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DEAD END WORDS IN LAMPLIGHTER GROUPS AND OTHER WREATH PRODUCTS

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Abstract

We explore the geometry of the Cayley graphs of the lamplighter groups and a wide range of wreath products. We show that these groups have dead end elements of arbitrary depth with respect to their natural generating sets. An element w in a group G with finite generating set X is a dead end element if no geodesic ray from the identity to w in the Cayley graph (G, X) can be extended past w. Additionally, we describe some non-convex behaviour of paths between elements in these Cayley graphs and seesaw words, which are potential obstructions to these graphs satisfying the k-fellow traveller property.

1. Introduction

The main goal of this paper is to investigate the metric properties of geodesics and balls in the Cayley graphs of lamplighter groups L_n, with respect to the presentation

$$L_n = \langle a, t | [t^i a t^{-i}, t^j a t^{-j}], a^n \rangle$$

as well as more general classes of wreath products. The lamplighter groups L_n have an easy geometric interpretation in which we can view an element of L_n as a set of instructions for changing the state of a bi-infinite string of light bulbs, each with n states. For example, in L_2, each light bulb is merely on or off.

Not much is known about the geometry of the Cayley graph of these groups in the above presentation, and we begin by describing normal forms for elements of L_n and discussing a special family of geodesics in this Cayley graph. We then describe an interesting phenomenon which occurs in this Cayley graph, namely the existence of *dead end* elements. Let G be a group with a finite generating set X, and denote by $= (G, X)$ the Cayley graph of G with respect to this finite generating set. An element $w \in G$ is a dead end element if no geodesic ray from the identity to w in can be extended past w.

Dead end behaviour occurs in a number of settings. Finite groups always have dead end elements, and even well-behaved groups such as the integers can have dead end elements with

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respect to contrived generating sets, as discussed in [7, Chapter 4, Example 5] of de la Harpe. Examples of infinite groups with dead end elements with respect to natural generating sets are rare. Thompson's group F is an example of a torsion-free infinite group with dead end elements with respect to its natural finite generating set, as described in [5]. In Theorem 4.3 below, we prove that the lamplighter groups L_n with the generating set given above have infinite families of dead end elements.

There is a fundamental difference between the dead end elements in Thompson's group F and the lamplighter groups L_n, characterized by the notion of depth. The *depth* of a dead end element w with $|w| = n$ is the length of the shortest path from w to a point in the sphere of radius $n + 1$ In F the depth of any dead element is two. We show in Theorem 4.3 that the lamplighter groups contain dead end words which require paths of arbitrary length to leave their containing balls.

We explore other families of words with interesting properties regarding which generators may decrease their word length. We define a family of *seesaw* words with the property that exactly two generators reduce word length, and furthermore that that initial choice of length-reducing generator determines a unique subsequent length-reducing generator for many iterations of reductions in length.

We continue our discussion of the Cayley graph of L_n in section 5 by considering convexity properties. Cannon defines the notion of almost convexity in [3]. This property guarantees algorithmic construction of the ball $B(n + 1)$ from the ball $B(n)$ by making it sufficient to consider only a finite set of possible ways that an element in $B(n + 1)$ can be obtained from different elements of $B(n)$. Geometrically, a group is *almost convex* with respect to a finite generating set if, for large enough n, any points at distance 2 in $B(n)$ can be connected by a path of uniformly bounded length which lies entirely inside $B(n)$.

The groups L_n are not finitely presented, and thus it follows from Cannon [3] that the Cayley graphs $= (L_n, \{a, t\})$ are not almost convex. Using the geometry of the groups L_n as well as our description of geodesics in , we give an explicit example of an infinite family of words which illustrates concretely the failure of the almost convexity property, and extend these examples to a more general class of wreath products.

Kapovich [13] defines a weaker convexity condition for Cayley graphs, called $K(2)$ or *minimal almost convexity*. Kapovich [13] and Riley [18] prove several consequences of minimal almost convexity; for example, if a group G is minimally almost convex then Kapovich [13] showed that it is necessarily finitely presented. It follows that the lamplighter groups are not minimally almost convex. We describe an explicit family of examples which show concretely why the groups L_n are not minimally almost convex.

Finally, in section 6 we describe dead end behaviour in more general wreath products. The lamplighter groups are algebraically the wreath products n . The proof of Theorem 4.3 uses certain properties of this wreath product structure, as well as metric properties of the groups. We extend Theorem 4.3 to a larger class of wreath products in Theorem 6.1 and show the existence of 'seesaw' words in more general wreath products.

2. Background

2.1. *Wreath products*

A wreath product is a standard algebraic construction which is a special case of a semi-direct product. Given groups G and H, we form the wreath product $G \; H$ by taking the direct product of copies of G, one for each element of H. The generators of G act on the copy of G indexed by the identity, and the generators of H act on the indexing elements of H, with the effect of translating the copy of G indexed by the identity to the appropriate conjugate copy of G. The direct sum $_{h \in H} G$ is, a subgroup of $G \ \ H$ given by conjugate copies of G, and there is a natural projection homomorphism from $G \ \ H$ to H which merely deletes occurrences of the generators of G.

