A_\lambda^{L)} t_\lambda^{L)}, \quad \text{where } A_{\lambda+6k}^{L)} = \begin{cases} A_\lambda^{L,2k)}, & \lambda = 1,2,3, \\ A_\lambda^{L,2k+1)}, & \lambda = 4,5,6. \end{cases} \tag{3.2}$$

The decomposition (2.6) of color factors c_i into the trace basis $\{t_\lambda^{L)}\}$ shows that the number of independent L-loop color factors cannot exceed $3L+3$. The dimension of the space of color factors is actually less than this, being 2-dimensional at tree level, 3-dimensional at one loop, and $3L-1$)-dimensional for $L \geqslant 2$ (only proven for L 4). As we will illustrate below, this implies the existence of right null eigenvectors (2.9) of the transformation matrix $M_{i\lambda}^{L)}$ and corresponding constraints (2.10) among the color-ordered amplitudes $A_\lambda^{L)}$.

At tree level, the space of color factors is spanned by the t-channel exchange diagram

$$C_{st}^{0)} = \tilde{f}^{a_1 a_4 b} \tilde{f}^{a_3 a_2 b} = t_1^{0)} - t_3^{0)} \tag{3.3}$$

and the corresponding s-channel exchange diagram

$$C_{ts}^{0)} = \tilde{f}^{a_1 a_2 b} \tilde{f}^{a_3 a_4 b} = t_1^{0)} - t_2^{0)}. \tag{3.4}$$

The u-channel diagram is related to these by the Jacobi identity. With $\{c_1, c_2\} = \{C_{st}^{(0)}, C_{ts}^{(0)}\}$, the transformation matrix (2.6) and its right null eigenvector (2.9) are

$$M_{i\lambda}^{(0)} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad r^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.5}$$

which implies the U(1) decoupling identity among color-ordered tree amplitudes [11,12]

$$A_1^{(0)} + A_2^{(0)} + A_3^{(0)} = 0. \tag{3.6}$$

This is Eq. (1.5) for $L = 0$.

The color factor of the one-loop box diagram

$$C_{st}^{(1)} = C_{ts}^{(1)} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{ca_3d}\tilde{f}^{da_4e} = t_1^{(1)} + 2\big(t_4^{(1)} + t_5^{(1)} + t_6^{(1)}\big) \tag{3.7}$$

and its independent permutations $C_{us}^{(1)}$ and $C_{tu}^{(1)}$ span the space of one-loop color factors, giving

$$M_{i\lambda}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}. \tag{3.8}$$

Alternatively, we can choose[2] for our basis $NC_{st}^{(0)}$ and $NC_{ts}^{(0)}$, together with $C_{st}^{(1)}$, to give

$$M_{i\lambda}^{(1)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}. \tag{3.9}$$

In either case, the transformation matrix has three independent right null eigenvectors

$$r^{(1)} = \begin{pmatrix} 6u \\ -u \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \text{where } u \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x \equiv \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad y \equiv \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{3.10}$$

implying three relations among the one-loop color-ordered amplitudes [14]

$$A_4^{(1)} = A_5^{(1)} = A_6^{(1)} = 2\big(A_1^{(1)} + A_2^{(1)} + A_3^{(1)}\big). \tag{3.11}$$

These are Eqs. (1.7) and (1.8) for $L = 1$.

At two loops, the ladder and non-planar diagrams[3] yield the color factors

$$C_{st}^{(2L)} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{cgd}\tilde{f}^{dfe}\tilde{f}^{ga_3h}\tilde{f}^{ha_4f} = t_1^{(2)} + 6t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)}, \tag{3.12}$$

$$C_{st}^{(2NP)} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{cgd}\tilde{f}^{hfe}\tilde{f}^{ga_3h}\tilde{f}^{da_4f} = -2t_4^{(2)} - 2t_5^{(2)} + 4t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)}. \tag{3.13}$$

The non-planar color factors can be expressed in terms of the planar ones,

$$3C_{st}^{(2NP)} = C_{st}^{(2L)} - C_{ts}^{(2L)} - C_{us}^{(2L)} + C_{su}^{(2L)}, \tag{3.14}$$

and a linear relation exists among the planar color factor and its permutations,

$$0 = C_{st}^{(2L)} - C_{ts}^{(2L)} + C_{us}^{(2L)} - C_{su}^{(2L)} + C_{tu}^{(2L)} - C_{ut}^{(2L)}. \tag{3.15}$$

