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# General relativistic magnetohydrodynamics for the numerical construction of dynamical spacetimes

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# GENERAL RELATIVISTIC MAGNETOHYDRODYNAMICS FOR THE NUMERICAL CONSTRUCTION OF DYNAMICAL SPACETIMES

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## ABSTRACT

We assemble the equations of general relativistic magnetohydrodynamics (MHD) in $3+1$ form. These consist of the complete coupled set of Maxwell's equations for the electromagnetic field, Einstein's equations for the gravitational field, and the equations of relativistic MHD for a perfectly conducting ideal gas. The adopted form of the equations is suitable for evolving numerically a relativistic MHD fluid in a dynamical spacetime characterized by a strong gravitational field.

*Subject headings:* MHD — relativity

## 1. INTRODUCTION

Magnetic fields play a crucial role in determining the evolution of many relativistic objects. In any highly conducting astrophysical plasma, a frozen-in magnetic field can be amplified appreciably by gas compression or shear. Even when an initial seed field is weak, the field can grow in the course of time to significantly influence the gasdynamical behavior of the system. If, in addition, the gravitational field is strong and dynamical, magnetic fields can even affect the entire geometry of spacetime, according to general relativity. In this situation, terms involving magnetic and electric fields are important not only as electromagnetic forces acting on the matter in the equations of relativistic hydrodynamics but also as stress-energy sources governing the metric in Einstein's gravitational field equations.

In this paper we assemble the complete set of Maxwell-Einstein magnetohydrodynamics (MHD) equations that determines the self-consistent evolution of a relativistic, ideal MHD fluid in a dynamical spacetime. Our goal is to set down a formulation of the equations that is suitable for numerical integration in full $3+1$ dimensions. Subsets of these equations have appeared elsewhere, but we recompile the complete set here for convenience and future reference. We reconcile several seemingly different, but equivalent, forms of the equations that have appeared in the existing literature. We also correct some errors in previously published results. In a companion paper (Baumgarte & Shapiro 2003, hereafter Paper II) we use these general relativistic MHD equations to follow the gravitational collapse of a magnetized star to a black hole.

We are motivated in part by the growing list of important, unsolved problems that involve hydromagnetic effects in strong-field dynamical spacetimes. The final fate of many of these astrophysical systems, and their distinguishing observational signatures, hinges on the role that magnetic fields may play during the evolution. Some of these systems are promising sources of gravitational radiation for detection by laser interferometers now under design and construction, like LIGO, VIRGO, TAMA, GEO, and *LISA*. Others may be responsible for gamma-ray bursts (GRBs). Recent examples of astrophysical scenarios involving strong-field dynamical spacetimes in which MHD effects may play a decisive role include the following:

1. *The merger of binary neutron stars*.—The merger can lead to the formation of a *hypermassive* star supported by differential rotation (Baumgarte, Shapiro, & Shibata 2000; Shibata & Uryū 2000). While such a star may be dynamically stable against gravitational collapse and bar formation, the radial stabilization due to differential rotation is likely to be temporary. Magnetic braking and viscosity, driven by differential rotation, combine to drive the star to uniform rotation, even if the seed magnetic field and the viscosity are small (Shapiro 2000). This process inevitably leads to delayed collapse, which will be accompanied by a delayed gravitational wave burst.

2. *Core collapse in a supernova*.—Core collapse may again induce differential rotation, even if the rotation of the progenitor at the onset of collapse is only moderately rapid and almost uniform (see, e.g., Zwerger & Müller 1997; Rampp, Müller, & Ruffert 1998 and references therein). Differential rotation can wind up a frozen-in magnetic field to high values, at which point it may provide a significant source of stress. Hypermassive neutron stars may form and survive until the fields are weakened by magnetic braking or other instabilities.

3. *The generation of GRBs*.—Typical models for GRB formation involve the collapse of rotating massive stars to a black hole (MacFadyen & Woosley 1999), the merger of binary neutron stars (Narayan, Paczynski, & Piran 1992), or the tidal disruption of a neutron star by a black hole (Ruffert & Janka 1999). In current scenarios, the burst is powered by the extraction of rotational energy from the neutron star or black hole or from the remnant disk material formed about the black hole (Vlahakis & Königl 2001). Strong magnetic fields provide the likely mechanism for extracting this energy on the required timescale and driving collimated GRB outflows in the form of relativistic jets (Mészáros & Rees 1997; Sari, Piran, & Halpern 1999; Piran 2003). Even if the initial magnetic fields are weak, they can be amplified to the required values by differential motions or dynamo action.

4. *Supermassive star (SMS) collapse*.—SMSs may form in the early universe, and their catastrophic collapse may

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provide the origin of supermassive black holes (SMBHs) observed in galaxies and quasars (see Rees 1984 and Baumgarte & Shapiro 1999a for discussion and references). If an SMS is uniformly rotating, cooling and secular contraction will ultimately trigger its coherent dynamical collapse to an SMBH, giving rise to a burst of gravitational waves (Saijo et al. 2002; Shibata & Shapiro 2002). If an SMS is differentially rotating, cooling and contraction will instead lead to the unstable formation of bars or spiral arms prior to collapse and the production of quasi-periodic waves (New & Shapiro 2001a, 2001b). Magnetic fields and turbulence provide the principal mechanisms that can damp differential rotation in such stars (Zeldovich & Novikov 1971; Shapiro 2000) and thus determine their ultimate fate.

5. *The r-mode instability in rotating neutron stars.*—This instability has recently been proposed as a possible mechanism for limiting the angular velocities in neutron stars and producing observable quasi-periodic gravitational waves (Andersson 1998; Friedman & Morsink 1998; Andersson, Kokkotas, & Stergioulas 1999; Lindblom, Owen, & Morsink 1998). However, preliminary calculations (Rezzolla, Lamb, & Shapiro 2000; Rezzolla et al. 2001a, 2001b and references therein) suggest that if the stellar magnetic field is strong enough, *r*-mode oscillations will not occur. Even if the initial field is weak, fluid motions produced by these oscillations may amplify the magnetic field and eventually distort or suppress the *r*-modes altogether (*r*-modes may also by suppressed by nonlinear mode coupling [Arras et al. 2002; Schenk et al. 2002]).