One of the simplest infinite wreath products is $_2$, referred to as the lamplighter group. This group has been studied for its remarkable spectral measure **[8, 11, 12]**, its strange random walks **[17, 15]** and its interest as an unusual amenable group **[14]**.

Wreath products often provide interesting examples of algebraic or geometric results. For example, Erschler **[9]** proved that is quasi-isometric to (S), where S is any finite group. This gave the first example of a finitely generated, although not finitely presented, solvable group quasi-isometric to a finitely generated group which is not virtually solvable.

2.2. *Convexity conditions*

We now define the convexity conditions which we use below.

Definition 2.1 A group G is *almost convex*(k), or AC(k), with respect to a finite generating set X if the following condition holds. There is a number $N(k)$ such that for all positive integers $n > N_0$, any two elements y and z in the ball $B(n)$ of radius n with $d_X(y, z) \leqslant k$ can be connected by a path of length at most $N(k)$ that lies entirely in $B(n)$.

Cannon **[3]** showed that the parameter k is unnecessary; if (G, X) is AC(2) then it is also AC(k) for all integral $k > 2$ Thus we may refer to (G, X) simply as almost convex, ignoring the parameter k. Furthermore, if the condition holds for all finite generating sets X of G, we say that G is almost convex.

Definition 2.2 A group G is *minimally almost convex*, or MAC, with generating set X if the following condition holds. Given any two elements x and y in the ball $B(n)$ of radius $n > N_0$ which are distance 2 apart, there is a path between them of length at most $2n - 1$ that remains inside $B(n)$.

Any two points x, $y \in B(n)$ can be connected by a path of length at most $2n$ that remains entirely inside $B(n)$ by connecting each point to the identity and joining those paths, so minimal almost convexity is the weakest of a family of weak convexity conditions described by Kapovich **[13]**.

3. Normal forms and geodesics in L_n and wreath products

3.1. *Lamplighter groups*

To understand normal forms of elements in L_n, we begin with the case of $L_2 =$ $_2$, and the standard geometric interpretation of elements of L_2. Consider a bi-infinite string of light bulbs, each of which has two states, 0 corresponding to 'off' and 1 corresponding to 'on', and a cursor which indicates the current bulb under consideration. A word in the lamplighter group L_2 is a sequence of movements of the cursor and commands to turn the bulb at the current location of the cursor on or off. The identity word is represented by all bulbs being off and the cursor at the origin. Using the presentation given in section 1, the generator t moves the cursor to the right, t^{-1}

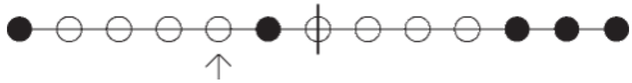

Fig. 1 A typical element of the lamplighter group L_2, $w = a_4 a_5 a_6 a_{-1} a_{-6} t^{-2}$ We use a solid circle to represent a bulb which has been turned on, and the vertical bar denotes the identity in .

moves the cursor to the left, and $a = a^{-1}$ changes the state of the bulb at the cursor's current location. Thus, a word $w \in L_2$ is represented by a configuration of bulbs and the final location of the cursor. This final cursor location is easily computed; if w is written in terms of the generators a and t in the presentation above, the final position of the cursor is just the exponent sum of t.

We begin our description of a normal form for elements of L_2 by defining $a_k = t^k a t^{-k}$ for conjugates of a. Beginning with the bi-infinite string of light bulbs which are all turned off, we see that a_n moves the cursor to the nth bulb, turns it on, and returns the cursor to the origin. It is clear that the a_n all commute. Repeated occurrences of a_n cancel in pairs, corresponding to turning on a bulb, and later turning it off.

In a product of the generators a and t, we can easily move all instances of t to the end of the word, changing the occurrences of a to the appropriate a_k to get a word of the form

$$a_{i_1} a_{i_2} \ldots a_{i_k} t^m,$$

where $i_k \in$ Since the a_{i_n} commute, we can order them as we choose. We consider two possible normal forms for a word $w \in L_2$, separating the word into parts which correspond to bulbs indexed by negative and non-negative integers:

$$rf(w) = a_{i_1} a_{i_2} \ldots a_{i_k} a_{-j_1} a_{-j_2} \ldots a_{-j_l} t^m$$

or

$$lf(w) = a_{-j_1} a_{-j_2} \ldots a_{j_l} a_{i_1} a_{i_2} \ldots a_{i_k} t^m$$

with $i_k > \cdots i_2 > i_1 \geqslant 0$ and $j_l > \cdots j_2 > j_1 > 0$ An example is shown in Fig. 1.