We could therefore choose five of the six permutations of the ladder diagram to span the space of two-loop color factors; alternatively, we can use $N^2C_{st}^{(0)}$, $N^2C_{ts}^{(0)}$, and $NC_{st}^{(1)}$, together with $C_{st}^{(2L)}$ and $C_{ts}^{(2L)}$, to obtain

$$M_{i\lambda}^{(2)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 \end{pmatrix}. \tag{3.16}$$

The two-loop transformation matrix has four independent right null eigenvectors

$$r^{(2)} = \begin{pmatrix} 6u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \tag{3.17}$$

implying four two-loop group-theory relations [16]

[2] This makes sense since we can use the Jacobi identity to replace the one-loop box diagram with another box diagram with permutated legs plus a tree diagram with one of the vertices replaced by a triangle diagram. The latter is proportional to a tree diagram since $\tilde{f}^{da_1b}\tilde{f}^{ba_2c}\tilde{f}^{ca_3d} = N\tilde{f}^{a_1a_2a_3}$.

[3] It can be easily shown that any other two-loop diagram is related to these ones by Jacobi relations.

$$
\begin{aligned}
0 &= A_4^{2)} + A_7^{2)} - 2\left(A_1^{2)} + A_2^{2)} + A_3^{2)}\right), \\
0 &= A_5^{2)} + A_8^{2)} - 2\left(A_1^{2)} + A_2^{2)} + A_3^{2)}\right), \\
0 &= A_6^{2)} + A_9^{2)} - 2\left(A_1^{2)} + A_2^{2)} + A_3^{2)}\right), \\
0 &= A_7^{2)} + A_8^{2)} + A_9^{2)}
\end{aligned}
\tag{3.18}
$$

equivalent to Eqs. (1.3)–(1.5) for $L=2$.

We now employ a recursive procedure to obtain null eigenvectors for higher-loop color factors. An $L+1)$-loop diagram may be obtained from an L-loop diagram by attaching a rung between two of its external legs, i and j. This corresponds to contracting its color factor with $\tilde{f}^{a_i a'_i b}\tilde{f}^{b a'_j a_j}$. Note that if i and j are not adjacent, this will convert a planar diagram into a non-planar diagram. First consider the effect of this procedure [25] on the trace basis (1.1)

$$
T_\lambda - \sum_{\kappa=1}^{6} G_{\lambda\kappa} T_\kappa, \quad \text{where } G = \begin{pmatrix} NA & B \\ C & ND \end{pmatrix}
\tag{3.19}
$$

with

$$
A = \begin{pmatrix} e_{12}+e_{14} & 0 & 0 \\ 0 & e_{12}+e_{13} & 0 \\ 0 & 0 & e_{13}+e_{14} \end{pmatrix}, \qquad
B = \begin{pmatrix} 0 & 2e_{14}-2e_{13} & 2e_{12}-2e_{13} \\ 2e_{13}-2e_{14} & 0 & 2e_{12}-2e_{14} \\ 2e_{13}-2e_{12} & 2e_{14}-2e_{12} & 0 \end{pmatrix},
$$

$$
C = \begin{pmatrix} 0 & e_{12}-e_{14} & e_{14}-e_{12} \\ e_{12}-e_{13} & 0 & e_{13}-e_{12} \\ e_{14}-e_{13} & e_{13}-e_{14} & 0 \end{pmatrix}, \qquad
D = \begin{pmatrix} 2e_{13} & 0 & 0 \\ 0 & 2e_{14} & 0 \\ 0 & 0 & 2e_{12} \end{pmatrix},
\tag{3.20}
$$

where the coefficient of e_{1j} corresponds to connecting legs 1 and j. On the expanded basis (3.1), the same procedure yields

$$
t_\lambda^{L)} \quad \sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} t_\kappa^{L+1)},
\tag{3.21}
$$

where g is the $3L+3) \times 3L+6)$ matrix

$$
g = \begin{pmatrix} A & B & 0 & 0 & 0 & \cdots \\ 0 & D & C & 0 & 0 & \cdots \\ 0 & 0 & A & B & 0 & \cdots \\ 0 & 0 & 0 & D & C & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\tag{3.22}
$$