This paper is partitioned as follows: In §§ 2 and 3 we review Einstein's field equations and Maxwell's equations in $3+1$ form. In § 4 we discuss the approximation of ideal magnetohydrodynamics and in § 5 the equations of general relativistic hydrodynamics. In § 6 we then develop the equations of general relativistic MHD. We derive the MHD source terms that appear in Einstein equations in § 7. We compare our results with those of previous treatments in § 8. Finally, we briefly summarize our analysis in § 9.

We adopt geometrized units throughout, setting $G = 1 = c$, where $G$ is the gravitation constant and $c$ is the speed of light.

## 2. EINSTEIN'S FIELD EQUATIONS IN $3+1$ FORM

The spacetime geometry (i.e., metric) is determined by integrating Einstein's field equations of general relativity. Most algorithms for performing this integration numerically are based on a $3+1$ decomposition of Einstein's equations, which is ideally suited for solving the general initial value problem. Below we briefly summarize the key equations that result from recasting Einstein's equations in $3+1$ form. More detailed discussions may be found, for example, in Misner, Thorne, & Wheeler (1973), York (1979, p. 83), Evans (1984), and references therein.

In a $3+1$ decomposition of Einstein's field equations, the four-dimensional spacetime $M$ is foliated into a family of nonintersecting spacelike three-surface $\Sigma$, which arise, at least locally, as level surfaces of a scalar time function $t$. The spatial metric $\gamma_{ab}$ on the three-dimensional hypersurfaces $\Sigma$ is induced by the spacetime metric $g_{ab}$ according to

$$\gamma_{ab} = g_{ab} + n_a n_b \ , \tag{1}$$

where $n^a$ is the unit normal vector $n_a = \alpha \nabla_a t$ to the slices. Here the normalization factor $\alpha$ is called the lapse function. The time vector $t^a$ is constructed so that it is dual to the foliation one-form $\nabla_a t$:

$$t^a = \alpha n^a + \beta^a \ , \tag{2}$$

where the shift vector $\beta^a$ is spatial, i.e., $n_a \beta^a = 0$, but otherwise arbitrary. In a coordinate system that is aligned with $t^a$ and $\Sigma$, the components of $n^a$ are

$$n_a = (-\alpha, 0, 0, 0) \text{ and } n^a = \alpha^{-1}\left(1, -\beta^i\right) \ . \tag{3}$$

We adopt the convention that latin indices $a, b, c, d, \ldots$, denote spacetime components, while $i, j, k, l, \ldots$, denote spatial components.

The spacetime metric can now be written in the ADM form (Arnowitt, Deser, & Misner 1962):

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \ . \tag{4}$$

The lapse function $\alpha$ determines by how much proper time advances along the normal vector from one time slice to the next, and the shift vector $\beta^i$ determines by how much spatial coordinates are shifted on the new slice. The lapse function and three components of the shift vector constitute gauge potentials that may be freely specified. Together, $\alpha$ and $\beta^i$ thus embody the four degrees of coordinate freedom inherent in general relativity.

Einstein's equation,

$$G_{ab} = 8\pi T_{ab} \ , \tag{5}$$

where $G_{ab}$ is the Einstein tensor associated with $g_{ab}$ and $T_{ab}$ is the stress-energy tensor, can be projected both along the normal direction $n^a$ and into the spatial slice $\Sigma$. The spatial projection yields two constraint equations, which constrain the fields within each slice $\Sigma$; they contain at most one time derivative of the spatial metric. The projection along the normal vector yields an evolution equation that describes how the fields propagate from one slice to the next; it contains second-order time derivatives of the spatial metric. The constraint equations consist of the Hamiltonian constraint,

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \ , \tag{6}$$

and the momentum constraint,

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i \ , \tag{7}$$

and the evolution equation is

$$\begin{aligned} \partial_t K_{ij} = & - D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) \\ & - 8\pi\alpha\left[S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)\right] + \mathscr{L}_\beta K_{ij} \ . \end{aligned} \tag{8}$$

Here $K_{ij}$ is the extrinsic curvature, and its definition in terms of the time derivative of the spatial metric is usually considered the second evolution equation,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathscr{L}_\beta \gamma_{ij} \ . \tag{9}$$

Here $D_i$, $R_{ij}$, and $R = \gamma^{ij}R_{ij}$ are the covariant derivative, Ricci tensor, and scalar curvature associated with $\gamma_{ij}$, respectively, while $K = \gamma^{ij}K_{ij}$ is the trace of the extrinsic curvature. The symbol $\mathscr{L}$ denotes a Lie derivative. The matter and nongravitational field sources $\rho$, $S_i$, and $S_{ij}$ are the projections of the stress-energy tensor into $n^a$ and $\Sigma$ and are given by

$$\rho = n_a n_b T^{ab} , \tag{10}$$

$$S_i = -\gamma_{ia} n_b T^{ab} , \tag{11}$$

$$S_{ij} = \gamma_{ia}\gamma_{jb} T^{ab} . \tag{12}$$

The quantity $\rho$ is the total mass-energy density as measured by a normal observer, $S_i$ is the momentum density, and $S_{ij}$ is the stress. Finally, $S$ is defined as the trace of $S_{ij}$:

$$S = \gamma^{ij} S_{ij} . \tag{13}$$

We remark that if the constraint equations are satisfied on an initial time slice $\Sigma$, the evolution equations guarantee that the constraints will be satisfied on all subsequent time slices.

Equations (6)–(9) are commonly referred to as the ADM equations (Arnowitt et al. 1962). Numerical implementations of these equations have revealed that their numerical stability can be improved dramatically by bringing them into a slightly different form. One such modification, now commonly referred to as "BSSN," is based on Shibata & Nakamura (1995) and Baumgarte & Shapiro (1999b). This system is widely used, and its enhanced stability properties have been analyzed by several authors (e.g., Alcubierre et al. 2000; see Knapp, Walker, & Baumgarte 2002 for an electromagnetic analogy). Alternatively, several authors have experimented with hyperbolic formulations of Einstein's equations (e.g., Anderson & York 1999). We refer the reader to these papers for further details and references.