In the 'right-first' form, $rf(w)$, the cursor moves first to the right from the origin, lighting the appropriate bulbs as it moves toward the bulb with the greatest positive index. Then the cursor moves back to the origin and works leftwards, again lighting the appropriate bulbs in that direction. Finally, the cursor moves to its ending location from the leftmost illuminated lamp.

The 'left-first' form is similar, but instead of initially moving to the right, the cursor begins by moving toward the left. We mention two specific cases where there are no bulbs illuminated on one side of the origin, calling these 'right-only' or 'left-only' words.

One, or possibly both, of these normal forms will lead to minimal-length representation for w, depending upon the final location of the cursor relative to the extreme positive and negative illuminated bulbs.

We now relate the normal form of an element to its word length in the group, where word length is computed with respect to the generators $\{a, t\}$ from the presentation for L_2 given in Section 1.

DEFINITION 3.1 If a word $w \in L_2$ is in either normal form given above, we define

$$D(w) = k + l + \min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}$$

We note that, geometrically, $D(w)$ is the sum of several quantities related to the geometric picture of the element w: the number of bulbs which are on, twice the distance of the furthest bulb from the origin in one direction, the distance of the furthest bulb from the origin in the other direction and the distance of the final position of the cursor from the furthest illuminated bulb in the second direction.

We begin by proving that in L_2, the quantity $D(w)$ is exactly the word length of w with respect to the generating set $\{a, t\}$

PROPOSITION 3.2 *Let L_2 be generated by a and t, as given above. The word length of $w \in L_2$ with respect to the generating set $\{a, t\}$ is given by $D(w)$.*

Proposition 3.2 is proved via the following two lemmas.

LEMMA 3.3 *The length of a word $w \in L_2$ with respect to the generating set $\{a, t\}$ is at most D w).*

Proof. We put w into the right-first form listed above and note that by cancelling adjacent opposite powers of t, we obtain the expression below for w, namely

$$w = t^{-j_1}at^{-(j_2-j_1)}a \ldots t^{-(j_l-j_{l-1})}at^{j_l+i_1}at^{i_2-i_1}a \ldots t^{i_k-i_{k-1}}at^{m-i_k}$$

which has $k+1$ occurrences of a and $2j_l + i_k + |\mathrm{m} - i_k|$ occurrences of t. Similarly, after cancellation the left-first form reduces to

$$w = t^{i_1}at^{i_2-i_1}a \ldots t^{i_k-i_{k-1}}at^{-(i_k+j_1)}at^{-(j_2-j_1)}a \ldots t^{-(j_l-j_{l-1})}at^{m+j_l}$$

which has $k + l$ occurrences of a and $2i_k + j_l + |m + j_l|$ occurrences of t. These bound the length of w above by $D(w)$.

LEMMA 3.4 *The length of a word $w \in L_2$ with respect to the generating set $\{a, t\}$ is at least D w).*

Proof. To obtain a lower bound on $|w|$, we view w geometrically, as a collection of light bulbs which are turned on, and a cursor at a final position $m \in$. We must relate this picture to the minimal number of generators a and t needed to create it. If w has n light bulbs which are turned on, then a minimal length representative for w must contain at least n occurrences of the generator $a = a^{-1}$, each occurrence of which turns on one bulb. If w is written in either of the normal forms given above, then $n = k + l$

When counting the total occurrences of the generator t in a minimal representative for w, we first consider words with bulbs illuminated at both positive and negative indices. Recalling that the exponent sum of t gives the final position of the cursor, we consider the partial exponent sums on t for a minimal length representative of w. For instance, at the moment the rightmost bulb in position i_k is lit, the exponent sum of the generator t in that prefix must be i_k. Similarly, when the leftmost bulb in position $-j_l$ is lit, the exponent sum of t must be $-j_l$. Additionally, the total t exponent sum for the entire word must be m.

We consider whether the rightmost or leftmost illuminated bulb is lit first. If the rightmost bulb is illuminated first, then the representative for w has prefixes with exponent sums of 0, i_k, $-j_l$ and m on the t generator. The total number of t or t^{-1} letters needed to accomplish this is $i_k + i_k + j_l + |m + j_l|$. Similarly, if the leftmost bulb is illuminated before the rightmost one, the word must have prefixes with exponent sums 0, $-j_l$, i_k and then m. This requires at least $j_l + j_l + i_k + |m - i_k|$ total occurrences of t and t^{-1}. We note that for 'right-only' and 'left-only' words one of k or l will be zero and the same bounds apply.

Combining the bounds on numbers of a and t generators appearing in a minimal length representative for w, we see that the lower of these two bounds is exactly $D(w)$.

It follows immediately that $|w| = D(w)$.

