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All-loop infrared-divergent behavior of most-subleading-color gauge-theory amplitudes

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Abstract: The infrared singularities of gravitational amplitudes are one-loop exact, in that higher-loop divergences are characterized by the exponential of the one-loop divergence. We show that the contributions to SU(N) gauge-theory amplitudes that are most-subleading in the $1/N$ expansion are also one-loop exact, provided that the dipole conjecture holds. Possible corrections to the dipole conjecture, beginning at three loops, could violate one-loop-exactness, though would still maintain the absence of collinear divergences. We also demonstrate a relation between L-loop four-point $\mathcal{N} = 8$ supergravity and most-subleading-color $\mathcal{N} = 4$ SYM amplitudes that holds for the two leading IR divergences, $\mathcal{O}(1/\epsilon^L)$ and $\mathcal{O}(1/\epsilon^{L-1})$, but breaks down at $\mathcal{O}(1/\epsilon^{L-2})$.

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Contents

1 Introduction

The structure of infrared divergences in scattering amplitudes of massless particles has been an object of much study over past decades. The IR behavior of gravity amplitudes, in particular, has a remarkable simplicity [1], traceable to the absence of collinear divergences, and also to the fact that divergences only arise from soft graviton exchange between external particles; non-abelian-like interactions among virtual gravitons ultimately do not contribute to the IR-divergent behavior. s a result, in dimensional regularization (in $D = 4 - 2\epsilon$ dimensions), the leading IR divergence at L loops [2, 3] goes as $1/\epsilon^L$, and further, the IR behavior is one-loop exact; that is, all L-loop divergences arise from the exponential of the one-loop divergence [1–6].

By contrast, the structure of IR divergences of non-abelian gauge theories is a richer subject; both collinear and soft divergences appear. The IR divergences of a gauge theory amplitude can be factored into a product of jet functions and a soft function acting on an IR-finite hard function [7, 8]. The soft function depends on a soft anomalous dimension matrix $\mathbf{\Gamma}^{(L)}$ at each loop level. Recently, strong constraints on the form of $\mathbf{\Gamma}^{(L)}$ were derived using soft collinear effective theory [9] and Sudakov factorization and momentum rescaling [10]. The simplest solution to these constraints is the sum-over-color-dipoles formula [9–11], which essentially states that $\mathbf{\Gamma}^{(L)}$ is proportional to $\mathbf{\Gamma}^{(1)}$ for all L. (This proportionality had previously been established at two loops in ref. [8], and conjectured to

JHEP04(2013)114

be true for all L in ref. [12].) lthough departures from the dipole formula are not ruled out at three loops and beyond, the kinematical dependence of such corrections is highly constrained [9–11, 13–15].

One can organize the scattering amplitudes of an SU(N) gauge theory in a combined loop and $1/N$ expansion. The leading-color (planar) L-loop n-point amplitude $^{(L,0)}$ is proportional to $g^{2}(g^2N)^L$, while the subleading-color amplitudes $^{(L,k)}$, with $k = 1, \cdots, L$, are down by $1/N^k$ relative to the planar amplitude. The most-subleading-color amplitudes $^{(L,L)}$ are independent of N. While the leading divergence of L-loop planar gauge theory amplitudes goes as $\mathcal{O}(1/\epsilon^{2L})$, the subleading-color amplitudes $^{(L,k)}$ are less divergent, with a leading divergence of $\mathcal{O}(1/\epsilon^{2L-k})$ [16–19]. Consequences of the dipole formula for the IR behavior of subleading-color amplitudes were derived in ref. [17, 19].

The fact that the most-subleading-color amplitudes $^{(L,L)}$ only go as $\mathcal{O}(1/\epsilon^L)$ suggests that they, like gravity amplitudes, have no collinear IR divergences, only soft IR divergences. In this paper we will explore whether the IR divergences of most-subleading-color amplitudes also have the property of being one-loop exact, as are gravity amplitudes. We will show that, provided the soft anomalous dimension matrices obey the dipole formula, this is indeed the case. That is, in a given trace basis, the IR divergences can be written in terms of the exponential of a matrix that describes the one-loop divergence. We also provide explicit expressions for the IR divergences of the most-subleading-color four-point amplitude for arbitrary L.

Corrections to the dipole formula, if present, begin at three loops. We compute the $1/N$ expansion of a possible three-loop correction term to the dipole formula and show that it would affect the most-subleading-color three-loop four-point amplitude $^{(3,3)}$ at $\mathcal{O}(1/\epsilon)$, spoiling the one-loop exactness of its IR behavior, although collinear IR divergences remain absent.

Finally, the similarity between gravity and most-subleading-color gauge-theory amplitudes can be used to deduce a relation between L-loop four-point $\mathcal{N} = 8$ supergravity and most-subleading-color $\mathcal{N} = 4$ SYM amplitudes that holds for the two leading IR divergences, $\mathcal{O}(1/\epsilon^L)$ and $\mathcal{O}(1/\epsilon^{L-1})$, but breaks down at $\mathcal{O}(1/\epsilon^{L-2})$.

In section 2, we review the one-loop exactness of gravity amplitudes. In section. 3, we demonstrate that most-subleading-color SU(N) gauge theory amplitudes are similarly one-loop exact, provided the dipole conjecture holds. In section 4, we derive an expression for the full IR divergences of the L-loop four-point most-subleading-color amplitude in terms of lower-loop amplitudes. In section 5, we examine the effect of a possible three-loop correction to the dipole conjecture. In section 6 we deduce a relation between four-point $\mathcal{N} = 8$ supergravity and most-subleading-color $\mathcal{N} = 4$ SYM amplitudes. Various technical details may be found in three appendices.

2 Infrared divergences of gravity amplitudes

The pioneering study of the IR singularities of gravitational theories by Weinberg [1] showed that these are one-loop exact in the sense that all IR divergences are characterized by the exponential of the one-loop divergence. Dunbar and Norridge later revisited this issue in

JHEP04(2013)114

the context of string theory [2]. Recently two of us [4] reformulated this problem in analogy with the modern treatment of IR singularities in gauge theories, and several other authors have studied and extended the subject from this point of view [5, 6, 20, 21].

In ref. [4] it was proposed that the n-graviton scattering amplitude can be written as

$$= S \cdot H \tag{2.1}$$

where S is the gravitational soft function, an IR-divergent factor describing the effects of soft graviton exchange between the n external particles, and H is the IR-finite hard function. Contrary to gauge theories, there are no jet functions, as collinear singularities are absent after summing over diagrams. We expand each of the quantities in eq. (2.1) in a loop expansion in powers of $\lambda = (\kappa/2)^2(4\pi e^{-\gamma})^{\epsilon}$, where $\kappa^2 = 32\pi G$:

$$= \sum_{L=0}^{\infty} {}^{(L)}, \qquad S = 1 + \sum_{L=1}^{\infty} S^{(L)}, \qquad H = \sum_{L=0}^{\infty} H^{(L)}. \tag{2.2}$$

IR divergences are regulated using dimensional regularization in $D = 4 - 2\epsilon$, with $\epsilon < 0$. Then, due to the fact that all IR singularities are associated with single graviton exchanges between pairs of external particles, the soft function is given by the exponential of the one-loop IR divergence [1–6]

$$S = \exp\left[\frac{\sigma}{\epsilon}\right], \qquad \sigma = \frac{\lambda}{16\pi^2} \sum_{j=1} \sum_{i<j} s_{ij} \log\left(\frac{-s_{ij}}{\mu^2}\right), \qquad s_{ij} = (k_i + k_j)^2. \tag{2.3}$$

(ny IR-finite contributions from these exchanges can be absorbed into H .) Hence the IR divergences of the gravitational amplitude are one-loop exact

$$= \exp\left[\frac{\sigma}{\epsilon}\right] H \tag{2.4}$$

and the L-loop amplitude can be expressed as

$${}^{(L)} = \sum_{\ell=0}^{L} \frac{1}{(L-\ell)!} \left[\frac{\sigma}{\epsilon}\right]^{L-\ell} H^{(\ell)}(\epsilon), \tag{2.5}$$

that is, the L-loop IR divergences are determined by σ together with the IR-finite contributions (including terms proportional to positive powers of ϵ) of all the lower-loop amplitudes. By keeping the first two terms

$${}^{(L)} = \frac{1}{L!}\left[\frac{\sigma}{\epsilon}\right]^{L} H^{(0)}(\epsilon) + \frac{1}{(L-1)!}\left[\frac{\sigma}{\epsilon}\right]^{L-1} H^{(1)}(\epsilon) + \mathcal{O}(1/\epsilon^{L-2}) \tag{2.6}$$

we observe that the two leading IR divergences of the L-loop amplitude are completely determined by the tree and one-loop amplitudes. Moreover, since $H^{(1)}(L\epsilon) = H^{(1)}(\epsilon) + \mathcal{O}(\epsilon)$, we see that the two leading divergences of the L-loop amplitude can be related to the one-loop amplitude evaluated in $D = 4 - 2L\epsilon$ dimensions:

$${}^{(L)}(\epsilon) = \frac{1}{(L-1)!}\left[\frac{\sigma}{\epsilon}\right]^{L-1} {}^{(1)}(L\epsilon) + \mathcal{O}(1/\epsilon^{L-2}). \tag{2.7}$$

JHEP04(2013)114

In section 4, we will find an analogous relationship for the most-subleading-color YM amplitude.