## 3. MAXWELL'S EQUATIONS

We decompose the Faraday tensor $F^{ab}$ as

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abc} B_c \tag{14}$$

so that $E^a$ and $B^a$ are the electric and magnetic fields, respectively, observed by a normal observer $n^a$. Both fields are purely spatial, whereby

$$E^a n_a = 0 \text{ and } B^a n_a = 0 , \tag{15}$$

and the three-dimensional Levi-Civita symbol $\epsilon_{abc}$ is defined by

$$\epsilon^{abc} = \epsilon^{abcd} n_d \text{ or } \epsilon_{abc} = n^d \epsilon_{dabc} . \tag{16}$$

Note that $\epsilon^{abc}$ is nonzero only for spatial indices, while $\epsilon_{abc}$ may be nonvanishing even if one index is timelike (see eq. [32]). We also decompose the electromagnetic current four-vector $\mathscr{J}^a$ according to

$$\mathscr{J}^a = n^a \rho_e + J^a , \tag{17}$$

where $\rho_e$ and $J^a$ are the charge density and current, respectively, as observed by a normal observer $n^a$. Note that $J^a$ is purely spatial, $J^a n_a = 0$.

With these definitions, Maxwell's equations,

$$\nabla_b F^{ab} = 4\pi \mathscr{J}^a \tag{18}$$

and

$$\nabla_{[a} F_{bc]} = 0 , \tag{19}$$

where $\nabla$ is the four-dimensional covariant derivative operator associated with $g_{ab}$, can be brought into the 3 + 1 form:

$$D_i E^i = 4\pi \rho_e , \tag{20}$$

$$\partial_t E^i = \epsilon^{ijk} D_j(\alpha B_k) - 4\pi\alpha J^i + \alpha K E^i + \mathscr{L}_\beta E^i , \tag{21}$$

$$D_i B^i = 0 , \tag{22}$$

$$\partial_t B^i = -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i + \mathscr{L}_\beta B^i \tag{23}$$

(see, e.g., Thorne & MacDonald 1982). The charge conservation equation,

$$\nabla_a \mathscr{J}^a = 0 , \tag{24}$$

which is implied by equation (18), becomes

$$\partial_t \rho_e = -D_i(\alpha J^i) + \alpha K \rho_e + \mathscr{L}_\beta \rho_e . \tag{25}$$

The special relativistic Maxwell's equations can be recovered very easily by evaluating the above equations for a Minkowski spacetime with $\gamma_{ij} = f_{ij}$, where $f_{ij}$ is the flat spatial metric in an arbitrary coordinate system, $\alpha = 1$, $K = 0$, and $\beta^i = 0$.

It is convenient to introduce a four-vector potential $\mathscr{A}_a$, which can be decomposed into

$$\mathscr{A}_a = \Phi n_a + A_a , \tag{26}$$

where $A_a$ is purely spatial: $A_a n^a = 0$. Inserting equation (3), this implies

$$A_t = \beta^i A_i , \tag{27}$$

while $A^t = 0$, as for any spatial vector. In terms of the vector potential, the Faraday tensor can be written

$$F_{ab} = \mathscr{A}_{b,a} - \mathscr{A}_{a,b} = n_a E_b - n_b E_a + \epsilon_{abc} B^c . \tag{28}$$

Contracting this equation with $\epsilon^{abc}$ yields

$$\epsilon^{abc}(\mathscr{A}_{b,a} - \mathscr{A}_{a,b}) = \epsilon^{abc}\epsilon_{abd} B^d = 2B^c \tag{29}$$

or

$$B^i = \epsilon^{ijk} A_{k,j} . \tag{30}$$

Note that with this identification, the magnetic field $B^i$ automatically satisfies the constraint equation (22).

It is possible, and often convenient (see Paper II), to rewrite Maxwell's equations completely in terms of $E_i$ and $A_i$, thereby eliminating $B_i$. Evaluating equation (28) for the components $a = t$ and $b = i$ with $\mathscr{A}_i = A_i$ and $\mathscr{A}_t = -\alpha\Phi + \beta^i A_i$ yields

$$\partial_t A_i = -\alpha E_i + \epsilon_{tij} B^j - (\alpha\Phi - \beta^j A_j)_{,i} . \tag{31}$$

Using equation (16), we can rewrite

$$\epsilon_{tij} = n^d \epsilon_{dtij} = -\alpha^{-1}\beta^k \epsilon_{ktij} = -\alpha^{-1}\beta^k \epsilon_{tikj}$$
$$= -\beta^k n^d \epsilon_{dikj} = -\beta^k \epsilon_{ikj} , \tag{32}$$

so that

$$\partial_t A_i = -\alpha E_i - \epsilon_{ijk}\beta^j B^k - (\alpha\Phi - \beta^j A_j)_{,i} . \tag{33}$$

With equation (30), $\epsilon_{ijk}\beta^j B^k$ can be expressed in terms of

$A_i$ as

$$\begin{aligned}\epsilon_{ijk}\beta^j B^k &= \epsilon_{ijk}\epsilon^{klm}\beta^j A_{m,l} \\ &= \left(\delta^l_i\delta^m_j - \delta^l_j\delta^m_i\right)\beta^j A_{m,l} \\ &= \beta^j A_{j,i} - \beta^j A_{i,j} \ . \end{aligned} \tag{34}$$

Inserting this into equation (33) yields

$$\partial_t A_i = -\alpha E_i - \partial_i(\alpha\Phi) + \mathscr{L}_\beta A_i \ . \tag{35}$$

In equation (21), the magnetic field $B_i$ can be eliminated similarly:

$$\begin{aligned}\epsilon^{ijk}D_j(\alpha B_k) &= \epsilon^{ijk}D_j(\alpha\epsilon_{klm}D^l A^m) \\ &= \epsilon^{ijk}\epsilon_{klm}D_j(\alpha D^l A^m) \\ &= (\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)D_j(\alpha D^l A^m) \\ &= D_j(\alpha D^i A^j) - D_j(\alpha D^j A^i) \ . \end{aligned} \tag{36}$$

Inserting this into equation (21) yields

$$\partial_t E^i = D_j(\alpha D^i A^j) - D_j(\alpha D^j A^i) - 4\pi\alpha J^i + \alpha K E^i + \mathscr{L}_\beta E^i \ . \tag{37}$$

Equations (35) and (37) form a system of equations for $E^i$ and $A_i$ alone. In the special relativistic limit, they again reduce to familiar expressions. We also note that in terms of partial derivatives, equation (37) can be expanded to yield

$$\begin{aligned}\partial_t E^i = {} & \gamma^{-1/2}[\alpha\gamma^{1/2}(\gamma^{il}\gamma^{jm} - \gamma^{im}\gamma^{jl})A_{m,l}]_{,j} \\ & - 4\pi\alpha J^i + \alpha K E^i + \mathscr{L}_\beta E^i \ , \end{aligned} \tag{38}$$

where $\gamma$ is the determinant of the spatial metric $\gamma_{ij}$. This form of the electric field evolution equation will be useful for applications in Paper II.