For the lamplighter groups $L_n = \langle a, t | [t^i a t^{-i}, t^j a t^{-j}], a^n \rangle$ with $n > 2$, we note that there are similar normal forms and geodesics. The difference is that the occurrences of a in the normal form above are replaced by a^k, for $k \in \{-h, -h+1, \ldots, -1, 0, 1, 2, \ldots h\}$ where h is the integer part of $\frac{n}{2}$ When n is even, we omit a^{-h} to ensure uniqueness, since $a^h = a^{-h}$ in $_{2h}$

We replace the definition of $D(w)$ given above with the following more general definition, based on the left and right first normal forms for elements $w \in L_n$:

$$rf(w) = a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-f_2}^{f_2} \ldots a_{-j_l}^{f_l} t^m$$

or

$$lf(w) = a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} a_{i_1}^{e_i} a_{i_2}^{e_i} \ldots a_{i_k}^{e_k} t^m$$

with $i_k > \cdots i_2 > i_1 \quad 0$ and $j_l > \cdots j_2 > j_1 > 0$ and e_i, f_j lie in the range $\{-h, \ldots, h\}$, as does the exponent k discussed above. Using these normal forms we make the following definition.

DEFINITION 3.5 If a word $w \in L_n$ is in normal form as given above, we define

$$D(w) = \quad e_i + \quad f_j + \min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}$$

With this definition, $D(w)$ again computes the word length of $w \in L_n$.

PROPOSITION 3.6 *Let L_n be generated by a and t, as given above. The word length of $w \in L_n$ with respect to the generating set $\{a, t\}$ is exactly D w).*

The proof is analogous to that of Proposition 3.2, additionally taking into account the number of possible states for each light bulb.

We note that these normal forms by no means give unique minimal length representatives of elements of L_n. For example, a word in right-first normal form turns on the bulbs while moving right, then moves leftwards to the origin back across bulbs already traversed without changing the state of any bulbs, then continues moving leftwards, illuminating bulbs while travelling, and then travels to the final position. There are potentially many other minimal-length representatives which, for example, move first to the rightmost bulb in an 'on' state, then turn on bulbs while moving leftwards back to the origin and so on. Suppose that a bulb in position k is illuminated, and the cursor passes position k twice. Then there is always a choice whether to illuminate the bulb on the first visit or the second. We note that each bulb is visited at most twice, so it is clear that all minimal length representatives will be of one of these forms, up to the choices described above. For each minimal length form of an element of L_2 (right-first or left-first) there will be 2^p minimal length representatives where p is the number of 'on' bulbs at positions visited twice during the application of that normal form.

In the case of L_n for $n > 2$, there are even more possibilities, since a bulb which is visited twice may be moved partway to its final state during one visit to the bulb, and then changed from that state into its final state during the subsequent visit.

3.2. *Wreath products*

We note that in wreath products of finitely generated groups with , there is metric structure and normal forms similar to those described above.

We consider groups of the form G , in which we have a family of minimal length representatives for elements of G. Let X be a finite set of elements which generate G, and let be generated by t. We view an element of L_n as a collection of light bulbs in different positions turned on to a variety of states; algebraically, this is a collection of elements of the form a^k in $_n$ translated to different conjugate copies of $_n$ by different powers of the generator t of . In the wreath products we consider here, we view an element $w \in G$ again as a collection $\{v_m\}$ of elements of G, which we then conjugate into the different copies of G indexed by elements of , together with the final cursor position. More precisely, let $v_m \in G$ be a minimal length representative for v_m, and $u_m = t^{i_m} v_m t^{-i_m}$ the conjugate of v_{k_m} into the correct conjugate copy of G in the wreath product. Using this notation, we again consider right-first and left-first normal forms for words $w \in G$ Let

$$rf(w) = u_{i_1} u_{i_2} \ldots u_{i_k} u_{-j_1} u_{-j_2} \ldots u_{-j_l} t^m$$

and

$$lf(w) = u_{-j_1} u_{-j_2} \ldots u_{j_l} u_{i_1} u_{i_2} \ldots u_{i_k} t^m$$

with $i_k > \cdots i_2 > i_1 \quad 0$ and $j_l > \cdots j_2 > j_1 > 0$, and where, as above, u_m is the conjugate by t^m of a minimal length representative of an element of G. Again we decide which form is minimal by considering the final position of the cursor relative to the endpoints.

These normal forms differ from those considered in the language-theoretic context by Baumslag, Shapiro and Short **[1]**; here we choose the order of the u_k to ensure minimal length representatives in the wreath product.

We identify the group G with the copy of G in the wreath product indexed by the identity in and measure the lengths of elements in any conjugate copy of G by translating them to this copy. Thus we have a metric quantity analogous to $D(w)$ defined above, where we indicate word length in G with respect to the generating set X by $|\cdot|_G$.

DEFINITION 3.7 Let $w \in G$ be in either normal form given above, and let v_m, v_m, i_m and u_m be as previously defined. Namely, let v_m be a minimal length representative of $v_m \in G$, and u_m a conjugate of v_{m_i} by power of t^{i_m} Then we define

$$D(w) = \sum_{r=1}^{k} |v_{i_r}|_G + \sum_{s=1}^{l} |v_{-j_s}|_G + \min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}$$

The quantity $D(w)$ is merely the sum of all the lengths of the elements in the conjugate copies together with the total number of instances of t needed to move between those conjugate copies and leave the cursor where required.