For the remainder of this section, we restrict ourselves to the four-point amplitude of $\mathcal{N} = 8$ supergravity. In this case, the all-loop-orders amplitude is proportional to the tree-level amplitude [22, 23], allowing us to define the helicity-independent ratios

$$M_4 = \quad _4/ \quad ^{(0)}_4, \qquad\qquad M_4^{(f)} = H_4/ \quad ^{(0)}_4 . \tag{2.8}$$

Then eqs. (2.3) and (2.4) imply

$$M_4 = \exp\left[\frac{\sigma_4}{\epsilon}\right] M_4^{(f)}, \qquad\qquad \sigma_4 = \frac{\lambda}{8\pi^2}\left[s\log\left(\frac{-s}{\mu^2}\right) + t\log\left(\frac{-t}{\mu^2}\right) + u\log\left(\frac{-u}{\mu^2}\right)\right] \tag{2.9}$$

where $s = s_{12}$, $t = s_{14}$, and $u = s_{13}$. Consequently, the logarithm of the ratio of the full amplitude to the tree amplitude

$$\Lambda_4 \equiv \log M_4 = \frac{\sigma_4}{\epsilon} + \log M_4^{(f)} \tag{2.10}$$

only has an IR divergence at one loop

$$\Lambda_4^{(1)} = M_4^{(1)} = \frac{\sigma_4}{\epsilon} + M_4^{(1f)} . \tag{2.11}$$

By expanding

$$M_4 = 1 + \sum_{L=1}^{\infty} M_4^{(L)} = \exp\left(\Lambda_4\right) = \exp\left(\sum_{L=1}^{\infty} \Lambda_4^{(L)}\right) \tag{2.12}$$

we can obtain explicit expressions at each loop order

$$\begin{aligned}
M_4^{(2)} &= \tfrac{1}{2}(\Lambda_4^{(1)})^2 + \Lambda_4^{(2)} , \\
M_4^{(3)} &= \tfrac{1}{6}(\Lambda_4^{(1)})^3 + \Lambda_4^{(1)}\Lambda_4^{(2)} + \Lambda_4^{(3)} , \\
M_4^{(4)} &= \tfrac{1}{24}(\Lambda_4^{(1)})^4 + \tfrac{1}{2}(\Lambda_4^{(1)})^2\Lambda_4^{(2)} + \Lambda_4^{(1)}\Lambda_4^{(3)} + \tfrac{1}{2}(\Lambda_4^{(2)})^2 + \Lambda_4^{(4)} , \\
M_4^{(5)} &= \tfrac{1}{120}(\Lambda_4^{(1)})^5 + \tfrac{1}{6}(\Lambda_4^{(1)})^3\Lambda_4^{(2)} + \tfrac{1}{2}(\Lambda_4^{(1)})^2\Lambda_4^{(3)} + \tfrac{1}{2}\Lambda_4^{(1)}(\Lambda_4^{(2)})^2 + \Lambda_4^{(1)}\Lambda_4^{(4)} + \Lambda_4^{(2)}\Lambda_4^{(3)} + \Lambda_4^{(5)} , \\
&\vdots
\end{aligned} \tag{2.13}$$

Since $\Lambda_4^{(1)}$ diverges as $1/\epsilon$, and $\Lambda_4^{(L)}$ are IR-finite for all $L \geq 2$, we see that the two leading IR-divergent terms of the L-loop amplitude can be expressed in terms of the one-loop amplitude

$$M_4^{(L)}(\epsilon) = \frac{1}{L!}\left[M_4^{(1)}(\epsilon)\right]^L + \mathcal{O}(1/\epsilon^{L\ \ 2}) . \tag{2.14}$$

s in the general case (2.7), we can also write this as a relation between the L-loop amplitude and the one-loop amplitude evaluated in $D = 4 - 2L\epsilon$ dimensions:

$$M_4^{(L)}(\epsilon) = \frac{1}{(L-1)!}\left[\frac{\sigma_4}{\epsilon}\right]^{L\ \ 1} M_4^{(1)}(L\epsilon) + \mathcal{O}(1/\epsilon^{L\ \ 2}) . \tag{2.15}$$

JHEP04(2013)114

3 Infrared divergences of the most-subleading-color YM amplitudes

In this section we will explore the IR divergences of n-gluon amplitudes that are most-subleading in the $1/N$ expansion. These amplitudes are similar to the n-graviton amplitudes discussed in the previous section in two respects: (1) although the leading IR divergence of an n-gluon amplitude at L loops goes as $1/\epsilon^{2L}$, the leading divergence of the most-subleading-color amplitude is milder, only going as $\mathcal{O}(1/\epsilon^L)$, due to the absence of collinear divergences, and (2) if the dipole conjecture, described below, holds, then the IR divergences of the most-subleading-color amplitudes are one-loop exact; that is, all IR divergences at L loops are determined by the exponential of the one-loop IR divergence, as we will show below. If the dipole conjecture is not valid, then the first property (lack of collinear divergences) continues to hold but the second does not: additional IR divergences unrelated to the one-loop divergence could be present, potentially beginning at three loops. We will describe the form of a potential three-loop correction to the most-subleading-color four-point function in section 5.

The n-point amplitude of particles transforming in the adjoint representation (e.g., gluons) can be expanded in a trace basis $\{T_\lambda\}$, consisting of single and multiple traces of generators in the fundamental representation,

$$= \sum_\lambda T_\lambda \quad _\lambda \tag{3.1}$$

where the coefficients $_\lambda$ are referred to as color-ordered amplitudes. It is convenient to organize [24, 25] the color-ordered amplitudes into a vector $|\ \rangle$. In an SU(N) gauge theory, this vector can be decomposed in a simultaneous loop and $1/N$ expansion[1]

$$|\ \rangle = \sum_{L=0}^{\infty} \sum_{k=0}^{L} \frac{a(\mu^2)^L}{N^k} \left|\ ^{(L,k)}\right\rangle \tag{3.2}$$

where

$$a(\mu^2) = \frac{g^2(\mu^2)N}{8\pi^2}(4\pi e^{\ -\gamma})^\epsilon \tag{3.3}$$

is the 't Hooft coupling and μ is the renormalization scale. Our interest in this paper is in the IR behavior of the most-subleading-color amplitudes, that part of the amplitude that depends only on $g^2(\mu^2)$ with no powers of N. Hence, we are interested in the terms $|\ ^{(L,L)}\rangle$ in the expansion (3.2), which carry as many powers of $1/N$ as of $a(\mu^2)$.

We follow refs. [7, 8] by organizing the IR divergences of a gauge theory amplitude as

$$\left|\ \left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon\right)\right\rangle = J\left(a(\mu^2),\epsilon\right)\ \mathbf{S}\left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon\right) \left|H\left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon\right)\right\rangle \tag{3.4}$$

The prefactors J ("jet function") and $\mathbf{S}$ ("soft function") characterize the long-distance IR-divergent behavior, while the short-distance behavior of the amplitude is characterized by $|H\rangle$ ("hard function"), and is finite as $\epsilon \to 0$. (Quantities in boldface act as matrices on the color space vectors.)

[1] We have omitted a overall factor of $g^{\ 2}$ for a -poi t fu ctio .

The jet function has leading IR behavior of $\mathcal{O}(1/\epsilon^{2L})$ at L-loops (although the poles of $\log J$ only go up through $1/\epsilon^{L+1}$ in a generic gauge theory [8], and $1/\epsilon^2$ in $\mathcal{N}=4$ SYM theory [26]). The jet function, however, is irrelevant to the IR divergences of the most-subleading-color amplitude because it carries no factors of $1/N$ to accompany the factors of $a(\mu^2)$.