## 4. IDEAL MAGNETOHYDRODYNAMICS APPROXIMATION

Ohm's law can be written (see, e.g., problem [11.16] in Jackson 1999)

$$\mathscr{J}_a - \tilde{\rho}_e u_a = \sigma F_{ab}u^b \ , \tag{39}$$

where $\sigma$ is the electrical conductivity and $\tilde{\rho}_e = -\mathscr{J}^a u_a$ is the charge density as seen by an observer comoving with the fluid four-velocity $u^a$ (in contrast to $\rho_e$, which was defined as the charge density as observed by a normal observer $n^a$).

A $3+1$ decomposition of Ohm's law can be derived by contracting equation (39) with $n^a$ and $\gamma^b_a$. The former yields

$$W\tilde{\rho}_e = \rho_e - \sigma u_a E^a \ , \tag{40}$$

where we have defined $W$ as the Lorentz factor between normal and fluid observers:

$$W \equiv -n_a u^a = \alpha u^t \ . \tag{41}$$

Projecting equation (39) into $\Sigma$, or equivalently, evaluating the spatial components $a = i$ of equation (39), yields

$$\begin{aligned} J_i - \tilde{\rho}_e u_i &= \sigma F_{ia}u^a = \sigma(F_{it}u^t + F_{ij}u^j) \\ &= \sigma(\alpha E_i u^t + \epsilon_{itk}B^k u^t + \epsilon_{ijk}B^k u^j) \\ &= \sigma[W E_i + \epsilon_{ijk}(\beta^j B^k u^t + B^k u^j)] \\ &= \sigma[W E_i + \epsilon_{ijk}(v^j + \beta^j)B^k u^t] \ . \end{aligned} \tag{42}$$

Here we have defined

$$v^i \equiv \frac{u^i}{u^t} \tag{43}$$

and have used equation (32) to relate $\epsilon_{itk}$ to $\epsilon_{ijk}$, giving rise to the shift term in equation (42) (the shift term is missing in some previous treatments; see § 8).

Dividing Ohm's law (eq. [39]) by $\sigma$ and allowing $\sigma \to \infty$ yields the perfect conductivity condition:

$$F_{ab}u^b = 0 \ . \tag{44}$$

According to equations (40) and (42), this result is equivalent to the condition that the electric field vanish in the fluid rest frame:

$$u_a E^a = 0 \ , \tag{45}$$

or

$$\alpha E_i = -\epsilon_{ijk}(v^j + \beta^j)B^k \ , \tag{46}$$

which is often called the ideal MHD relation. When evaluated in a Minkowski spacetime, the last equation reduces to the familiar expression $E_i = -\epsilon_{ijk}v^j B^k$ or $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$.

We can now evaluate Faraday's law (eq. [23]) under the assumption of perfect conductivity. Taking the trace of equation (9) yields

$$\alpha K = -\partial_t \ln\gamma^{1/2} + D_i\beta^i \ . \tag{47}$$

The above expression can be combined with the Lie derivative $\mathscr{L}_\beta B^i$ to give

$$\alpha K B^i + \mathscr{L}_\beta B^i = D_j(\beta^j B^i - \beta^i B^j) - B^i\partial_t \ln\gamma^{1/2} \ , \tag{48}$$

where we have used equation (22). Inserting equation (48) together with the ideal MHD equation (46) into Faraday's law (eq. [23]) reveals that all the shift terms cancel, leaving

$$\frac{1}{\gamma^{1/2}}\partial_t(\gamma^{1/2}B^i) = D_j(v^i B^j - v^j B^i) \ . \tag{49}$$

It is convenient to introduce the magnetic vector density

$$\mathscr{B}^i \equiv \gamma^{1/2}B^i \ , \tag{50}$$

in terms of which equations (49) and (22) reduce to the particularly simple forms

$$\partial_t\mathscr{B}^i = \partial_j(v^i\mathscr{B}^j - v^j\mathscr{B}^i) \tag{51}$$

and

$$\partial_i\mathscr{B}^i = 0 \ . \tag{52}$$

In § 8 we compare our results with previous treatments and correct errors in some previously published equations.

## 5. GENERAL RELATIVISTIC HYDRODYNAMICS

For a perfect fluid, the stress-energy tensor $T^{ab}_{\rm fluid}$ can be written

$$T^{ab}_{\rm fluid} = \rho_0 h u^a u^b + P g^{ab} \ . \tag{53}$$

Here $\rho_0$ is the rest-mass density as observed by an observer comoving with the fluid $u^a$, $P$ is the pressure, and $h$ is the specific enthalpy:

$$h = 1 + \epsilon + P/\rho_0 \ , \tag{54}$$

where $\epsilon$ is the specific internal energy density.

In the absence of electromagnetic fields, the equations of motion for the fluid can be derived from the local conservation of energy momentum,

$$\nabla_b T^{ab}_{\rm fluid} = 0 \ , \tag{55}$$

and the conservation of baryons,

$$\nabla_a(\rho_0 u^a) = 0 \ . \tag{56}$$

The resulting equations can be cast in various forms, depending on how the primitive fluid variables are chosen (see, e.g., Font 2000 for a recent review). The most frequently adopted relativistic formalism was originally developed by Wilson (1972; see also Hawley, Smarr, & Wilson 1984), who defined a rest-mass density variable,

$$D \equiv \rho_0 W \ , \tag{57}$$

an internal energy density variable,

$$E \equiv \rho_0 \epsilon W \ , \tag{58}$$

and a momentum variable,

$$S_a \equiv \rho_0 h W u_a = (D + E + PW) u_a \ . \tag{59}$$

Note that the spatial vector $S_i$ defined above is the fluid contribution to the source term $S_i$ appearing in Einstein's field equations (see eq. [11]). In terms of these variables, the equation of continuity becomes

$$\partial_t(\gamma^{1/2} D) + \partial_j(\gamma^{1/2} D v^j) = 0 \ . \tag{60}$$

Contracting equation (55) with $u^b$ yields the energy equation

$$\partial_t(\gamma^{1/2} E) + \partial_j(\gamma^{1/2} E v^j) = -P[\partial_t(\gamma^{1/2} W) + \partial_i(\gamma^{1/2} W v^i)] \ , \tag{61}$$

while the spatial components of equation (55) yield the Euler equation

$$\partial_t(\gamma^{1/2} S_i) + \partial_j(\gamma^{1/2} S_i v^j) = -\alpha\gamma^{1/2}\left(\partial_i P + \frac{S_a S_b}{2\alpha S^t}\partial_i g^{ab}\right) \ . \tag{62}$$