The computation of the word metric in G is analogous to the case of the lamplighter groups.

PROPOSITION 3.8 *Let G be a finitely-generated group. The word length of $w \in G$ with respect to the generating set X $\{t\}$ is given by D w).*

This proposition is proved by two lemmas identical to Lemmas 3.3 and 3.4, which we omit here.

Again, these right and left normal forms are by no means unique; there are choices to make analogous to those for the normal forms in the lamplighter groups. Let $w \in G$ and consider those conjugate copies of G containing a non-trivial element u_m as part of the word w. The choice made in the formulation of a normal form is between the following three possibilities, as we imagine the cursor traversing the indexed conjugate copies of G.

- Represent u_m by a minimal length form from G entire during the first visit to the appropriate conjugate copy of G and do nothing the second visit.
- Do nothing the first visit and represent u_m by a minimal length form from G entirely during the second visit.
- Represent u_m by a prefix of a minimal length form from G during the first visit and the remaining part of that minimal length representative during the second visit.

We note that in more general wreath products, such as $_2$ ( ), the computation of the metric becomes much more complicated because the question of finding minimal length paths between a collection of points in the integer lattice is more difficult than in the integer line. In the plane, finding a minimal length path between a specified set of points is equivalent to the travelling salesman problem. Thus, though the word problem for $_2$ ( ) is straightforward, the question of determining minimal length representatives is NP-hard, as shown by Parry **[16]**.

4. Properties of the Cayley graphs of lamplighter groups

4.1. *Dead end words in L_n*

We begin by formally defining dead end words and their depth.

DEFINITION 4.1 A word w in a finitely generated group G is a dead end word with respect to a finite generating set X for G if $|w| = n$ and $|wx| \leqslant n$ for all generators x in X X^{-1}

Such words are called dead end words because a geodesic ray in the Cayley graph (G, X) from the origin to w cannot be extended beyond w. Note that in groups such as L_2, where all relators are of even length, $|wx|$ will be necessarily $n - 1$ for dead end words.

There are different forms of dead end behaviour, characterized by depth.

DEFINITION 4.2 A word w in a finitely generated group G is a *dead end word of depth k with respect to a finite generating set X if* k is the greatest integer with the following property. If the word length of w is n, then $|wx_1x_2\ldots x_l| \leqslant n$ for $1 \leqslant l \leqslant k$ and all choices of generators $x_i \in X$ X^{-1}

A dead end word is a place from which is impossible to make short-term progress away from the identity in the Cayley graph (G, X) The depth of a dead end word reflects how far it may be necessary to 'back up' before being able to move away from the identity. Dead end words of large depth show that the balls in the Cayley graph are significantly folded inward upon themselves. Note that in groups where all relators are of even length, there are no dead end words of depth one.

We show in [5] that there are dead end words of depth two in Thompson's group F with respect to the standard finite generating set, but none of depth greater than two. There is, however, no bound on depth of dead end words in the lamplighter groups L_n.

THEOREM 4.3 *The lamplighter groups* L_n *contain dead end words of arbitrary depth with respect to the generating set* $\{a, t\}$

Proof. We first prove that L_n contains dead end elements.

We consider a word d_m for $m \geqslant 1$ in which all of the bulbs at positions within m of the origin are turned to their setting furthest from off, and the cursor returns to the origin. For example, in L_2 the bulbs are merely turned on, and we have the words $d_m = a_0\, a_1\, a_2 \ldots a_m\, a_{-1} a_{-2} \ldots a_{-m} = (at)^m at^{-m}(t^{-1}a)^m t^m$; see Fig. 2 for an illustration of d_5. In L_4, for example, to put each bulb into the state farthest from off, we apply the generator a^2 to each bulb. If h is the integer part of $\frac{1}{2}n$, we see that in L_n, we want to consider words $d_m = a_0^h\, a_1^h\, a_2^h \ldots a_m^h\, a_{-1}^h a_{-2}^h \ldots a_{-m}^h = (a^h\, t)^m a^h t^{-m}(t^{-1} a^h)^m t^m$

It is easily seen that these words axe of length $D_m = 4m + h(2m+1)$ and that the right-first and left-first normal forms are both of minimal length.

In order for d_m to be a dead end element, each generator of L_n must decrease the word length of d_m. Writing d_m in the right-first normal form, we see that the generator t^{-1} reduces word length. Similarly, writing d_m in the left-first normal form, we see the generator t must also reduce the word length of d_m. Applying the generator a or a^{-1} turns off the bulb at the origin in the $n = 2$ case. For even $n > 2$, both a and a^{-1} will decrease the length of the word by making the bulb at the origin of lower intensity. For odd $n > 2$, one of a and a^{-1} will decrease the length and the other will keep the length the same. In all cases except odd n, the resulting words can then be expressed in either normal form using one less term; in the case of n odd, all of the generators will reduce length by one except one of either a or a^{-1} which will keep the length unchanged.