The soft function [7, 8]

$$\mathbf{S}\left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon\right) = \mathrm{P}\,\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\mathbf{\Gamma}\left(\frac{s_{ij}}{\mu^2}, \bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a(\mu^2), \epsilon\right)\right)\right] \tag{3.5}$$

depends on the soft anomalous dimension matrix, which can be expanded as

$$\mathbf{\Gamma}\left(\frac{s_{ij}}{\mu^2}, a(\mu^2)\right) = \sum_{L=1}^{\infty} a(\mu^2)^L\, \mathbf{\Gamma}^{(L)}\left(\frac{s_{ij}}{\mu^2}\right). \tag{3.6}$$

The one-loop soft anomalous dimension matrix is given by [8]

$$\mathbf{\Gamma}^{(1)} = \frac{1}{N}\sum_{j=1}\sum_{i<j}\mathbf{T}_i\cdot\mathbf{T}_j\log\left(\frac{\mu^2}{-s_{ij}}\right) \tag{3.7}$$

where $\mathbf{T}_i$ are the SU(N) generators in the adjoint representation. Diagrammatically, the operators $\mathbf{T}_i\cdot\mathbf{T}_j$ act by attaching a gluon rung between the legs of the ith and jth external particles. In terms of the color-ordered expansion (3.1), $\mathbf{\Gamma}^{(L)}$ acts on a given element T_λ of the trace basis (3.1) to yield a linear combination

$$\mathbf{\Gamma}^{(L)}T_\lambda = \sum_\kappa T_\kappa \mathbf{\Gamma}^{(L)}_{\kappa\lambda} \tag{3.8}$$

and it is the matrix $\mathbf{\Gamma}^{(L)}_{\kappa\lambda}$ that then acts on the ket $|H\rangle$.

t this point, we invoke the dipole conjecture [9–11], according to which the soft anomalous dimension matrix $\mathbf{\Gamma}^{(L)}$ is proportional to $\mathbf{\Gamma}^{(1)}$ for all L (with the proportionality constants given by the coefficients of the cusp anomalous dimension). This had previously been proven for $\mathbf{\Gamma}^{(2)}$ in ref. [8], and hypothesized to be valid for all L in ref. [12]. Corrections at three loops and above, however, have not (yet) been ruled out completely, although they are highly constrained [9–11, 13–15]. We assume the validity of the dipole formula for the remainder of this section, but in section 5 we will consider the possibility of a violation at three loops.

If the dipole conjecture holds, then $\mathbf{\Gamma}^{(L)}$ all commute with one another, so that path ordering of the exponential in eq. (3.5) is irrelevant. We can then integrate the terms to obtain

$$\mathbf{S}\left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon\right) = \exp\left[\sum_{L=1}^{\infty}\frac{a(\mu^2)^L}{2L\epsilon}\mathbf{\Gamma}^{(L)}\left(1+\mathcal{O}\left(\frac{a(\mu^2)}{\epsilon}\right)\right)\right] \tag{3.9}$$

where the leading form of the running coupling is given by [7, 8]

$$\bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a(\mu^2), \epsilon\right) = a(\mu^2)\left(\frac{\mu^2}{\tilde{\mu}^2}\right)^{\epsilon}\sum_{\ell=0}^{\infty}\left[\frac{\beta_0}{4\pi\epsilon}\left(\left(\frac{\mu^2}{\tilde{\mu}^2}\right)^{\epsilon}-1\right)a(\mu^2)\right]^\ell. \tag{3.10}$$

JHEP04(2013)114

The omitted terms in eq. (3.9), which depend on β_0, the one-loop coefficient of the beta function, will not contribute to the most-subleading-color amplitudes because there are no factors of $1/N$ to accompany the powers of $a(\mu^2)$.

Generically, one would expect the soft anomalous dimension matrices $\mathbf{\Gamma}^{(L)}_{\kappa\lambda}$ to contain terms of $\mathcal{O}(1)$ through $\mathcal{O}(1/N^L)$ in the $1/N$-expansion. If the dipole conjecture is valid, however, then $\mathbf{\Gamma}^{(L)}_{\kappa\lambda}$ is proportional to $\mathbf{\Gamma}^{(1)}_{\kappa\lambda}$, and hence only contains terms of $\mathcal{O}(1)$ and $\mathcal{O}(1/N)$. Since $\mathbf{\Gamma}^{(L)}$ is multiplied by $a(\mu^2)^L$ but carries at most one power of $1/N$, only $\mathbf{\Gamma}^{(1)}$ can contribute to the most-subleading-color amplitude, which now simplifies to

$$| \ \rangle \Big|_{\text{most subleading color}} = \exp\left[\frac{a(\mu^2)}{2\epsilon}\mathbf{\Gamma}^{(1)}_{\text{sub}}\right] |H(\epsilon)\rangle \Big|_{\text{most subleading color}} \tag{3.11}$$

where $\mathbf{\Gamma}^{(1)}_{\text{sub}}$ denotes the $1/N$ contribution of the one-loop soft anomalous dimension matrix.

Eq. (3.11) is parallel to the gravitational analog (2.4). It demonstrates that, provided the dipole conjecture is valid, the IR divergences of the most-subleading-color amplitudes are one-loop exact, that is, determined by the one-loop soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}_{\text{sub}}$ and the finite contributions (including terms proportional to positive powers of ϵ) of lower loop amplitudes, just as in the case of gravitational amplitudes.

4 IR behavior of the most-subleading-color four-point amplitude

In the previous section, we showed that, subject to the validity of the dipole conjecture, the IR divergences of the most-subleading color amplitudes are one-loop exact, given by the exponential of the one-loop soft anomalous dimension matrix. In this section, we will write the IR divergences of the most-subleading-color L-loop amplitude explicitly in the case of the four-point function, using the group-theory relations among four-point color-ordered amplitudes [27].

For the four-point amplitude, the one-loop soft anomalous dimension matrix (3.7) becomes

$$\begin{aligned}\mathbf{\Gamma}^{(1)} = \frac{1}{N}\Bigg[&(\mathbf{T}_1\cdot\mathbf{T}_2+\mathbf{T}_3\cdot\mathbf{T}_4)\log\left(\frac{\mu^2}{-s}\right)+(\mathbf{T}_1\cdot\mathbf{T}_3+\mathbf{T}_2\cdot\mathbf{T}_4)\log\left(\frac{\mu^2}{-u}\right)\\ &+(\mathbf{T}_1\cdot\mathbf{T}_4+\mathbf{T}_2\cdot\mathbf{T}_3)\log\left(\frac{\mu^2}{-t}\right)\Bigg].\end{aligned} \tag{4.1}$$

s described in the previous section, to evaluate this operator, we choose a specific four-point trace basis, consisting of single and double traces of SU(N) generators[2]

$$\begin{aligned} T_1 &= \mathrm{Tr}(1234)+\mathrm{Tr}(1432), & T_4 &= 2\,\mathrm{Tr}(13)\,\mathrm{Tr}(24),\\ T_2 &= \mathrm{Tr}(1243)+\mathrm{Tr}(1342), & T_5 &= 2\,\mathrm{Tr}(14)\,\mathrm{Tr}(23),\\ T_3 &= \mathrm{Tr}(1324)+\mathrm{Tr}(1423), & T_6 &= 2\,\mathrm{Tr}(12)\,\mathrm{Tr}(34). \end{aligned} \tag{4.2}$$

[2]The basis specified here is that of ref. [28], which differs from refs. [19, 27] by a factor of two i the double-trace terms.

JHEP04(2013)114

The six-dimensional ket $|\ \rangle$ then consists of the coefficients $\ _\lambda$ of T_λ in the amplitude (3.1). In this basis, the (subleading-color piece of the) one-loop soft anomalous dimension matrix takes the form

$$\mathbf{\Gamma}^{(1)}_{\rm sub} = \frac{2}{N}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & -2Y & 2X \\ 2Z & 0 & -2X \\ -2Z & 2Y & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & -X & Y \\ X & 0 & -Z \\ -Y & Z & 0 \end{pmatrix} \tag{4.3}$$

where

$$X = \log\left(\frac{t}{u}\right), \qquad Y = \log\left(\frac{u}{s}\right), \qquad Z = \log\left(\frac{s}{t}\right). \tag{4.4}$$

Hence, eq. (3.11) becomes

$$|\ \rangle\Big|_{\rm most\ \ subleading\ \ color} = \exp\left[\frac{a(\mu^2)}{N\epsilon}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right] |H(\epsilon)\rangle\Big|_{\rm most\ \ subleading\ \ color}. \tag{4.5}$$

Expanding both sides in a loop expansion, we can write

$$\Big|\ ^{(L,L)}\Big\rangle = \sum_{\ell=0}^{L} \frac{1}{(L-\ell)!\epsilon^{L\ \ \ell}} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{L\ \ \ell} \Big|H^{(\ell,\ell)}(\epsilon)\Big\rangle, \tag{4.6}$$

analogous to eq. (2.5) for gravitational amplitudes. This expression, valid to all orders in the ϵ expansion, was previously obtained in ref. [17] for $\mathcal{N}=4$ SYM theory, but here we see that it remains valid for the four-gluon amplitude in a general gauge theory provided the dipole conjecture holds.

