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Tame combing and almost convexity conditions

**Sean Cleary   Susan Hermiller   Melanie Stein
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Abstract   We give the first examples of groups which admit a tame combing with linear radial tameness function with respect to any choice of finite presentation, but which are not minimally almost convex on a standard generating set. Namely, we explicitly construct such combings for Thompson's group F and the Baumslag–Solitar groups $BS(1, p)$ with $p \geq 3$. In order to make this construction for Thompson's group F, we significantly expand the understanding of the Cayley complex of this group with respect to the standard finite presentation. In particular we describe a quasigeodesic set of normal forms and combinatorially classify the arrangements of 2-cells adjacent to edges that do not lie on normal form paths.

1 Introduction

This paper has two goals: to study the relationships between the hierarchies of convexity conditions and tame combing conditions on a Cayley complex corresponding to a given group, and to significantly expand the understanding of the Cayley complex of Thompson's

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group F with respect to the standard finite presentation with two generators and two relators.

Several notions of almost convexity for groups have been developed in geometric group theory, from the most restrictive property defined by Cannon [4] to the weakest notion of minimal almost convexity introduced by Kapovich [12]. For a group G with finite generating set A, almost convexity conditions for different classes of functions measure, in terms of the given function, how close balls in the Cayley graph for (G, A) are to being convex sets (see Sect. 2.1 for the formal definition). Results of Thiel [18] and Elder and Hermiller [6], respectively, show that Cannon's almost convexity and minimal almost convexity, respectively, are not quasi-isometry invariants.

Mihalik and Tschantz [13] introduced the notion of a tame 1-combing of a 2-complex, and in particular of the Cayley complex of a group presentation, in the context of studying properties of 3-manifolds. Hermiller and Meier [10] refined the definition of tame combing to differentiate between types of tameness functions, analogous to almost convexity conditions. For a group G with finite presentation $\mathcal{P}$, intuitively the radial tameness function measures the relationship, for any loop, between the size of the ball in the Cayley complex containing the loop and the size of the ball needed to contain a disk filling in that loop (see Sect. 2.2 for the formal definition). Hermiller and Meier [10] showed that the advantage of studying balls in a Cayley complex from the viewpoint of tame combings and radial tameness functions is that the classes of tame combable groups are, up to Lipschitz equivalence of radial tameness functions (for example, linear functions or exponential functions), invariant under quasi-isometry, and hence under change of presentation. In the same paper they also showed that groups which are almost convex with respect to several classes of functions are contained in the quasi-isometry invariant class of groups admitting a 1-combing with a linear radial tameness function. In Sect. 2.3, we give a more complete discussion of the hierarchies of almost convexity conditions and tame combing functions, and their interconnections.

In seeking to further understand the correspondence between these two hierarchies, we use geometric information from the Cayley complex to construct tame 1-combings with linear tameness functions for two groups: Thompson's group F and the solvable Baumslag–Solitar group $BS(1, p)$ for $p \geq 3$. Belk and Bux [1] showed that Thompson's group F is not minimally almost convex with respect to the standard finite generating set. Elder and Hermiller [6] showed that the groups $BS(1, p)$ for $p \geq 7$ are also not minimally almost convex with respect to their usual generating set; moreover, Miller and Shapiro [14] showed shown that the group $BS(1, p)$ does not satisfy Cannon's almost convexity condition for any generating set. Combining these then provides the first examples of groups which admit a combing with a linear radial tameness function (with respect to any choice of finite presentation), but which are not minimally almost convex on a particular finite generating set. These also provide the first examples of groups which admit a combing with a linear radial tameness function but which do not satisfy Cannon's almost convexity condition with respect to every finite presentation. In the case of Thompson's group F, our proof also gives significant new insight into the Cayley complex of this group.

Despite the prevalence of F in geometric group theory, a detailed understanding of the Cayley complex for the standard finite presentation

$$\langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$$

has been elusive. In an intricate analysis, Guba [9] showed that Thompson's group F has a quadratic isoperimetric function, but it is as yet unknown if Thompson's group is automatic, nor even if it is asynchronously combable. The tame 1-combing we construct for F utilizes the nested traversal paths defined by Cleary and Taback in [5]. We show that these paths

yield a set of quasigeodesic normal forms for the group. Extending these paths to a tame 1-combing of the Cayley 2-complex X for the presentation above then requires a careful, detailed classification of the edges and 2-cells of X. In particular we analyze which edges do not lie on these normal form paths and for each such edge, we characterize which 2-cells of X adjacent to that edge have the property that their other boundary edges lie "closer" to the identity vertex ϵ. Our analysis and measure of closeness to the identity use combinatorial properties computed from the group elements labeling the vertices adjacent to that edge.