We now show that the elements d_m have depth m, providing examples of dead end words of arbitrary depth in L_n. Let w be a word in which all bulbs at distance greater than m from the origin are turned off, a collection of bulbs at distance at most m from the origin are turned on to some possible state, and the cursor lies somewhere between $-m$ and m. It is clear from $D(w)$ in L_n that the word length of w is at most the word length of d_m. It follows that an element which is in the ball $B(k)$ for $k > D_m = |d_m|$ must have a bulb at least distance $m+1$ from the origin turned on to some state. If b is any such element, the minimal length of a path from d_m to b is $m+1$, because in d_m the cursor is at the origin. Thus we have found dead end words of depth at least m for arbitrary m.

4.2. *Seesaw words in* L_n

We again consider the question of when certain generators decrease the word length of elements of L_n. Dead end words provide an example of a class of words for which all generators

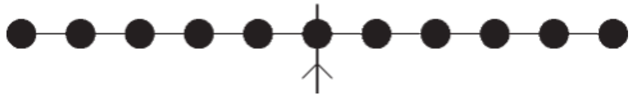

Fig. 2 The word d_5, which has the cursor at the origin, and is a dead end element in L_2. A solid circle represents a bulb that has been turned on.

$\{a^{\ 1}, t^{\ 1}\}$ decrease word length. We now describe another class of words with specific conditions on which generators decrease their word length. We call these words seesaw words, because there is a single generator and its inverse which reduce word length, and for which successive applications of that generator also reduce length.

DEFINITION 4.4 A word w in a finitely generated group G with $|w| = n$ *is a seesaw word of swing k with respect to a generator* g in a generating set X if the following conditions hold.

(1) The generator g and its inverse g^{-1} reduce the length of w and all other generators do not reduce the length of w; that is, $|wg^{\ 1}| = |w| - 1$, and for all other $h \in X$, we have $|wh^{\ 1}| \geqslant |w|$.
(2) Additionally, $|wg^{l}| = |wg^{l-1}| - 1$ for integral $l \in [1, k-1]$ and $|wg^{l-1}h^{\ 1}| \geqslant |wg^{l-1}|$ for all $h \quad g \in X$, and the analogous condition for wg^{-1} is also satisfied.

Thus $w \in G$ is a seesaw word of swing k if exactly two generators $g^{\ 1}$ of G reduce the word length of w and additionally, $g^{\ 1}$ is the only generator which reduces the word length of $wg^{\ 1}$ for $k-1$ further iterations. Seesaw words are of interest as they demonstrate one potential difficulty in choosing geodesic paths which satisfy the 'fellow traveller' property. Namely, the paths $g^{\ k}$ in the Cayley graph $\ (G, X)$ from w to $wg^{\ k}$ are part of minimal length representatives for w which remain far apart in this graph.

In a finite cyclic group $\ _{2k}$, the element k is a seesaw word of swing k with respect to the standard generating set $\{1\}$. We prove below that the lamplighter groups contain seesaw words of arbitrary swing in the generating set used above. The only other examples of seesaw words of this type known to the authors occur in Thompson's group F, and are described in [4].

THEOREM 4.5 *The lamplighter groups L_n contain seesaw words of arbitrary swing with respect to the generator t in the standard generating set* $\{a, t\}$

Proof. We consider the words $w_n = a_n^{e_1} a_{-n}^{e_2}$, for $e^{i} \in \{-h, -h+1, \ldots, h-1, h\}$, where h is the integer part of $\frac{1}{2}n$. These words have length $4n+2$ The word w_5 is shown in Fig. 3. For a given n, this word consists of turning on two bulbs to any non-trivial state, a single bulb in each direction at distance n from the origin and then returning the cursor to origin. Right multiplication by both t and t^{-1} moves the cursor one away from the origin, in each case shortening the length of w. The words $w_n t^k$ are reduced in length only by further applications of t until the lit bulb is reached and a^{-1} reduces the length. Similarly, the words $w_n t^{-k}$ are reduced in length only by further applications of t^{-1} for $k = 1, \ldots, n-1$, so we have seesaw words of swing k, as desired.

Since the exponent sum of t in the relators is zero, words with t-exponent sum differing by m are at least distance m apart. So though these two paths from a seesaw word of swing k are both headed toward the identity, they immediately separate as fast as possible to distance $2k$.

5. Convexity properties of some wreath products

5.1. *Lamplighter groups are not almost convex*

Cannon [3] showed that almost convex groups are finitely presented, so non-trivial wreath products cannot satisfy the almost convexity condition AC(2) with respect to any generating set. Kapovich [13] showed that groups satisfying the minimal almost convexity condition are finitely presentable, so it is clear again that non-trivial wreath products cannot satisfy the minimal almost convexity condition MAC either. For certain wreath products, we can illustrate the failures of these convexity conditions directly using a natural family of words.