We rewrite eq. (4.6) separately for even- and odd-loop cases:

$$\begin{aligned}\Big|\ ^{(2\ell,2\ell)}\Big\rangle = \sum_{k=0}^{\ell\ \ 1} \frac{b(cb)^{\ell\ \ k\ \ 1}}{(2\ell-2k)!\ \epsilon^{2\ell\ \ 2k}} \Big[c\Big|H^{(2k,2k)}(\epsilon)\Big\rangle + (2\ell-2k)\epsilon\Big|H^{(2k+1,2k+1)}(\epsilon)\Big\rangle\Big] \\ + \Big|H^{(2\ell,2\ell)}\Big\rangle, \end{aligned} \tag{4.7}$$

$$\begin{aligned}\Big|\ ^{(2\ell+1,2\ell+1)}\Big\rangle = \sum_{k=0}^{\ell} \frac{(cb)^{\ell\ \ k}}{(2\ell-2k+1)!\epsilon^{2\ell\ \ 2k+1}} \\ \times\Big[c\Big|H^{(2k,2k)}(\epsilon)\Big\rangle + (2\ell-2k+1)\epsilon\Big|H^{(2k+1,2k+1)}(\epsilon)\Big\rangle\Big], \end{aligned} \tag{4.8}$$

where now the kets denote 3-dimensional vectors

$$\Big|\ ^{(2\ell,2\ell)}\Big\rangle = \begin{pmatrix} {}^{(2\ell,2\ell)}_{1} \\ {}^{(2\ell,2\ell)}_{2} \\ {}^{(2\ell,2\ell)}_{3} \end{pmatrix}, \qquad \Big|\ ^{(2\ell+1,2\ell+1)}\Big\rangle = \begin{pmatrix} {}^{(2\ell+1,2\ell+1)}_{4} \\ {}^{(2\ell+1,2\ell+1)}_{5} \\ {}^{(2\ell+1,2\ell+1)}_{6} \end{pmatrix}. \tag{4.9}$$

JHEP04(2013)114

We are able to further simplify the expressions (4.7) and (4.8) by using the group-theory constraints satisfied by the four-point amplitude [27, 28]. For $L = 2\ell$ even, there are four independent group-theory relations[3] (only one when $L = 0$)

$$\begin{aligned}
\overset{(2\ell,2\ell)}{1} &= -\tfrac{4}{3}\ \overset{(2\ell,2\ell\ \ 1)}{4} + \tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{5} + \tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{6}\,, \\
\overset{(2\ell,2\ell)}{2} &= +\tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{4} - \tfrac{4}{3}\ \overset{(2\ell,2\ell\ \ 1)}{5} + \tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{6}\,, \\
\overset{(2\ell,2\ell)}{3} &= +\tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{4} + \tfrac{2}{3}\ \overset{(2\ell,2\ell\ \ 1)}{5} - \tfrac{4}{3}\ \overset{(2\ell,2\ell\ \ 1)}{6}\,, \\
0 &= \overset{(2\ell,2\ell\ \ 2)}{1} + \overset{(2\ell,2\ell\ \ 2)}{2} + \overset{(2\ell,2\ell\ \ 2)}{3} - \tfrac{1}{3}\left[\overset{(2\ell,2\ell\ \ 1)}{4} + \overset{(2\ell,2\ell\ \ 1)}{5} + \overset{(2\ell,2\ell\ \ 1)}{6}\right]
\end{aligned} \tag{4.10}$$

which implies

$$\overset{(2\ell,2\ell)}{1} + \overset{(2\ell,2\ell)}{2} + \overset{(2\ell,2\ell)}{3} = 0 \tag{4.11}$$

and similarly for the IR-finite $H_\lambda^{(2\ell,2\ell)}$.

For $L = 2\ell + 1$ odd, we also have four independent relations[3] (only three when $L = 1$)

$$\begin{aligned}
\overset{(2\ell+1,2\ell+1)}{4} &= \overset{(2\ell+1,2\ell)}{1} + \overset{(2\ell+1,2\ell)}{2} + \overset{(2\ell+1,2\ell)}{3}\,, \\
\overset{(2\ell+1,2\ell+1)}{5} &= \overset{(2\ell+1,2\ell)}{1} + \overset{(2\ell+1,2\ell)}{2} + \overset{(2\ell+1,2\ell)}{3}\,, \\
\overset{(2\ell+1,2\ell+1)}{6} &= \overset{(2\ell+1,2\ell)}{1} + \overset{(2\ell+1,2\ell)}{2} + \overset{(2\ell+1,2\ell)}{3}\,, \\
0 &= \overset{(2\ell+1,2\ell\ \ 2)}{1} + \overset{(2\ell+1,2\ell\ \ 2)}{2} + \overset{(2\ell+1,2\ell\ \ 2)}{3} \\
&\quad -\tfrac{1}{3}\left[\overset{(2\ell+1,2\ell\ \ 1)}{4} + \overset{(2\ell+1,2\ell\ \ 1)}{5} + \overset{(2\ell+1,2\ell\ \ 1)}{6}\right] \\
&\quad +\tfrac{1}{3}\left[\overset{(2\ell+1,2\ell)}{1} + \overset{(2\ell+1,2\ell)}{2} + \overset{(2\ell+1,2\ell)}{3}\right]
\end{aligned} \tag{4.12}$$

which implies

$$\overset{(2\ell+1,2\ell+1)}{4} = \overset{(2\ell+1,2\ell+1)}{5} = \overset{(2\ell+1,2\ell+1)}{6} \tag{4.13}$$

and similarly for the IR-finite $H_\lambda^{(2\ell+1,2\ell+1)}$.

By virtue of eqs. (4.11) and (4.13), together with eq. (4.4), one can show that the entries of both $c\left|H^{(2k,2k)}\right\rangle$ and $\left|H^{(2k+1,2k+1)}\right\rangle$ are all equal

$$c\left|H^{(2k,2k)}\right\rangle = (Y H_3^{(2k,2k)} - X H_2^{(2k,2k)}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \left|H^{(2k+1,2k+1)}\right\rangle = H_4^{(2k+1,2k+1)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4.14}$$

These together with

$$cb \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (2X^2 + 2Y^2 + 2Z^2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} X - Y \\ Z - X \\ Y - Z \end{pmatrix} \tag{4.15}$$

[3] These relations differ from those given in ref. [27] by some factors of two, due to the change in the trace basis.

JHEP04(2013)114

allow one to write the even- and odd-loop most-subleading-color amplitude (4.7) and (4.8) as

$$\begin{pmatrix} {}^{(2\ell,2\ell)}_{1} \\ {}^{(2\ell,2\ell)}_{2} \\ {}^{(2\ell,2\ell)}_{3} \end{pmatrix} = \sum_{k=0}^{\ell\ \ 1} \frac{2(2X^2+2Y^2+2Z^2)^{\ell\ \ k\ \ 1}}{(2\ell-2k)!\epsilon^{2\ell\ \ 2k}} \times \Big[(Y H_3^{(2k,2k)} - X H_2^{(2k,2k)}) + (2\ell-2k)\epsilon H_4^{(2k+1,2k+1)} \Big] \begin{pmatrix} X-Y \\ Z-X \\ Y-Z \end{pmatrix} + \begin{pmatrix} H_1^{(2\ell,2\ell)} \\ H_2^{(2\ell,2\ell)} \\ H_3^{(2\ell,2\ell)} \end{pmatrix}, \tag{4.16}$$

$$\begin{pmatrix} {}^{(2\ell+1,2\ell+1)}_{4} \\ {}^{(2\ell+1,2\ell+1)}_{5} \\ {}^{(2\ell+1,2\ell+1)}_{6} \end{pmatrix} = \sum_{k=0}^{\ell} \frac{(2X^2+2Y^2+2Z^2)^{\ell\ \ k}}{(2\ell-2k+1)!\epsilon^{2\ell\ \ 2k+1}} \times \Big[(Y H_3^{(2k,2k)} - X H_2^{(2k,2k)}) + (2\ell-2k+1)\epsilon H_4^{(2k+1,2k+1)} \Big] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4.17}$$

Provided the dipole conjecture holds, these expressions give the complete IR-divergent contribution to the most-subleading-color L-loop four-point amplitudes in terms of the IR-finite parts of lower-loop amplitudes, as in the case of gravitational amplitudes (2.5).