The paper is organized as follows. In Sect. 2, we provide an overview of the development of the notions of convexity and combings for groups, and the relations between them. In Sect. 3, we provide a brief introduction to Thompson's group F, and define the set of normal forms which will be used in the construction of the tame 1-combing. We also show that this set of normal forms satisfies a quasigeodesic property. In Sect. 4, we construct the 1-combing of F, and in Sect. 5 we show that this combing satisfies a linear radial tameness function, as stated in Theorem 5.4. In Sect. 6 we show that $G = BS(1, p)$ with $p \geq 3$ has a 1-combing which satisfies a linear radial tameness function, proving Theorem 6.1. Finally, Sect. 7 is devoted to the proof of Theorem 7.1, verifying that although $G = BS(1, p)$ with $p \geq 8$ and the standard presentation admits a 1-combing with a linear radial tameness function, this tameness function must have a multiplicative constant greater than 1.

2 Convexity and combings for groups

2.1 Almost convexity conditions on Cayley graphs

For a group G with a finite inverse-closed generating set A, we let $\Gamma(G, A)$ denote the Cayley graph of G with respect to A, and let d_A denote the word metric with respect to this generating set. The pair (G, A) satisfies the almost convexity condition AC_f for a function $f : \quad \rightarrow \mathbb{R}_+$ if there is an $r_0 \in \quad$ such that for every two points a, b in the sphere $S(r)$ (centered at the identity) with $d_A(a, b) \leq 2$ and every natural number $r > r_0$, there is a path inside the ball $B(r)$ from a to b of length no more than $f(r)$.

Every group satisfies the almost convexity condition AC_f for the function $f(r) = 2r$, as two points in the ball of radius r can always be connected by a path of length $2r$ which remains inside $B(r)$, simply by going to the identity and returning outward. Thus the weakest nontrivial almost convexity condition for a pair (G, A) is AC_f for the function $f(r) = 2r - 1$. Kapovich [12] and Riley [17] have shown that this minimal almost convexity condition (MAC) implies finite presentation of the group and the existence of an algorithm for constructing the Cayley graph.

At the other end of the spectrum, (G, A) is almost convex (AC) in the sense of Cannon [4] if it satisfies AC_f for a constant function f. Between the constant function and $f(r) = 2r - 1$, there are a number of other possible functions which give rise to a range of almost convexity conditions. For example, Poénaru [15,16] studied the property AC_f for sublinear functions f.

2.2 Tame combings of Cayley complexes

Let $G = \langle A \mid R \rangle$ be a finitely presented group, with A an inverse-closed generating set. Let X denote the Cayley complex corresponding to this presentation, with 0- and 1-skeletons $X^0 = G$ and $X^1 = \Gamma(G, A)$; that is, X^1 is the Cayley graph with respect to this presentation.

In order to have a notion of a ball centered at the identity ϵ in the 2-complex X, the notion of distance between the vertices of a Cayley graph is extended to a notion of level on the entire complex. The following definition is equivalent to that in [10].

Definition 2.1 (1) If g is a vertex in X^0, the *level* lev(g) is defined to be the word length $l_A(g)$ with respect to the generating set A.

(2) If $x \in X^1 - X^0$, then x is in the interior of some edge with vertices $g, h \in X^0$. Then let

$$\text{lev}(x) := \frac{\text{lev}(g) + \text{lev}(h)}{2} + \frac{1}{4}$$

(3) If $x \in X - X^1$, then x is in the interior of some 2-cell with vertices $g_1, g_2, \ldots, g_n$ along the boundary, and

$$\text{lev}(x) := \frac{\text{lev}(g_1) + \text{lev}(g_2) + \quad + \text{lev}(g_n)}{n} + \frac{1}{4} + \frac{1}{c}$$

where if $R = \{r_1, r_2, \ldots, r_k\}$ is the set of relators, and for each $1 \leq i \leq k, n_i$ is the number of letters in the relator r_i, then $c := 4n_1 n_2 \quad n_k + 1$.