Next, given some L-loop diagram with color factor $c_i^{L)}$, we can connect two of its external legs with a rung to obtain an $L+1)$-loop diagram with color factor

$$
c_i^{L+1)} = \sum_{\kappa=1}^{3L+6} M_{i\kappa}^{L+1)} t_\kappa^{L+1)}
\tag{3.23}
$$

where

$$
M_{i\kappa}^{L+1)} = \sum_{\lambda=1}^{3L+3} M_{i\lambda}^{L)} g_{\lambda\kappa}, \quad \text{with } c_i^{L)} = \sum_{\lambda=1}^{3L+3} M_{i\lambda}^{L)} t_\lambda^{L)}.
\tag{3.24}
$$

Now, suppose we possess a complete set of L-loop color factors $\{c_i^{L)}\}$ and a maximal set of right null eigenvectors $\{r_\lambda^{L)}\}$:

$$
\sum_{\lambda=1}^{3L+3} M_{i\lambda}^{L)} r_\lambda^{L)} = 0.
\tag{3.25}
$$

Then the color factors of all $L+1)$-loop diagrams obtained by connecting two external legs of any L-loop diagram will have a right null eigenvector

$$
\sum_{\kappa=1}^{3L+6} M_{i\kappa}^{L+1)} r_\kappa^{L+1)} = 0
\tag{3.26}
$$

provided that $r_\kappa^{L+1)}$ satis es

$$
\sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} r_\kappa^{L+1)} = \text{linear combination of } \left\{r_\lambda^{L)}\right\}.
\tag{3.27}
$$

We can now solve Eq. (3.27) recursively, beginning with the set of $L=2$ right null eigenvectors (3.17), the rst case with four independent eigenvectors. The maximal set of right null eigenvectors satisfying Eq. (3.27) is

$$\{r^{2\ell+1)}\} = \begin{pmatrix} \vdots \\ 0 \\ 6u \\ -u \\ 2u \\ 0 \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 6u \\ -u \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ y \end{pmatrix}, \qquad \{r^{2\ell)}\} = \begin{pmatrix} \vdots \\ 0 \\ 6u \\ -u \\ 0 \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ x \\ x \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ y \\ y \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ u \end{pmatrix}. \tag{3.28}$$

The constraints on color-ordered amplitudes

$$\sum_\lambda A_\lambda^{L)} r_\lambda^{L)} = 0 \tag{3.29}$$

that follow from the set of right null eigenvectors (3.28) can be written in terms of Eq. (3.2) to yield the constraints (1.3)–(1.8) given in the introduction.

Since there are generally four[4] linearly-independent null eigenvectors in a $3L+3)$-dimensional trace space, the space of L-loop color factors satisfying Eq. (3.25) is generally $3L-1)$-dimensional.[5] Since there are no further independent solutions of Eq. (3.27), we have shown that the full space of L-loop color factors is *at least* $3L-1)$-dimensional.

We have not strictly shown that Eq. (3.28) are null eigenvectors for *any* possible color factor associated with an L-loop diagram, but rather only for those that can be obtained from an $L-1)$-loop diagram by attaching a rung between two external legs. It is therefore conceivable (but we think unlikely) that the space of *all* L-loop color factors could be greater than $3L-1)$-dimensional. However, for $L=3$ and $L=4$, it has been shown [24] that, despite the fact that many diagrams cannot be obtained by attaching a rung to the external legs of lower-loop diagrams, all color factors can be related to these using Jacobi relations (see Appendix A for further discussion of $L=3$ and $L=4$). It would be nice to have a proof of this for all L, however.