Finally we turn our attention to the two leading IR divergences of the most-subleading-color amplitudes. These are given by the $k=0$ terms in eqs. (4.16) and (4.17),

$$\begin{pmatrix} {}^{(2\ell,2\ell)}_{1} \\ {}^{(2\ell,2\ell)}_{2} \\ {}^{(2\ell,2\ell)}_{3} \end{pmatrix} = \frac{2(2X^2+2Y^2+2Z^2)^{\ell\ \ 1}}{(2\ell)!\epsilon^{2\ell}} \Big[(Y\ {}^{(0)}_{3} - X\ {}^{(0)}_{2}) + (2\ell)\epsilon H_4^{(1,1)} \Big] \times \begin{pmatrix} X-Y \\ Z-X \\ Y-Z \end{pmatrix} + \mathcal{O}(1/\epsilon^{2\ell\ \ 2}), \tag{4.18}$$

$$\begin{pmatrix} {}^{(2\ell+1,2\ell+1)}_{4} \\ {}^{(2\ell+1,2\ell+1)}_{5} \\ {}^{(2\ell+1,2\ell+1)}_{6} \end{pmatrix} = \frac{(2X^2+2Y^2+2Z^2)^{\ell}}{(2\ell+1)!\epsilon^{2\ell+1}} \Big[(Y\ {}^{(0)}_{3} - X\ {}^{(0)}_{2}) + (2\ell+1)\epsilon H_4^{(1,1)} \Big] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}(1/\epsilon^{2\ell\ \ 1}). \tag{4.19}$$

Using the fact that

$$\begin{pmatrix} {}^{(1,1)}_{4}(\epsilon) \\ {}^{(1,1)}_{5}(\epsilon) \\ {}^{(1,1)}_{6}(\epsilon) \end{pmatrix} = \left[\frac{Y\ {}^{(0)}_{3} - X\ {}^{(0)}_{2}}{\epsilon} + H_4^{(1,1)} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.20}$$

JHEP04(2013)114

we see that the two leading IR divergences can be expressed in terms of the one-loop subleading-color amplitude evaluated in $D = 4 - 2L\epsilon$ dimensions:

$$\begin{pmatrix} {}^{(2\ell,2\ell)}_{1}(\epsilon) \\ {}^{(2\ell,2\ell)}_{2}(\epsilon) \\ {}^{(2\ell,2\ell)}_{3}(\epsilon) \end{pmatrix} = \frac{2(2X^2+2Y^2+2Z^2)^{\ell\ \ 1}}{(2\ell-1)!\epsilon^{2\ell\ \ 1}} \ {}^{(1,1)}_{4}((2\ell)\epsilon) \begin{pmatrix} X-Y \\ Z-X \\ Y-Z \end{pmatrix} + \mathcal{O}(1/\epsilon^{2\ell\ \ 2}) \,, \tag{4.21}$$

$$\begin{pmatrix} {}^{(2\ell+1,2\ell+1)}_{4}(\epsilon) \\ {}^{(2\ell+1,2\ell+1)}_{5}(\epsilon) \\ {}^{(2\ell+1,2\ell+1)}_{6}(\epsilon) \end{pmatrix} = \frac{(2X^2+2Y^2+2Z^2)^{\ell}}{(2\ell)!\epsilon^{2\ell}} \ {}^{(1,1)}_{4}((2\ell+1)\epsilon) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}(1/\epsilon^{2\ell\ \ 1}) \,. \tag{4.22}$$

In the case of $\mathcal{N} = 4$ SYM theory, these relations were previously conjectured in ref. [16] and proved in ref. [17]. Here we point out that this is another point of similarity with gravity amplitudes, which obey the analogous eq. (2.7).

5 Possible three-loop corrections to the dipole conjecture

The results of the previous two sections were contingent on the validity of the dipole formula for the soft anomalous dimension matrix. The dipole formula holds through at least two loops [8], but could break down beginning at three loops. Possible forms of a three-loop correction term were considered in refs. [11, 13], including a term of the form

$$\Delta\mathbf{\Gamma}^{(3)} = \frac{1}{N^3}\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \left[f^{ade}f^{cbe}P_t(s_{ij}) + f^{cae}f^{dbe}P_u(s_{ij}) + f^{bae}f^{cde}P_s(s_{ij})\right] \tag{5.1}$$

which might be generated by the purely gluonic diagram shown in figure 1 of ref. [13], and which contributes at $\mathcal{O}(1/\epsilon)$ [11]. Other correction terms, involving d^{abc}, were also discussed in ref. [13]. In refs. [11, 13–15], strong constraints were put on the possible kinematical dependence of the functions $P(s_{ij})$ appearing in eq. (5.1). In this section, we discuss the effect of a term of the form (5.1) on the most-subleading-color four-point amplitude ${}^{(3,3)}$.

cting with $\Delta\mathbf{\Gamma}^{(3)}$ on the four-point trace basis (4.2), as in eq. (3.8), we extract the matrix

$$\Delta\mathbf{\Gamma}^{(3)} = \frac{1}{N^3}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.2}$$

where

$$a = \begin{pmatrix} 0 & 2N(3P_t - P_s - 2P_u) & 2N(2P_u - 3P_s + P_t) \\ 2N(P_s + 2P_t - 3P_u) & 0 & 2N(3P_s - 2P_t - P_u) \\ 2N(3P_u - 2P_s - P_t) & 2N(2P_s - 3P_t + P_u) & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 8(P_t-P_s) & 2N^2(P_t-P_s)+4(P_u-P_s) & 2N^2(P_t-P_s)+4(P_t-P_u) \\ 2N^2(P_s-P_u)+4(P_s-P_t) & 8(P_s-P_u) & 2N^2(P_s-P_u)+4(P_t-P_u) \\ 2N^2(P_u-P_t)+4(P_s-P_t) & 2N^2(P_u-P_t)+4(P_u-P_s) & 8(P_u-P_t) \end{pmatrix},$$

JHEP04(2013)114

$$c = \begin{pmatrix} 2(P_t - P_s) & (N^2+2)(P_s - P_u) & (N^2+2)(P_u - P_t) \\ (N^2+2)(P_t - P_s) & 2(P_s - P_u) & (N^2+2)(P_u - P_t) \\ (N^2+2)(P_t - P_s) & (N^2+2)(P_s - P_u) & 2(P_u - P_t) \end{pmatrix},$$

$$d = \begin{pmatrix} 6N(P_s - P_t) & 0 & 0 \\ 0 & 6N(P_u - P_s) & 0 \\ 0 & 0 & 6N(P_t - P_u) \end{pmatrix}. \tag{5.3}$$

One can see that $\Delta\mathbf{\Gamma}^{(3)}$ is subleading in the $1/N$ expansion and hence cannot contribute to the planar amplitude $^{(3,0)}$. However, $\mathcal{O}(1/\epsilon)$ corrections to all the subleading-color amplitudes $^{(3,1)}$, $^{(3,2)}$, and $^{(3,3)}$ are possible.[4] In particular, by keeping only the most-subleading-color contribution of $(a(\mu^2)^3/\epsilon)\Delta\mathbf{\Gamma}^{(3)} \left|\ ^{(0)}\right\rangle$, where

$$\left|\ ^{(0)}\right\rangle = -\frac{4iK}{stu} \begin{pmatrix} u \\ t \\ s \end{pmatrix} \tag{5.4}$$

is the tree-level amplitude, we obtain the following three-loop contribution to the most-subleading-color amplitude

$$\begin{pmatrix} \Delta\ ^{(3,3)}_4 \\ \Delta\ ^{(3,3)}_5 \\ \Delta\ ^{(3,3)}_6 \end{pmatrix} = -\frac{8iK}{\epsilon} \frac{[(u-s)P_t + (s-t)P_u + (t-u)P_s]}{stu} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\epsilon^0\right). \tag{5.5}$$

This is an example of a possible IR-divergent contribution to the most-subleading-color amplitude that does not arise from the exponentiation of the one-loop divergence. Hence, if the dipole formula is modified by a term of the form (5.1), then the one-loop-exactness of this class of amplitudes breaks down.

6 -loop supergravity/SYM relations

In the previous sections, we saw that gravity and most-subleading-color gauge-theory amplitudes are one-loop exact, i.e. higher-loop divergences can be expressed in terms of one-loop divergences. In this section, we use this result to derive a relation between the two leading divergences of the L-loop four-point $\mathcal{N} = 8$ supergravity amplitude and the most-subleading-color $\mathcal{N} = 4$ SYM amplitudes.

n exact relation between the one-loop four-point $\mathcal{N} = 8$ supergravity and subleading-color $\mathcal{N} = 4$ SYM amplitudes has long been known [16, 19, 29]. In the notation of the current paper, this relation is

$$M_4^{(1)}(\epsilon) = \left(-\frac{\lambda}{8\pi^2}\right) \frac{^{(1,1)}(\epsilon)}{(\ ^{(0)}_1/u)} \tag{6.1}$$

[4] I appe dix B of ref. [9], it was stated that the three-loop correctio term (5.1) co tributes at (N). This is i deed true for the matrix eleme t a co ecti g si gle-trace terms. However, the off-diago al matrix eleme ts b, c, which co ect si gle- a d double-trace terms, have a (N^2) co tributio , as the authors of ref. [9] have co firmed (private commu icatio).