Intuitively, a 0-combing of a group G with generating set A is a choice of path in the Cayley graph $\Gamma(G, A)$ from the identity ϵ to each group element. To obtain a 1-combing for $(G, \langle A \mid R \rangle)$, a 0-combing is extended continuously through the 1-skeleton of the Cayley complex. Viewing the ball of radius q in X as the set of points of level at most q, a radial tameness function $\rho : \mathbb{Q} \to \mathbb{R}_+$ for a 1-combing ensures that once a combing path leaves the ball of radius $\rho(q)$, it never returns to the ball of radius q.

Definition 2.2 The pair $(G, \langle A \mid R \rangle)$ satisfies the tame combing condition TC_ρ for a function $\rho : \mathbb{Q} \to \mathbb{R}_+$ if there is a continuous function $\Psi : X^1 \times [0, 1] \to X$ satisfying:

(1) For all $x \in X^1$, $\Psi(x, 0) = \epsilon$ and $\Psi(x, 1) = x$,
(2) $\Psi(X^0 \times [0, 1]) \subseteq X^1$, and
(3) For all $x \in X^1, 0 \leq s < t \leq 1$, and $q \in \mathbb{Q}$, if lev$(\Psi(x, s)) > \rho(q)$, then lev$(\Psi(x, t)) > q$.

The function Ψ is a 1-*combing* of X, and ρ is a *radial tameness function* for Ψ.

A continuous function $\Psi : X^0 \times [0, 1] \to X^1$ with $\Psi(x, 0) = \epsilon$ and $\Psi(x, 1) = x$ for all $x \in X^0$ is called a 0-*combing* for the pair (G, A). The restriction of a 1-combing to the vertices of X is a 0-combing.

In [10], Hermiller and Meier show that the condition TC_ρ is quasi-isometry invariant, and thus is independent of the choice of presentation for the group, up to Lipschitz equivalence on the radial tameness functions. Hence it makes sense to define the class of groups admitting a tame 1-combing with a linear radial tameness function, and also classes with polynomial and exponential radial tameness functions.

2.3 Hierarchies of convexity and combing functions

A pair $(G, \langle A \mid R \rangle)$ may satisfy a variety of almost convexity and tame combing conditions. In Fig. 1, we illustrate what is known about the relevant relationships between the different classes of almost convexity functions, and the different classes of possible radial tameness functions which arise from tame 1-combings. In particular, all descending vertical arrows in Fig. 1 follow immediately from the definitions.
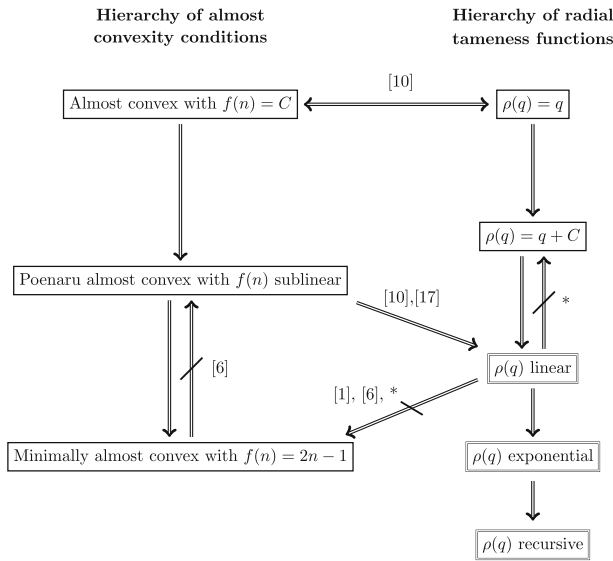
Fig. 1 The relationships between the hierarchies of convexity conditions and degrees of radial tameness functions for a pair $(G, \langle A \mid R \rangle)$. A slash across an arrow indicates that it is known that there exists a counterexample to the implication in that direction. Numbers in brackets are bibliographic references; the two instances marked by a * are established in this paper. The tameness properties contained in double boxes are independent of the choice of the finite presentation for G

These two chains of conditions are tied together at the base by the results of Hermiller and Meier [10] which show that a pair (G, A) is almost convex if and only if there is a set of defining relations R such that the pair $(G, \langle A \mid R \rangle)$ admits a 1-combing satisfying the radial tameness function $\rho(q) = q$.