For gamma-law equations of state,

$$P = (\Gamma - 1)\rho_0 \epsilon \ , \tag{63}$$

the right-hand side of the energy equation (61) can be eliminated to yield

$$\partial_t(\gamma^{1/2} E_*) + \partial_j(\gamma^{1/2} E_* v^j) = 0 \ , \tag{64}$$

where we have introduced the energy variable $E_*$, defined as

$$E_* \equiv (\rho_0 \epsilon)^{1/\Gamma} W \tag{65}$$

(see Shibata 1999, who also absorbed the determinant $\gamma^{1/2}$ into the definition of the fluid variables; see also Shibata, Baumgarte, & Shapiro 1998). This simplification has great computational advantages since the time derivatives on the right-hand side of equation (61) are difficult to handle in strongly relativistic fluid flow (compare Norman & Winkler 1986).

For given values of $u_i$, $W$ can be found from the normalization relation $u_a u^a = -1$,

$$W = \alpha u^t = \left(1 + \gamma^{ij} u_i u_j\right)^{1/2} \ , \tag{66}$$

and $v^i$ from

$$v^i = \frac{\alpha \gamma^{ij} u_j}{W} - \beta^i \ . \tag{67}$$

Finite difference implementations of the above equations must be adapted to handle the appearance of shock discontinuities. Two strategies are commonly adopted.

The more traditional approach is to add an artificial viscosity term to the equations (von Neuman & Richtmyer 1950). Typically, the artificial viscosity term $Q$ is nonzero only where the fluid is compressed and is added to the pressure on the right-hand sides of both the energy equation (61) and the Euler equation (62). The artificial viscosity spreads the shock discontinuity over several grid zones. For shocks occurring in Newtonian fluids with modest Mach numbers, artificial viscosity generates the Rankine-Hugoniot jump conditions to reasonable accuracy. Artificial viscosity has also been used successfully in relativistic applications (see, e.g., Wilson 1972; Shibata 1999), but it leads to less satisfactory results for highly relativistic flows or high Mach numbers (Norman & Winkler 1986).

An alternative approach to handling shocks is a high-resolution shock capturing (HRSC) scheme (see, e.g., Martí & Müller 1999 for a recent review). In such a scheme, one treats all fluid variables as constant in each grid cell. The discontinuous fluid variables at the grid interfaces serve as initial conditions for a local Riemann shock-tube problem, which can be solved either exactly or approximately. Allowing for discontinuities, including shocks, lies at the core of these schemes and does not require any additional artificial viscosity. Constructing Riemann solvers for HRSC requires knowledge of the local characteristic structure of the equations to be solved. This has motived the development of several flux-conservative hydrodynamics schemes, which do not contain any derivatives of the fluid variables in the source terms and for which this characteristic structure can be determined (see, e.g., Font 2000; Font et al. 2002).

## 6. GENERAL RELATIVISTIC MAGNETOHYDRODYNAMICS

To derive the equations of general relativistic magnetohydrodynamics, we now add the electromagnetic stress-energy tensor $T^{ab}_{\rm em}$ to the fluid stress-energy tensor:

$$T_{ab} = T^{ab}_{\rm fluid} + T^{ab}_{\rm em} \ . \tag{68}$$

Local conservation of energy momentum demands that the divergence of the sum $T_{ab}$ vanish. The divergence of the individual stress-energy tensors $T^{ab}_{\rm fluid}$ and $T^{ab}_{\rm em}$ do not vanish in general since the fluid and electromagnetic fields may exchange energy and momentum. In particular, one finds

$$\nabla_b T^{ab}_{\rm fluid} = -\nabla_b T^{ab}_{\rm em} = F^{ab}\mathscr{J}_b \tag{69}$$

(see eq. [5.40] in Misner et al. 1973). The right-hand side of equation (69) now includes the Lorentz force in the equations of relativistic hydrodynamics. Note that the baryon conservation equations (56) and (60) remain unchanged.

In the energy equation (61), which was derived from the contraction $u_b\nabla_a T^{ab}$, the addition of the Lorentz force yields

$$\begin{aligned}\partial_t(\gamma^{1/2}E) + \partial_j(\gamma^{1/2}Ev^j) \\ = -P[\partial_t(\gamma^{1/2}W) + \partial_i(\gamma^{1/2}Wv^i)] - \alpha\gamma^{1/2}u_a F^{ab}\mathscr{J}_b \,.\end{aligned} \tag{70}$$

For a gamma-law equation of state, the above equation may be written in terms of $E_*$ according to

$$\partial_t(\gamma^{1/2}E_*) + \partial_j(\gamma^{1/2}E_*v^j) = -u_a F^{ab}\mathscr{J}_b\left(\frac{E_*}{W}\right)^{1-\Gamma}\frac{\alpha\gamma^{1/2}}{\Gamma}\,, \tag{71}$$

which now takes the place of equation (64) in general. The Euler equation (62) now becomes

$$\begin{aligned}\partial_t(\gamma^{1/2}S_i) + \partial_j(\gamma^{1/2}S_iv^j) \\ = -\alpha\gamma^{1/2}\left(\partial_i P + \frac{S_aS_b}{2\alpha S^t}\partial_i g^{ab}\right) + \alpha\gamma^{1/2}F_{ia}\mathscr{J}^a \,.\end{aligned} \tag{72}$$

In the case of ideal MHD, the new terms on the right-hand sides of the energy equations (70) and (71) vanish because of equation (44). This result is understandable since it corresponds to the absence of Joule heating in the limit of infinite conductivity.