JHEP04(2013)114

where $^{(1,1)}$ refers to any of the four-point subleading-color amplitudes $^{(1,1)}_{4} = {}^{(1,1)}_{5} = {}^{(1,1)}_{6}$, and we recall that the tree-level amplitude $^{(0)}$ is given by eq. (5.4).

In eq. (2.15) we showed that the two leading IR divergences of the L-loop four-point supergravity amplitude can be expressed in terms of the one-loop supergravity amplitude. In eqs. (4.21) and (4.22), we derived similar expressions for the two leading IR divergences of the L-loop most-subleading-color four-point SYM amplitudes. Combining these with eq. (6.1), we obtain for odd $L = 2\ell + 1$ the relation

$$\begin{aligned} M_4^{(2\ell+1)}(\epsilon) &= \left(-\frac{\lambda}{8\pi^2}\right)^{2\ell+1} \frac{(sY - tX)^{2\ell}}{(2X^2+2Y^2+2Z^2)^{\ell}} \quad \frac{{}^{(2\ell+1,2\ell+1)}(\epsilon)}{({}^{(0)}_{1}/u)} + \mathcal{O}\left(\frac{1}{\epsilon^{2\ell}\ {}^{1}}\right) \\ &= \left(-\frac{\lambda}{8\pi^2}\right)^{2\ell+1} \left[\frac{(s\log s + t\log t + u\log u)^2}{2(\log^2(t/u)+\log^2(u/s)+\log^2(s/t))}\right]^{\ell} \frac{{}^{(2\ell+1,2\ell+1)}(\epsilon)}{({}^{(0)}_{1}/u)} \\ &\quad + \mathcal{O}\left(\frac{1}{\epsilon^{2\ell}\ {}^{1}}\right). \end{aligned} \tag{6.2}$$

gain, $^{(2\ell+1,2\ell+1)}$ refers to any of the most-subleading-color four-point amplitudes $^{(2\ell+1,2\ell+1)}_{4} = {}^{(2\ell+1,2\ell+1)}_{5} = {}^{(2\ell+1,2\ell+1)}_{6}$ (cf. eq. (4.13)).

For even $L = 2\ell$, a similar relation holds, namely

$$M_4^{(2\ell)}(\epsilon) = \left(\frac{\lambda}{8\pi^2}\right)^{2\ell} \frac{(sY - tX)^{2\ell}\ {}^{1}}{(2X^2+2Y^2+2Z^2)^{\ell}\ {}^{1}} \quad \frac{{}^{(2\ell,2\ell)}_{1}(\epsilon)}{2(X-Y)({}^{(0)}_{1}/u)} + \mathcal{O}\left(\frac{1}{\epsilon^{2\ell}\ {}^{2}}\right) \tag{6.3}$$

for $\ell \geq 1$. The factor $^{(2\ell,2\ell)}_{1}/(X-Y)$ can of course be replaced with $^{(2\ell,2\ell)}_{2}/(Z-X)$ or $^{(2\ell,2\ell)}_{3}/(Y-Z)$, or in fact with $\left({}^{(2\ell,2\ell)}_{1} - {}^{(2\ell,2\ell)}_{2}\right)/3X$ (since $X+Y+Z=0$) giving

$$M_4^{(2\ell)}(\epsilon) = \left(\frac{\lambda}{8\pi^2}\right)^{2\ell} \frac{(sY - tX)^{2\ell}\ {}^{1}}{(2X^2+2Y^2+2Z^2)^{\ell}\ {}^{1}} \frac{\left({}^{(2\ell,2\ell)}_{1}(\epsilon) - {}^{(2\ell,2\ell)}_{2}(\epsilon)\right)}{6X({}^{(0)}_{1}/u)} + \mathcal{O}\left(\frac{1}{\epsilon^{2\ell}\ {}^{2}}\right). \tag{6.4}$$

To repeat, these relations are immediate consequences of eqs. (2.15), (4.21), (4.22), and (6.1).

n interesting question is whether the relations (6.2) and (6.3) remain valid beyond the leading two orders in the Laurent expansion. Unfortunately, the answer will turn out to be no.

To see this, observe that for $L = 2$, eq. (6.3) states that

$$M_4^{(2)}(\epsilon) = \left(\frac{\lambda}{8\pi^2}\right)^2 \frac{(sY - tX)}{2(X-Y)} \frac{{}^{(2,2)}_{1}(\epsilon)}{({}^{(0)}_{1}/u)} + \mathcal{O}\left(\epsilon^0\right). \tag{6.5}$$

We know this to be valid at $\mathcal{O}(1/\epsilon^2)$ and $\mathcal{O}(1/\epsilon)$, and the question is whether it continues to hold at $\mathcal{O}(\epsilon^0)$. To answer this, we recall the *exact* two-loop supergravity/SYM relation derived in [16, 19]

$$M_4^{(2)}(\epsilon) = \left(\frac{\lambda}{8\pi^2}\right)^2 \frac{\left(u\ {}^{(2,2)}_{1}(\epsilon) + t\ {}^{(2,2)}_{2}(\epsilon) + s\ {}^{(2,2)}_{3}(\epsilon)\right)}{6({}^{(0)}_{1}/u)}. \tag{6.6}$$

JHEP04(2013)114

short calculation using $s+t+u=0$, $X+Y+Z=0$, and eq. (4.11) shows that eqs. (6.5) and (6.6) are consistent provided that

$$\frac{\overset{(2,2)}{1}}{X-Y} \overset{?}{=} \frac{\overset{(2,2)}{2}}{Z-X} \overset{?}{=} \frac{\overset{(2,2)}{3}}{Y-Z}\,. \tag{6.7}$$

In fact, were eq. (6.7) to hold, then eq. (6.2) would also hold at $\mathcal{O}(1/\epsilon)$ for $L=3$ (provided that the dipole conjecture is also valid at three loops), and in fact for the $\mathcal{O}(1/\epsilon^{L\ \ 2})$ term at higher loops as well.

While eq. (6.7) evidently holds for the IR-divergent parts of the amplitude (cf. eq. (4.16)), we have verified that it fails at $\mathcal{O}(\epsilon^0)$, that is

$$(Z-X)\ \ \overset{(2,2)}{1} - (X-Y)\ \ \overset{(2,2)}{2} \neq 0 \tag{6.8}$$

using the explicit expressions for the two-loop most-subleading-color $\mathcal{N}=4$ SYM four-point amplitudes (see appendix). To ensure that the complicated expression obtained for the left hand side of eq. (6.8) does not vanish due to polylogarithmic identities, we evaluated it numerically for various values of the kinematic variables, obtaining nonzero results. Finally, we checked that the symbol [30] for the expression on the left hand side of eq. (6.8) does not vanish (sometimes a non-obvious polylog identity reduces a long expression to a simple one, which can be made explicit by the calculation of the symbol [30–32], and moreover an identity could be valid only up to terms with zero symbol); see appendices B and C for details.

Consequently, eqs. (6.2) and (6.3) are valid for the two leading terms, but break down at the next order in the Laurent expansion.

7 Conclusions

In this paper, we explored parallels between the IR behavior of gravitational amplitudes and that of the most-subleading-color gauge-theory amplitudes. Both sets of amplitudes have a leading IR divergence of $\mathcal{O}(1/\epsilon^L)$ at L loops, due to the absence of collinear divergences. We have shown that, if the dipole conjecture for the IR behavior of gauge-theory amplitudes is valid, then the most-subleading-color amplitudes, like gravity amplitudes, are one-loop-exact; that is, higher-loop divergences are determined by the one-loop result. Specifically, the all-loop amplitude is given by the exponential of the one-loop soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}$ acting on the IR-finite hard function.

ssuming the validity of the dipole conjecture, we computed an expression for the complete IR behavior of the L-loop most-subleading-color four-point amplitude in terms of the finite parts of lower-loop amplitudes. Similar expressions could be derived for five- and higher-point amplitudes using the explicit form for the one-loop soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}$. We note that $\mathbf{\Gamma}^{(1)}$ is essentially equivalent to the (transpose of the) iterative matrix G specified in refs. [27, 28, 33], defined by attaching a rung between two external legs i and j of an element of the trace basis T_λ. In the present context, each rung corresponds to the exchange of a soft gluon between the corresponding external particles, accompanied by a factor of $\log(\mu^2/-s_{ij})$.