In Theorem D of [10], Hermiller and Meier showed that the property AC_f with f sublinear, together with a linear isodiametric function (a combination of properties motivated by work of Poénaru in [15]), imply the existence of a 1-combing with a linear radial tameness function. For a pair (G, A) satisfying AC_f for any function $f :\ \to \mathbb{R}_+$ such that $f(n) \leq n - 2$ for all n, it follows from Riley [17, Equation 3.2] and induction that for all $n \geq 2r_0 + 2$, we have $Diam(n) \leq n + D$, where $D = Diam(2r_0 + 2)$ is a constant. Hence the property AC_f with f sublinear implies a linear isodiametric function, and so this extra assumption was redundant. As a consequence, it follows that AC_f with f sublinear implies the condition TC_ρ with ρ linear.

Tantalizing questions to consider, given these results, involve the potential connections between weaker notions of almost convexity and radial tameness functions. As yet, there are few examples known, other than for groups satisfying the condition AC_f with f sublinear, of groups with 1-combings admitting restricted tameness functions.

In this paper the results of Theorems 5.4 and 6.1 show that the quasi-isometry independent class TC_{linear} of groups with a 1-combing satisfying a linear radial tameness function contains groups which are not even minimally almost convex for some particular generating set, giving the diagonal non-implication in Fig. 1. However, this leaves open the question of whether every group in TC_{linear}, and in particular whether F and $BS(1, p)$ with $p \geq 3$, might have some generating set with respect to which it is minimally almost convex.

Other intriguing questions involve the possibility of upward implications in either of the two hierarchies. In Theorem 7.1 we show the vertical non-implication for tameness functions drawn in Fig. 1. For almost convexity, Elder and Hermiller [6] have exhibited a pair (G, A) which is minimally almost convex but does not satisfy the condition AC_f with f sublinear. It is still an open question whether there can be a pair (G, A) satisfying the Poénaru AC_f condition with f sublinear that does not also satisfy Cannon's AC property.

3 An introduction to Thompson's group F

We present a brief introduction to Thompson's group F and refer the reader to [3] for a more detailed discussion, with historical background. In addition, in Sect. 3.1 we define a the set of normal forms for F which will be used in our construction of a 1-combing.

Thompson's group F has a standard infinite presentation:

$$\langle x_k, \ k \geq 0 | x_i^{-1} x_j x_i = x_{j+1} \text{ if } i < j \rangle.$$

The elements x_0 and x_1 are sufficient to generate the entire group, since powers of x_0 conjugate x_1 to x_i for $i \geq 2$. Only two relators are required for a presentation with the generating set $A := \{x_0, x_1\}$, resulting in the finite presentation for F:

$$\langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

This is the most commonly used finite generating set and presentation for Thompson's group F, and in this paper we will build the 1-combing for F using the Cayley complex for this presentation.

With respect to the infinite presentation given above, each element $w \in F$ can be written in normal form as

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k} x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

with $r_i, s_i > 0, 0 \leq i_1 < i_2 \quad < i_k$ and $0 \leq j_1 < j_2 \quad < j_l$. Furthermore, we require that when both x_i and x_i^{-1} occur, so does x_{i+1} or x_{i+1}^{-1}, as discussed by Brown and Geoghegan [2]. We will use the term *infinite normal form* to mean this normal form, and write $w = w_p w_n$ where w_p is the maximal subword of this normal form with positive exponents, and w_n is the maximal subword with negative exponents.

Elements of F can be viewed combinatorially as pairs of finite binary rooted trees, each with the same number of edges and vertices, called *tree pair diagrams*. Let T be a finite rooted binary tree. We define a *caret* of T to be a vertex of the tree together with two downward oriented edges, which we refer to as the left and right edges of the caret. The *right (respectively left) child* of a caret c is defined to be a caret which is attached to the right (resp. left) edge of c. If a caret c does not have a right (resp. left) child, we call the right (resp. left) leaf of c *exposed*. The caret itself is *exposed* if both of its leaves are also leaves of the tree; that is, the caret has no children.

For a given tree T, let $N(T)$ denote the number of carets in T. We number the carets from 1 through $N(T)$ in infix order. The infix ordering is carried out by numbering the left descendants (the left child and all descendants of the left child) of a caret c before numbering c, and the right descendants of c afterward. We use the infix numbers as names for the carets, and the statement $p < q$ for two carets p and q simply expresses the relationship between their infix numbers. In a tree pair diagram (T, S), we refer to the pair of carets with infix number p, one in each tree, as the *caret pair p*.