In Example 5.1, we construct a a family of pairs of words in increasingly large balls in L_2 which lie distance 2 apart. These words have the property that any path connecting them which stays inside the ball B_n has length depending on n, contradicting the definition of almost convexity.

We show directly by example that L_2 is not minimally almost convex in the generating set $\{a, t\}$, and note that the example extends immediately to L_n. We use the relation $a^2 = 1$ to refer to a bulb as either on or off, but the proof is not affected by the bulbs having additional 'states' arising from the relation $a^n = 1$

We begin by considering the seesaw words $w_n = a_n\ a_{-n}$, which were described in section 4.2. These words light bulbs at distance n from the origin and return the cursor to the origin, as shown in Fig. 3. Right multiplication by both t and t^{-1} moves the cursor one away from the origin, in each case shortening the length of w. The pairs of words, we consider below are always of the form $w_n t$ and $w_n t^{-1}$, each of which has word length $4n + 1$ in the generating set $\{a, t\}$ of L_2.

Example 5.1 The lamplighter groups L_n do not satisfy Cannon's almost convexity condition AC(2) with respect to the generating set $\{a, t\}$

We let γ be a path in the Cayley graph $= (L_2, \{a, t\})$ from $w_n t$ to $w_n t^{-1}$ which stays entirely inside $B(4n + 1)$, where w_n is defined above. Since the cursor is to the right of the origin in w_n and to the left of the origin in $w^n t^{-1}$, there must be at least one prefix of γ in which the cursor is at the origin. Call the first prefix where this occurs α.

Let B_+ and B_- denote the two light bulbs which are turned on in both of the words $w_n t$ and $w_n t^{-1}$, with B_+ at position n and B_- at position $-n$. The following lemma characterizes the length of words in L_n with these two bulbs turned on.

LEMMA 5.2 *Let $v \in L_2$ be a word with B_+ and B_- turned on, and any additional but possibly empty collection of light bulbs turned on, and the cursor at the origin. Then $|v| \geqslant 4n + 2$*

Proof. Given such a word v, we can write v in either one of the two normal forms defined above: the right-first normal form $a_{-j_1} a_{-j_2} \ldots a_{j_l} a_{i_1} a_{i_2} \ldots a_{i_k} t^m$ or the left-first normal form

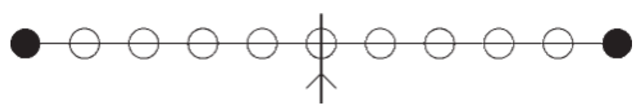

Fig. 3 The seesaw word $w_5 \in L_n$, which has the cursor at the origin. A solid circle represents a bulb which has been turned on to any state.

$a_{i_1}a_{i_2}\ldots a_{i_k}a_{-j_1}a_{-j_2}\ldots a_{-j_l}t^m$ In either case, $D(v) \geqslant 2 + 2(i_k + j_l) = 4n + 2 = |w|$, since $i_k \geqslant n$ and $j_k \geqslant n$ and k and l are at least 1 and $m = m = 0$

To finish this example, we note that since the path γ is contained in $B(4n+1)$, at the point α we must have one of B_- or B_+ turned off, otherwise by Lemma 5.2, we have $|w\alpha| \geqslant 4n+2$ The minimum length of a path turning off one of these bulbs is $n+1$, depending upon our original choice for n. Thus we see that L_2 is not almost convex in the generating set $\{a, t\}$

For lamplighter groups L_k with $k > 2$, the situation is identical, we need only that the bulbs B_+ and B_- are in non-trivial states.

5.2. *Lamplighter groups are not minimally almost convex*

We can use these same examples to show that L_n is not minimally almost convex by analysing the connecting paths more carefully. Again, since the groups L_n are not finitely presentable, they are not minimally almost convex by the work of Kapovich **[13]**; here, we illustrate that fact concretely. For finitely presented groups, there are not many examples known of groups which are not minimally almost convex. Belk and Bux **[2]** showed that Thompson's group F is not minimally almost convex, after we showed **[6]** that F is not almost convex. Few examples are known of groups which are minimally almost convex but not almost convex; Elder and Hermiller **[10]** show that the solvable Baumslag–Solitar group BS(1,2) has this property but some Baumslag–Solitar groups BS(1,n) with $n > 2$ are not minimally almost convex.

Example 5.3 The lamplighter groups L_n are not minimally almost convex with respect to the generating set $\{a, t\}$

Example 5.3 is a continuation of Example 5.1. As in Example 5.1, we consider the two group elements w_nt and w_nt^{-1} which lie in $B(4n+1)$ Let γ be a path from w_nt to w_nt^{-1} which is entirely contained in $B(4n+1)$ From Lemma 5.2 we know that at any points along this path which have the cursor at the origin, at least one of B_+ and B_- must be turned off.