We now proceed to determine the Lorentz force $F_{ia}\mathscr{J}^a$ in the Euler equation (72). Since $\mathscr{J}^a$ is not known a priori, we first use equation (18) to express $\mathscr{J}_a$ in terms of the electromagnetic fields. Note that $\mathscr{J}_a$ is the current four-vector as opposed to its spatial projection $J^i$. We could express $J^i$ immediately using the spatial Maxwell equation (21). Instead, we need to derive the four-dimensional equivalent of equation (21) to express $\mathscr{J}_a$, which we then can contract with the Faraday tensor (eq. [14]) to obtain the Lorentz force.

The divergence of the Faraday tensor in equation (69) is

$$\nabla_b F^{ab} = n^a\nabla_b E^b + E^b\nabla_b n^a - n^b\nabla_b E^a - E^a\nabla_b n^b + \nabla_b\epsilon^{abc}B_c\,. \tag{73}$$

We now decompose the four-dimensional derivatives $\nabla_a$ in each term above.

The Lie derivative of $E^a$ along $\alpha n^a$ is

$$\mathscr{L}_{\alpha\boldsymbol{n}}E^a = \alpha n^b\nabla_b E^a - \alpha E^b\nabla_b n^a - E^b n^a\nabla_b\alpha\,, \tag{74}$$

or, with equation (2),

$$\frac{1}{\alpha}(\partial_t - \mathscr{L}_\beta)E^a = n^b\nabla_b E^a - E^b\nabla_b n^a - E^b n^a\nabla_b\ln\alpha\,. \tag{75}$$

The four-dimensional divergence $\nabla_a E^a$ can be expressed in terms of the three-dimensional divergence $D_iE^i$:

$$\begin{aligned}D_aE^a = \gamma^b_a\nabla_b E^a = (g^b_a + n_a n^b)\nabla_b E^a \\ = \nabla_a E^a - E^a D_a\ln\alpha\,,\end{aligned} \tag{76}$$

where we have used $n_aE^a = 0$ and

$$n^b\nabla_b n_a = a_a = D_a\ln\alpha\,. \tag{77}$$

Here $a_a$ is the four-acceleration of a normal observer. Since the extrinsic curvature $K_{ab}$ can be written

$$K_{ab} = -\nabla_a n_b - n_a a_b\,, \tag{78}$$

the divergence of $n^a$ satisfies

$$\nabla_a n^a = -K\,. \tag{79}$$

Inserting these expressions into equation (73), we now find the intermediate result

$$\nabla_b F^{ab} = n^a D_b E^b + KE^a - \frac{1}{\alpha}(\partial_t - \mathscr{L}_\beta)E^a + \nabla_b\epsilon^{abc}B_c\,, \tag{80}$$

where we have also used $E^aD_a\ln\alpha = E^a\nabla_a\ln\alpha$.

The term involving $B_a$ in equation (80) can be rewritten as

$$\begin{aligned}\nabla_b\epsilon^{abc}B_c &= \epsilon^{abcd}\nabla_b B_c n_d = \epsilon^{abcd}(n_d\nabla_b B_c + B_c\nabla_b n_d) \\ &= \epsilon^{abcd}(n_d D_b B_c - B_c n_b a_d) \\ &= \alpha^{-1}\epsilon^{abcd}(\alpha n_d D_b B_c + n_d B_c D_b\alpha) \\ &= \alpha^{-1}\epsilon^{abc}D_b(\alpha B_c),\end{aligned} \tag{81}$$

where we have used equations (16), (77), and (78). Inserting this expression into equation (80) now yields

$$\begin{aligned}4\pi\alpha\mathscr{J}^a = \alpha\nabla_b F^{ab} = -(\partial_t - \mathscr{L}_\beta)E^a + \epsilon^{abc}D_b(\alpha B_c) \\ + \alpha n^a D_b E^b + \alpha KE^a\,.\end{aligned} \tag{82}$$

Not surprisingly, this is the four-dimensional version of equation (21), which can be found by taking the spatial projection of equation (82).

The next step is to contract equation (82) with the Faraday tensor (eq. [14]). Using $n_aE^a = 0$, $n_a\epsilon^{abc} = 0$, and $n_a\mathscr{L}_{\alpha\boldsymbol{n}}E^a = 0$, several terms cancel, and one finds

$$\begin{aligned}4\pi\alpha\mathscr{J}^b F_{ab} = {} & \alpha E_a D_b E^b \\ & + n_a\left[-E_b\left(\partial_t - \mathscr{L}_\beta - \alpha K\right)E^b + E_b\epsilon^{bcd}D_c(\alpha B_d)\right] \\ & - B^c\epsilon_{cab}\left[\left(\partial_t - \mathscr{L}_\beta - \alpha K\right)E^b - \epsilon^{bde}D_d(\alpha B_e)\right]\,.\end{aligned} \tag{83}$$

This expression can now be inserted into the Euler equation (72). For spatial components $n_i = 0$, so that the second line in equation (83) vanishes. The source term $\alpha\gamma^{1/2}F_{ia}\mathscr{J}^a$ can then be rewritten

$$\begin{aligned}\alpha\gamma^{1/2}F_{ia}\mathscr{J}^a = \frac{\gamma^{1/2}}{4\pi}\big[ & -B^j\epsilon_{jik}\left(\partial_t - \mathscr{L}_\beta - \alpha K\right)E^k \\ & + \alpha E_i\left(D_jE^j\right) + B^jD_j(\alpha B_i) - B^jD_i(\alpha B_j)\big]\,.\end{aligned} \tag{84}$$

If desired, the covariant derivatives in the last two terms can be converted into partial derivatives, which finally yields the

Euler equation

$$\begin{aligned}
\partial_t(\gamma^{1/2}S_i) + \partial_j(\gamma^{1/2}S_i v^j) = \\
- \alpha\gamma^{1/2}\left(\partial_i P + \frac{S_a S_b}{2\alpha S^t}\partial_i g^{ab}\right) + \alpha\frac{\gamma^{1/2}}{4\pi}E_i\left(D_j E^j\right) \\
- \frac{\gamma^{1/2}}{4\pi}B^j\left[\epsilon_{jik}\left(\partial_t E^k - \beta^l\partial_l E^k + E^l\partial_l\beta^k - \alpha K E^k\right)\right. \\
\left. + \partial_i(\alpha B_j) - \partial_j(\alpha B_i)\right] , \qquad (85)
\end{aligned}$$

where we have expanded the Lie derivative of $E^i$. Note that in this equation the electric field terms enter with the opposite sign from those in the corresponding equation (3.2) of Sloan & Smarr (1985, hereafter SS85), who further assume $\beta = 0 = K$.