JHEP04(2013)114

Corrections to the dipole conjecture may occur at three loops and beyond, although the possible form of such corrections is highly constrained. In recent work, Oxburgh and White [34] use BCJ duality and the double-copy property to study the IR behavior of gauge theory and gravity. They emphasize that the known IR structure of gravity is insensitive to possible corrections to the dipole conjecture in gauge theories. Therefore the presence or absence of such corrections at three loops will likely require a Laurent expansion of the three-loop non-planar diagrams contributing to the gauge amplitude. We showed that, though collinear IR divergences remain absent, corrections could spoil the one-loop-exactness of most-subleading-color amplitudes.

Finally, we showed that the similarities between gravity and most-subleading-color amplitudes allow us to deduce a relation between L-loop four-point $\mathcal{N}=8$ supergravity and most-subleading-color $\mathcal{N}=4$ SYM amplitudes that holds for the two leading IR divergences, $\mathcal{O}(1/\epsilon^L)$ and $\mathcal{O}(1/\epsilon^{L-1})$, but breaks down at $\mathcal{O}(1/\epsilon^{L-2})$.

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Two-loop most-subleading-color $=4$ SYM four-point amplitude

The two-loop most-subleading-color four-point amplitudes in $\mathcal{N}=4$ SYM theory are given in terms of two-loop planar and nonplanar scalar integrals [35]. Explicit expressions for these may be derived employing the Laurent expansions of the planar [26] and non-planar [36] integrals. nalytically continuing these integrals to the kinematic region $t>0$ and $s,\,u<0$, we obtain the expression

$$\frac{{}^{(2,2)}_{1}(\epsilon)}{({}^{(0)}_{1}/u)} = \left(\frac{\mu^2}{t}\right)^{2\epsilon}\Bigg\{\frac{(-s\log y - u\log(1-y) - i\pi(s+u))\,(X-Y)}{\epsilon^2} \qquad (\ .1)$$

$$+\frac{(2(s+u)\log y\log(1-y) + 2i\pi u\log y + 2i\pi s\log(1-y))\,(X-Y)}{\epsilon}$$

$$\begin{aligned}
&+(-20s-4u)\,S_{3,1}(y) + (4s-4u)\,S_{2,2}(y) + (-8s-4u)\,S_{1,3}(y)\\
&+\big[\,(10s-4u)\log y + (8s+10u)\log(1-y) + (14i\pi s + 10i\pi u)\,\big]S_{2,1}(y)\\
&+\big[\,(4s+8u)\log y + (-16s-8u)\log(1-y) + (-4i\pi s + 4i\pi u)\,\big]S_{1,2}(y)\\
&+\big[6u\log^2 y + (-8s-10u)\log y\log(1-y) + (-12s-6u)i\pi\log y\\
&+(8s-2u)i\pi\log(1-y) + (-4s+4u)\pi^2\big]S_{1,1}(y) + \frac{1}{2}s\log^4 y - \frac{4}{3}(s-u)\log^3 y\log(1-y)\\
&-(2s+4u)\log^2 y\log^2(1-y) + (4s+4u)\log y\log^3(1-y) - u\log^4(1-y)\\
&-i\pi(s+2u)\log^3 y + 5i\pi u\log^2 y\log(1-y) - 4i\pi u\log y\log^2(1-y)
\end{aligned}$$

$$
\begin{aligned}
&+4i\pi s\log^3(1-y)+\left(\frac{13}{6}s-2u\right)\pi^2\log^2 y+\left(-\frac{13}{3}s+\frac{37}{6}u\right)\pi^2\log y\log(1-y)\\
&+\left(-2s-\frac{13}{3}u\right)\pi^2\log^2(1-y)+\left(-2i\pi^3 s+\frac{1}{6}i\pi^3 u-2s\zeta_3\right)\log y\\
&+\left(-\frac{5}{3}i\pi^3 s-\frac{19}{6}i\pi^3 u+8s\zeta_3+2u\zeta_3\right)\log(1-y)\\
&+\frac{17\pi^4 s}{15}+\frac{\pi^4 u}{6}+2i\pi s\zeta_3-6i\pi u\zeta_3+\mathcal{O}(\epsilon)\Big\}
\end{aligned}
$$

where $y\equiv -s/t$ and $S_{,p}(y)$ denote the generalized polylogarithms of Nielsen [37]. In this region, the variables X, Y, and Z defined in eq. (4.4) become

$$
\begin{aligned}
X &= -\log(1-y)-i\pi\,,\\
Y &= -\log y+\log(1-y)\,,\\
Z &= \log y+i\pi\,.
\end{aligned}
\qquad (\ .2)
$$

(See appendix of ref. [3] for details on the performance of the analytic continuation.)

We also have

$$
\begin{aligned}
\frac{{}^{(2,2)}_{2}(\epsilon)}{({}^{(0)}_{1}/u)} &= \left(\frac{\mu^2}{t}\right)^{2\epsilon}\Big\{\frac{(-s\log y-u\log(1-y)-i\pi(s+u))\,(Z-X)}{\epsilon^2} \qquad (\ .3)\\
&+\frac{(2(s+u)\log y\log(1-y)+2i\pi u\log y+2i\pi s\log(1-y))\,(Z-X)}{\epsilon}\\
&+(16s-4u)\,S_{3,1}(y)+(-8s+8u)\,S_{2,2}(y)+(4s-16u)\,S_{1,3}(y)\\
&+\big[\,(-14s-4u)\log y+(-4s-2u)\log(1-y)+(-10i\pi s-14i\pi u)\,\big]S_{2,1}(y)\\
&+\big[\,(10s-4u)\log y+(8s-2u)\log(1-y)+(14i\pi s+10i\pi u)\,\big]S_{1,2}(y)\\
&+\big[(6s+6u)\log^2 y+(4s+2u)\log y\log(1-y)+(6s+18u)i\pi\log y\\
&+(-4s+4u)i\pi\log(1-y)+(8s-8u)\pi^2\big]S_{1,1}(y)+\frac{1}{2}s\log^4(y)+\frac{2}{3}(s+2u)\log^3(y)\log(1-y)\\
&-(2s+u)\log^2(y)\log^2(1-y)-(2s+u)\log(y)\log^3(1-y)+\frac{1}{2}u\log^4(1-y)\\
&+i\pi(s-2u)\log^3(y)+5i\pi u\log^2(y)\log(1-y)-i\pi(6s+u)\log(y)\log^2(1-y)\\
&+i\pi(-2s+u)\log^3(1-y)+\left(\frac{13}{6}s+4u\right)\pi^2\log^2 y+\left(\frac{13}{6}s-\frac{35}{6}u\right)\pi^2\log y\log(1-y)\\
&+\left(4s+\frac{13}{6}u\right)\pi^2\log^2(1-y)+\left(\frac{11}{2}i\pi^3 s+\frac{1}{6}i\pi^3 u-2s\zeta_3\right)\log y\\
&+\left(\frac{5}{6}i\pi^3 s+\frac{29}{6}i\pi^3 u-4s\zeta_3+2u\zeta_3\right)\log(1-y)\\
&-\frac{28\pi^4 s}{15}-\frac{\pi^4 u}{3}-10i\pi s\zeta_3-6i\pi u\zeta_3+\mathcal{O}(\epsilon)\Big\}\,.
\end{aligned}
$$

Finally, ${}^{(2,2)}_{3}$ is obtained using

$$
{}^{(2,2)}_{1}+{}^{(2,2)}_{2}+{}^{(2,2)}_{3}=0\,. \qquad (\ .4)
$$

JHEP04(2013)114

To our knowledge, explicit expressions for these amplitudes have not appeared previously in the published literature. The uniform transcendentality of these expressions, previously noted in ref. [16], is evident in these expressions, underlining the fact that this property of $\mathcal{N}=4$ SYM observables extends beyond the planar approximation.

Using the expressions above, one may verify that

$$(Z-X)\ {}^{(2,2)}_{1} - (X-Y)\ {}^{(2,2)}_{2} \neq 0\,. \qquad (\ \ .5)$$

For this purpose, it is simplest to examine the coefficient of ζ_3.

B Symbology

In this appendix we review the general features of symbols. For more details, see refs. [38] and [39].