At the first point along γ with the cursor at the origin, we must have B_+ turned off, otherwise the cursor would be at the origin with both bulbs illuminated. Since the path goes from w_nt to w_nt^{-1}, we cannot first have B_- turned off, for that would lead to a point where the cursor is at the origin and both bulbs are turned on. Thus the length of γ is at least $(n-1)+1 = n$, since that is the minimum number of generators required to move the cursor to position n and turn off the light bulb.

Since in w_nt^{-1} the bulb B_+ is turned on, somewhere along the path γ the bulb B_+ must be turned back on. Before this can happen, B_- must be turned off, otherwise the cursor would pass through the origin with both bulbs illuminated. Thus the path γ must contain, at least an additional $2n+1$ generators, those necessary to move the cursor to the position $-n$ and turn off the bulb.

We note that B_+ must be turned on again before B_- can be turned on again, since the path γ goes from w_nt to w_nt^{-1}, and in w_nt^{-1} the cursor is to the left of the origin. Thus γ must contain at least an additional $2n+1$ letters which move the cursor to the position n and turn on the bulb B_+, and then an additional $2n+1$ letters which move the cursor back to position $-n$ and turn on B_-. This brings the length of γ to at least $7n+3$, and we see that to finish the path there must be at least another t^{n-1}, although not contiguously, returning the cursor to the position^{-1}

from the position $-n$. Thus the length of γ is at least $8n+2$, which is the sum of the lengths of the original two words, illustrating that L_2 is not minimally almost convex.

6. More general wreath products

Some more general wreath products behave similarly to the lamplighter groups described above. In the case of dead end elements, we obtain the following theorem.

THEOREM 6.1 *Let G be any finitely generated non-trivial group which contains dead end elements of depth d with respect to a generating set X, and let be generated by t. Then* G *contains dead ends of depth d with respect to the generating set X {t}.*

Proof. Let a be a dead end element in G of depth d. Define a_n to be the conjugate $a_n = t^n a t^{-\mathrm{n}}$, similar to its definition for the lamplighter groups above in Theorem 4.3. The words $d_n = a_0 a_1 a_2 \ldots a_m a_{-1} a_{-2} \ldots a_{-m}$ will be dead end words of depth d with respect to the generating set of the wreath product according to an argument identical to that of Theorem 4.3. Namely, right multiplication by all the generators of G will not increase the word length since the cursor is at the origin in d_m, and a is a dead end word in G; right multiplication by t and t^{-1} will reduce the length of d_n as well.

We note that Example 5.1 does not rely on the relation $a^n = 1$ involving the generator a of L_n. It is merely necessary to have some generator which only affects the second factor of the wreath product. Thus, if we can mimic the roles of the generators a and t in a more general wreath product, the examples are parallel.

Example 6.2 Let F be a finitely generated group containing an isometrically embedded copy of , and G any finitely generated non-trivial group. Then G F is neither almost convex nor minimally almost convex in at least one generating set.

Again, these groups are not finitely presentable and thus not almost convex by Cannon [3] and not minimally almost convex by the work of Kapovich [13]. We illustrate this concretely in the following example. Let a be any generator of G, and t the generator of the isometrically embedded copy of inside, F. Then $\{a, t\}$ can be completed to a generating set of G F Again we define a_n to be $t^n a t^{-n}$, and consider the words $w_n = a_n a_{-n}$. The words w_n give rise to pairs of words $w_n t$ 1 of length $4n+1$ which are connected by a path of length 2, as in the case of the lamplighter groups described in Examples 5.1 and 5.3. The natural analog of Lemma 5.2 holds; we only use the fact that each word contains a non-trivial element in the conjugate subgroups $t^n G t^{-n}$ and $t^{-n} G t^n$. Thus we again have concrete examples illustrating the failure of minimal almost convexity for these groups.

More general wreath products also exhibit the seesaw word phenomenon described above for the lamplighter groups.

THEOREM 6.3 *Let F be a finitely generated group containing an isometrically embedded copy of* $\cong \langle t \rangle$ *with respect to a generating set Y, and G any finitely generated non-trivial group with generating set X. Then* G F *contains seesaw words of arbitrary swing with respect to at least one generating set.*

Proof. Let $a \in X$ be any generator of G, and t the generator of the isometrically embedded copy of inside F. Then $\{a, t\}$ can be completed to a generating set of G F Again we define

a_n to be $t^n a t^{-n}$, and consider the words of the form $w_n = a_n a_{-n}$ In these words, the cursor remains pointed at the conjugate copy of G indexed by the identity. As above, right multiplication by any generator of G will increase the length of w_n. Right multiplication by $t^{\ 1}$ will decrease the length of w_n just as it does in the lamplighter groups considered in Theorem 4.5. Thus these words are seesaw words in $G\ \ F$ The proof that they have arbitrary swing is identical to the proof that seesaw words in the lamplighter groups have this property.

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