For numerical implementations, the most challenging term in Euler's equation is probably the time derivative of the electric field. For ideal MHD, this term can be rewritten by first expressing $E^i$ in terms of the magnetic fields $B^i$ using the ideal MHD relation (eq. [46]) and then using equation (49) to eliminate the time derivative of $B^i$ (see Zhang 1989, hereafter Z89). This term is likely to be small in most applications; for example, it is $\mathcal{O}(v^2/c^2)$ times smaller than the last two terms on the right-hand side of equation (85). In such cases, extrapolating and iterating, or some other simple treatment, may be adequate to account for its contribution.

It is instructive to take the Newtonian limit of equation (85) and recover a familiar expression. With $g_{00} \to -(1+2\phi)$, where $\phi$ is the Newtonian potential, we find

$$\frac{1}{2}\frac{S_a S_b}{\alpha S^t}\partial_i g^{ab} \to -\frac{1}{2}\rho\partial_i g^{00} = \rho\partial_i\phi . \qquad (86)$$

In Cartesian coordinates ($\gamma^{1/2} = 1$), the Newtonian limit of equation (85) then becomes

$$\begin{aligned}
\partial_t S_i + \partial_j(S_i v^j) = -\partial_i P - \rho\partial_i\phi + \rho_e E_i \\
- \frac{1}{8\pi}\partial_i(B^j B_j) + \frac{1}{4\pi}B^j\partial_j B_i \qquad (87)
\end{aligned}$$

or, equivalently,

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla(P + P_M) - \rho\nabla\phi + \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \rho_e\boldsymbol{E} , \qquad (88)$$

where we have defined the magnetic pressure

$$P_M = \frac{\boldsymbol{B}^2}{8\pi} \qquad (89)$$

and where $\nabla$ is the spatial gradient operator. Note that for a neutral plasma $\rho_e = 0$, the electric field $E^a$ disappears entirely from the above Newtonian equation.

## 7. SOURCE TERMS FOR THE GRAVITATIONAL FIELD EQUATIONS

We now catalog the source terms $\rho$ (eq. [10]), $S_i$ (eq. [11]), $S_{ij}$ (eq. [12]), and $S$ (eq. [13]) that appear in the Hamiltonian constraint (eq. [6]), the momentum constraint (eq. [7]), and the evolution equation (8). Inserting the fluid stress-energy tensor (eq. [53]) into equations (10)–(13) yields the fluid contributions to the source terms:

$$\rho_{\rm fluid} = \rho_0 h W^2 - P , \qquad (90)$$

$$S_i^{\rm fluid} = \rho_0 h W u_i , \qquad (91)$$

$$S_{ij}^{\rm fluid} = P\gamma_{ij} + \frac{S_i^{\rm fluid}S_j^{\rm fluid}}{\rho_0 h W^2} , \qquad (92)$$

$$S_{\rm fluid} = 3P + \rho_0 h(W^2 - 1) . \qquad (93)$$

Next we assemble the electromagnetic contributions to the source terms. To do so, we first need to construct the electromagnetic stress-energy tensor $T^{ab}_{\rm em}$ from the Faraday tensor $F^{ab}$:

$$4\pi T^{ab}_{\rm em} = F^{ac}F^b{}_c - \tfrac{1}{4}g^{ab}F_{cd}F^{cd} . \qquad (94)$$

Inserting equation (14), we first find

$$F_{ab}F^{ab} = 2(B_i B^i - E_i E^i) , \qquad (95)$$

where we have used $\epsilon_{abc}\epsilon^{abd} = 2\gamma^d_c$. With $\epsilon^{abc}\epsilon_a^{\ de} = \gamma^{bd}\gamma^{ce} - \gamma^{be}\gamma^{cd}$, the first term in equation (94) becomes

$$\begin{aligned}
F^{ac}F^b{}_c = n^a n^b E_i E^i + 2n^{(a}\epsilon^{b)cd}E_c B_d \\
- E^a E^b - B^a B^b + \gamma^{ab}B_i B^i . \qquad (96)
\end{aligned}$$

Combining the last two equations then yields the electromagnetic stress-energy tensor in $3+1$ form

$$\begin{aligned}
4\pi T^{ab}_{\rm em} = \tfrac{1}{2}\left(n^a n^b + \gamma^{ab}\right)\left(E_i E^i + B_i B^i\right) \\
+ 2n^{(a}\epsilon^{b)cd}E_c B_d - \left(E^a E^b + B^a B^b\right) . \qquad (97)
\end{aligned}$$

This stress-energy tensor can now be inserted into equations (10)–(13) to obtain the electromagnetic source terms. For the mass-energy density $\rho_{\rm em}$, we find

$$4\pi\rho_{\rm em} = n_a n_b 4\pi T^{ab}_{\rm em} = \tfrac{1}{2}\left(E_i E^i + B_i B^i\right) = \tfrac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) , \qquad (98)$$

which is the energy density of the electromagnetic fields. The energy flux $S_i^{\rm em}$ reduces to the Poynting vector

$$\begin{aligned}
4\pi S_i^{\rm em} &= -\gamma_{ia}n_b 4\pi T^{ab}_{\rm em} = -\gamma_{ia}n_b n^b\epsilon^{acd}E_c B_d \\
&= \epsilon_{ijk}E^j B^k = (\boldsymbol{E}\times\boldsymbol{B})_i . \qquad (99)
\end{aligned}$$

The stress tensor $S_{ij}^{\rm em}$ is

$$4\pi S_{ij}^{\rm em} = \gamma_{ia}\gamma_{jb}4\pi T^{ab}_{\rm em} = -E_i E_j - B_i B_j + \tfrac{1}{2}\gamma_{ij}(\boldsymbol{E}^2 + \boldsymbol{B}^2) . \qquad (100)$$

Its trace equation (13), finally, is equal to the mass-energy density $\rho_{\rm em}$:

$$4\pi S_{\rm em} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) . \qquad (101)$$

The above results are not surprising: expressed in terms of the electromagnetic field components as measured by a normal observer, $n^a$, i.e., an observer who is at rest with respect to the slices $\Sigma$, the $3+1$ source terms have the same form as in flat space (compare exercise [5.1] in Misner et al. 1973).