The symbols are simply defined for Goncharov polynomials of one variable, defined recursively as

$$G(a_1,\ldots,a\ ;x) = \int_0^x \frac{dt}{t-a_1} G(a_2,\ldots,a\ ;t) \qquad \text{(B.1)}$$

with

$$G(x) = G(;x) = 1; G(0) = 0 \qquad \text{(B.2)}$$

Other functions are obtained from them as

$$\begin{aligned} G(\vec{0}\ ;x) &= \frac{1}{n!}\log\ \ x \\ G(\vec{a}\ ;x) &= \frac{1}{n!}\log\ \left(1-\frac{x}{a}\right) \\ G(\vec{0}\ \ {}_1,a;x) &= -\mathrm{Li}\ \ \left(\frac{x}{a}\right) \\ G(\vec{0}\ ,\vec{a}_p;x) &= (-1)^p S\ {}_{,p}\left(\frac{x}{a}\right) \end{aligned} \qquad \text{(B.3)}$$

The Goncharov polylogarithms of one variable are similarly defined with the harmonic polylogarithms [40] H with indices 0 and ± 1, which are related to the Nielsen polylogarithms by

$$S\ {}_{,p}(x) = H(\vec{0}\ ,\vec{1}_p;x). \qquad \text{(B.4)}$$

The symbol of a Goncharov polynomial is defined as sum of terms of the type tensor product of R_i's, understood as $d\log R_i = dR_i/R_i$'s, i.e. such that the rules the R_i's satisfy follow from this $d\log$ form. These tensor monomials are written as $R_1\otimes\ldots\otimes R\ \ $ and satisfy

$$\begin{aligned} \ldots\otimes(R_1\cdot R_2)\otimes\ldots &= \ldots\otimes R_1\otimes\ldots+\ldots\otimes R_2\otimes\ldots \\ \Rightarrow\otimes(R_1)\ \ \otimes &= n\ldots\otimes R_1\otimes\ldots \\ \ldots\otimes cR_1\otimes\ldots &= \ldots\otimes R_1\otimes\ldots \\ \ldots\otimes c\otimes\ldots &= 0 \\ \Rightarrow\ldots\otimes R_1\otimes\ldots &= -\ldots\otimes 1/R_1\otimes\ldots \end{aligned} \qquad \text{(B.5)}$$

JHEP04(2013)114

where $R_1, R_2, \ldots$ are variable monomials and c is a constant. The symbol of an object T_k, a priori a function of several variables, an extension of the simple Goncharov polynomials of one variable above and defined recursively as

$$T_k = \int_a^b d\log R_1 \circ \ldots \circ d\log R \ = \int_a^b \left(\int_a^t d\log R_1 \circ \ldots d\log R \quad _1 \right) d\log R \ (t), \quad \text{(B.6)}$$

is

$$S[T_k] = R_1 \otimes R_2 \otimes \ldots \otimes R \ . \quad \text{(B.7)}$$

From this definition we obtain immediately the symbol of a Li_k polylogarithm,

$$S[\mathrm{Li}_k(z)] = -(1-z) \otimes z \otimes \ldots z \quad \text{(B.8)}$$

(there are $k-1$ factors of z), as a particular case of the Goncharov polylogarithms. Note that $(1-z)$ and $(z-1)$ are the same, since they differ by multiplication by the constant -1, however, the overall minus sign is for the tensor monomial, it does not belong into any of the tensored factors.

We can also define the rule for multiplication of two symbol terms $S[F] = \otimes_{i=1} R_i$ and $S[G] = \otimes_{i=\ +1}^{m} R_i$, as

$$S[FG] = \sum_{\Pi} \otimes_{i=1}^{m+\ } R_{\Pi(i)} \quad \text{(B.9)}$$

where the permutations Π preserve the original order of the factors in $S[F]$ and in $S[G]$ within $S[FG]$. For example, if $n = m = 2$ we get

$$\begin{aligned} S[FG] = \ & R_1 \otimes R_2 \otimes R_3 \otimes R_4 + R_1 \otimes R_3 \otimes R_2 \otimes R_4 + R_1 \otimes R_3 \otimes R_4 \otimes R_2 \\ & + R_3 \otimes R_1 \otimes R_2 \otimes R_4 + R_3 \otimes R_1 \otimes R_4 \otimes R_2 + R_3 \otimes R_4 \otimes R_1 \otimes R_2. \end{aligned} \quad \text{(B.10)}$$

For logs and their products we obtain

$$\begin{aligned} & S[\log x] = x \\ & S[\log x \log y] = x \otimes y + y \otimes x. \end{aligned} \quad \text{(B.11)}$$

Finally, for the Nielsen polylogarithms

$$S_{\ ,p} = \frac{(-1)^{\ +p\ \ 1}}{(n-1)!p!} \int_0^1 dt \frac{\log \quad ^1(t) \log^p(1-xt)}{t}, \quad \text{(B.12)}$$

the symbol is given by

$$S[S_{\ ,p}(x) = H(\vec{0}\ , \vec{1}_p; x)] = (-1)^p (1-x) \otimes (1-x) \otimes \ldots \otimes (1-x) \otimes x \otimes x \ldots \otimes x \quad \text{(B.13)}$$

where there are p $1-x$'s and n x's.

JHEP04(2013)114

C Symbol relation

In this appendix, we describe how we tested the relation (6.8) using the symbol. ppendix B reviews some salient features of symbols of polylogarithms.

The amplitude $\overset{(2,2)}{1}$ from appendix , divided by $\overset{(0)}{1}/u$, has terms proportional to the independent variables s and u (where $s+t+u=0$), and these kinematic factors are not touched by the symbol. Therefore we will only check the s-terms in the desired relation,

$$(Z-X)\ \overset{(2,2)}{1} \overset{?}{=} (X-Y)\ \overset{(2,2)}{2}, \tag{C.1}$$

for the finite order pieces. (We know the IR divergent pieces satisfy this relation, and we have in fact explicitly checked this.)

Given that we are interested only in the relation between symbols, the analytical continuations become simpler. To find the first cyclic term, in the $t>0, s, u<0$ region, we need to first analytically continue to $s>0, t, u<0$ and then do the cyclic shift. The analytical continuation gives

$$\begin{aligned} y=-\frac{s}{t} &\to y e^{\ 2\pi i} \Rightarrow \log y \to \log y - 2\pi i \\ 1-y=-\frac{u}{t} &\to -(1-y) e^{\ \pi i} \\ \mathrm{Li}_k(y) &\to \mathrm{Li}_k(y e^{\ 2\pi i}) = \mathrm{Li}_k(y) \\ S_{1,k}(y) &\to S_{1,k}(y e^{\ 2\pi i}) = S_{1,k}(y) \end{aligned} \tag{C.2}$$

and then the change (s,t,u) into (t,u,s) leads to $y \to 1/(1-y)$ and $1-y \to -y/(1-y)$. ll in all, we obtain

$$\begin{aligned} T &\to V + 2\pi i \\ V &\to -T - V - \pi i \\ \mathrm{Li}_k(y) &\to \mathrm{Li}_k\left(\frac{1}{1-y}\right) \\ S_{1,k}(y) &\to S_{1,k}\left(\frac{1}{1-y}\right). \end{aligned} \tag{C.3}$$

To find the second cyclic term in the $t>0,\ s, u<0$ region, we first analytically continue to $u>0, s, t<0$, and then do the cyclic shift. The analytical continuation gives

$$\begin{aligned} y &\to -y e^{\ i\pi} \Rightarrow \log y \to \log(-y) - \pi i \\ 1-y &\to (1-y) e^{\ 2\pi i} \\ \mathrm{Li}_k(y) &\to \mathrm{Li}_k(y) + \text{terms of 0 symbol} \\ S_{1,k}(y) &\to S_{1,k}(y) + \text{terms of 0 symbol} \end{aligned} \tag{C.4}$$

and then the change (s,t,u) into (u,s,t) leads to $y \to -(1-y)/y$ and $1-y \to 1/y$. ll in all, we obtain

$$\begin{aligned} T &\to -V - T + \pi i \\ V &\to T - 2\pi i \\ \mathrm{Li}_k(y) &\to \mathrm{Li}_k\left(-\frac{1-y}{y}\right) + \text{terms of 0 symbol} \\ S_{1,k}(y) &\to S_{1,k}\left(-\frac{1-y}{y}\right) + \text{terms of 0 symbol} \end{aligned} \tag{C.5}$$

JHEP04(2013)114

and now we can ignore the terms with zero symbol, involving transcendental constants like π. That means that ignoring these terms, the relation we need to check is

$$\overset{(2,2)}{1}(\log y + \log(1-y)) = \overset{(2,2)}{2}(\log y - 2\log(1-y)) + \text{terms of 0 symbol} \qquad \text{(C.6)}$$

and as we mentioned, we will only check the s-terms.

The resulting symbol contains tensor products of y and $(1-y)$ monomials forming a 5-fold tensor product, so there are $2^5 = 32$ independent tensor structures which should have zero coefficient if this identity is to hold in symbol. We have checked 4 of these coefficients, and shown them to be nonzero.

In conclusion, the identity (C.1), and therefore also the SYM-supergravity relation at two-loops, does not hold to finite order, not even in symbol.

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JHEP04(2013)114