4. Conclusions

In this Letter, we have extended known group theory identities for four-point color-ordered amplitudes in SU(N) gauge theories to all loop orders. We have shown that color-ordered amplitude generally must satisfy four independent relations at each loop order (except for $L=0$ and $L=1$, where there are one and three constraints respectively). This was achieved via a recursive procedure that derives the constraints on L-loop color factors generated by attaching a rung between two external legs of an $L-1)$-loop color factor. Assuming that all L-loop color factors are linear combinations of those just described (i.e., via Jacobi relations), then the constraints derived apply to all L-loop color-ordered amplitudes. Although this has been established through four loops, it would clearly be desirable to have an all-orders proof of this assumption.

The recursive method employed in this Letter can also be extended to n-point functions with $n>4$ to yield constraints on the color-ordered amplitudes beyond those already known at tree- [13] and one-loop [14,15] level, although the size of the color basis grows quickly with n.

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Appendix A

In Appendix B of Ref. [24], bases for the space of all three- and four-loop color factors were identi ed. In this appendix, we explicitly check that the right null eigenvectors of these spaces coincide with our recursive solution (3.28), and therefore that all three- and four-loop color-ordered amplitudes indeed satisfy the group theory constraints (1.3)–(1.8).

The basis for three-loop color factors can be chosen as $N^3 C_{st}^{0)}$, $N^3 C_{ts}^{0)}$, $N^2 C_{st}^{1)}$, $N C_{st}^{2L)}$, and $N C_{ts}^{2L)}$, plus the color factor for the three-loop ladder diagram

$$C_{st}^{3L)} = t_1^{3)} + 14t_6^{3)} + 2t_7^{3)} + 2t_8^{3)} + 8t_{10}^{3)} + 8t_{11}^{3)} + 8t_{12}^{3)} \tag{A.1}$$

and two of its permutations,[6] $C_{ts}^{3L)}$ and $C_{us}^{3L)}$, yielding the transformation matrix

4 One for $L=0$ and three for $L=1$.

5 Two-dimensional for $L=0$ and three-dimensional for $L=1$.

6 Only $C_{st}^{3L)}$ is used in Ref. [24], but the authors also include $N C_{st}^{0)}$ and $N C_{ts}^{0)}$ in their basis, which in our approach are independent of $N^3 C_{st}^{0)}$ and $N^3 C_{ts}^{0)}$.

$$M_{i\lambda}^{3)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 8 & 8 & 8 \\ 1 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 2 & 8 & 8 & 8 \\ 0 & 1 & 0 & 14 & 0 & 0 & 0 & 2 & 2 & 8 & 8 & 8 \end{pmatrix}. \tag{A.2}$$

The four independent right null eigenvectors of this matrix

$$r^{\,3)} = \begin{pmatrix} 6u \\ -u \\ 2u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 6u \\ -u \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ y \end{pmatrix} \tag{A.3}$$

agree with those in Eq. (3.28), and imply the four constraints among the color-ordered amplitudes given by Eqs. (1.6)–(1.8) with $L = 3$.

The four-loop color basis can be chosen as (N times) the three-loop basis plus three color factors from the four-loop ladder diagram and two[7] permutations, $C_{st}^{4L)}$, $C_{ts}^{4L)}$, and $C_{us}^{4L)}$, yielding

$$M_{i\lambda}^{4)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 2 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 14 & 0 & 0 & 0 & 2 & 2 & 8 & 8 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 30 & 2 & 2 & 0 & 0 & 0 & 24 & 8 & 8 & 16 \\ 1 & 0 & 0 & 0 & 30 & 0 & 2 & 0 & 2 & 0 & 24 & 0 & 8 & 16 & 8 \\ 0 & 1 & 0 & 30 & 0 & 0 & 0 & 2 & 2 & 24 & 0 & 0 & 16 & 8 & 8 \end{pmatrix}. \tag{A.4}$$

The four independent right null eigenvectors of this matrix

$$r^{\,4)} = \begin{pmatrix} 0 \\ 0 \\ 6u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ y \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u \end{pmatrix}, \tag{A.5}$$

agree with those in Eq. (3.28). The right null eigenvalues imply the four relations among color-ordered amplitudes given by Eqs. (1.3)–(1.5) for $L = 4$.

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[7] Only $C_{st}^{4L)}$ and $C_{ts}^{4L)}$ are used in Ref. [24], but the authors also include $NC_{st}^{1)}$, which in our approach counts as independent from $N^3C_{st}^{1)}$.