## 8. COMPARISON WITH PREVIOUS TREATMENTS

In this section we compare our notation and findings with those of SS85, Evans & Hawley (1988, hereafter EH88), Hawley & Evans (1989, hereafter HE89), and Z89.

SS85 define the three-velocity $v^i_{\rm SS}$ by writing the four-velocity $u^a$ as

$$u^a = \alpha^{-1} W\big(1,\ \alpha v^i_{\rm SS} - \beta^i\big) \tag{102}$$

and

$$u_a = W\big(-\alpha + v_i^{\rm SS}\beta^i,\ v_i^{\rm SS}\big) \tag{103}$$

(see eq. [2.1] of SS85). Here $W$ is the Lorentz factor between $u^a$ and $n^a$:

$$W = -n_a u^a = \alpha u^t\ , \tag{104}$$

as in equation (41). The normalization $u^a u_a = -1$ leads to

$$W = \big(1 - v^i_{\rm SS} v_i^{\rm SS}\big)^{-1/2}\ , \tag{105}$$

which shows that $v^i_{\rm SS}$ is the velocity of the fluid with respect to a *normal* observer.

Z89 adopts the same formulation as SS85, but denotes $W$ with $\gamma$ (see eq. [2.11] of Z89).

HE89 adopt the same definition of three-velocity as we do, defining the three-velocity $v^i$ to be the velocity with respect to *coordinate* observers:

$$v^i_{\rm W} = \frac{u^i}{u^t} \tag{106}$$

(see eq. [43]). We use the subscript "W" since this definition is used in Wilson's equations of relativistic hydrodynamics (see Wilson 1972; Hawley et al. 1984). With equation (106), the four-velocity $u^a$ can be written

$$u^a = \alpha^{-1} W\big(1,\ v^i_{\rm W}\big)\ . \tag{107}$$

Comparing equations (102) and (107) shows that the two definitions of $v^i$ are related by

$$v^i_{\rm W} = \alpha v^i_{\rm SS} - \beta^i\ . \tag{108}$$

We can now compare the ideal MHD equation (46) in the different treatments. Since HE89 adopt the same definition for $v^i = v^i_{\rm W}$ as we do, their equations (eq. [A14] from EH88 and eq. [14] from HE89) should be identical to our equation (46). In their expression, however, the shift term is absent. This absent shift term can be traced back to equation (13) from HE89, which does not agree with our equation (42). It is likely that the shift term was missed by dropping the term $\epsilon_{itj}$. The alignment of indices is incorrect in SS85's equation (2.9) (which they express in terms of $u^i$ instead of $v^i$). Fixing it and utilizing equations (103) and (108) make their equation equivalent to equation (46).

To compare with Z89's ideal MHD equation (eq. [2.12] of Z89), we insert equation (108) into equation (46), which immediately yields the Z89 result,

$$E_i = -\epsilon_{ijk} v^j_{\rm SS} B^k\ , \tag{109}$$

showing that our result agrees with that of Z89.

We find similar errors in the Faraday equation. We found that the shift terms in equation (46) cancel all other shift terms when inserted into equation (23), ultimately yielding equations (49) and (51), which do not include any shift terms. With the shift terms being absent in equation (14) from HE89, the corresponding terms do not cancel, leading to the incorrect equations (17) and (18) from HE89 (see also eqs. [2.8] and [A17] from EH88).

Z89's expression for the Faraday equation (eq. [2.13] of Z89) can be recovered by inserting equation (108) into equation (51):

$$\partial_t \mathscr{B}^i = \partial_j[(\alpha v^i_{\rm SS} - \beta^i)\mathscr{B}^j - (\alpha v^j_{\rm SS} - \beta^j)\mathscr{B}^i]\ , \tag{110}$$

which can be rewritten as

$$\frac{1}{\gamma^{1/2}}\partial_t(\gamma^{1/2}B^i) = D_j[(\alpha v^i_{\rm SS} - \beta^i)B^j - (\alpha v^j_{\rm SS} - \beta^j)B^i] \tag{111}$$

or

$$\frac{1}{\gamma^{1/2}}\partial_t(\gamma^{1/2}\boldsymbol{B}) = \boldsymbol{\nabla}\times[(\alpha\boldsymbol{v}_{\rm SS} - \beta)\times\boldsymbol{B}]\ . \tag{112}$$

This shows that our equations (49) and (51) again agree with the expressions of Z89.

Interestingly, Z89 refers to EH88, and in fact their equations look quite similar in that they both contain the above shift terms. However, Z89 uses $v^i_{\rm SS}$ as the three-velocity, while EH88 use $v^i_{\rm W}$. Therefore, the shift terms are correct in the former but incorrect in the latter.

Finally, we show that our equation (42) is equivalent to Z89's equation (2.10). On the left-hand side of equation (42) we rewrite

$$\begin{aligned}\tilde\rho_e &= u_a\mathscr{J}^a = u_a(n^a\rho_e + J^a) = -W\rho_e + u_i\boldsymbol{J}^i\\ &= W(\boldsymbol{v}_{\rm SS}\cdot\boldsymbol{J} - \rho_e)\ ,\end{aligned} \tag{113}$$

where we have used the decomposition equation (17). Inserting this and equation (108) into equation (42) yields

$$\boldsymbol{J} + W^2(\boldsymbol{v}_{\rm SS}\cdot\boldsymbol{J} - \rho_e)\boldsymbol{v}_{\rm SS} = \sigma W(\boldsymbol{E} + \boldsymbol{v}_{\rm SS}\times\boldsymbol{B})\ , \tag{114}$$

which is identical to equation (2.10) of Z89.

## 9. SUMMARY

We have assembled a complete set of Maxwell-Einstein MHD equations, describing the structure and evolution of a relativistic, ideal MHD gas in a dynamical spacetime. We compare with previous treatments and correct some errors in the existing literature.

Our compilation of these equations is motivated by a large number of problems in relativistic astrophysics in which magnetic fields are likely to play an important role (see the incomplete list in § 1). Self-consistent solutions to the Maxwell-Einstein MHD equations will be necessary for a thorough understanding of these problems, and we therefore anticipate that relativistic MHD in dynamical spacetimes will attract much interest in the future. We hope that our compilation of these equations will be useful for such investigations, particularly for treatments that will rely on numerical simulations. In Paper II we use these equations to model the collapse of a magnetized star to a black hole.

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