The left (resp. right) side of a binary rooted tree T consists of the left (resp. right) edge of the root caret, together with the left (resp. right) side of the subtree consisting of all left (resp. right) descendants of the root caret. A caret in a tree T is said to be a *right* (resp. *left*) caret if one of its edges lies on the right (resp. left) side of T. The root caret can be considered either left or right. All other carets are called *interior* carets. We also number the leaves of the tree T from left to right, from 0 through $N(T)$.

An element $w \in F$ is represented by an equivalence class of tree pair diagrams, among which there is a unique reduced tree pair diagram. We say that a pair of trees is *unreduced* if, when the leaves are numbered from 0 through $N(T)$, there is a caret in both trees with two exposed leaves bearing the same leaf numbers. If so, we remove that pair of carets, and renumber the carets in both trees. Repeating this process until there are no such pairs produces the unique *reduced* tree pair diagram representing w.

The equivalence of these two interpretations of Thompson's group F is given using the infinite normal form for elements with respect to the standard infinite presentation, and the concept of leaf exponent. In a single tree T whose leaves are numbered from left to right beginning with 0, the *leaf exponent* $E_T(k)$ of leaf number k is defined to be the integral length of the longest path of left edges from leaf k which does not reach the right side of the tree.

Given the reduced tree pair diagram (T, S) representing $w \in F$, compute the leaf exponents $E_T(k)$ for all leaves k in T, numbered 0 through $n = N(T) = N(S)$. The negative part of the infinite normal form for w is then $x_n^{-E_T(n)} x_{n-1}^{-E_T(n-1)} \quad x_1^{-E_T(1)} x_0^{-E_T(0)}$. We compute the exponents $E_S(k)$ for the leaves of the tree S and thus obtain the positive part of the infinite normal form as $x_0^{E_S(0)} x_1^{E_S(1)} \quad x_n^{E_S(n)}$. Many of these exponents will be 0, and after deleting these, we can index the remaining terms to correspond to the infinite normal form given above, following [3] (see Fig. 2). As a result of this process, we often denote the unique reduced tree pair diagram for w by $w = (T_-(w), T_+(w))$, since the first tree in the pair determines the terms in the infinite normal form with negative exponents, and the second tree determines those terms with positive exponents. We refer to $T_-(w)$ as the negative tree in the pair, and $T_+(w)$ as the positive tree.

Group multiplication is defined as follows when multiplying two elements represented by tree pair diagrams. Let $w = (T_-, T_+)$ and $z = (S_-, S_+)$. To form the product wz, we take unreduced representatives of both elements, (T'_-, T'_+) and (S'_-, S'_+), respectively, in which $S'_+ = T'_-$. The product is then represented by the (possibly unreduced) pair of trees (S'_-, T'_+). If the fewest possible carets are added to the tree pairs for g and h in order to make $S'_+ = T'_-$, and yet the pair (S'_-, T'_+) is unreduced, we say that a caret must be *removed* to reduce the tree pair diagram for wz.

Given any $w = (T_-(w), T_+(w))$ in F, let $N(w) := N(T_-(w)) = N(T_+(w))$ denote the number of carets in either tree of a reduced tree pair diagram representing w. For any natural number k, let R_k (respectively L_k) denote the tree with k right (respectively left) carets, and no other carets; if $k = 0$, R_0 (or L_0) denotes the empty tree. For $w = w_p w_n$, where as above w_p and w_n are the positive and negative subwords of the infinite normal form, the tree pair diagram $(R_{N(w)}, T_+(w))$ represents $w_p(w)$, and $(T_-(w), R_{N(w)})$ represents w_n. However, one of these tree pair diagrams may not be reduced. If the last k carets of $T_-(w)$ (respectively

Fig. 2 The reduced tree pair diagrams representing (respectively) x_0^{-1} and x_1^{-1}

$T_+(w)$) are all right carets, then at least $k - 1$ of them must be removed in order to produce the reduced tree pair diagram for w_n (respectively w_p). The inverse of w is represented by the reduced tree pair diagram $w^{-1} = (T_+(w), T_-(w))$.

For a word $y \in A^* = \{x_0^{\pm 1}, x_1^{\pm 1}\}^*$, let $l(y)$ denote the number of letters in the word y, and for an element $w \in F$, let $l_A(w)$ be the length of the shortest word over the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$ that represents w. Following the notation of Horak et al. [11], the length $l_A(w)$ can be described in terms of the reduced tree pair diagram $(T_-(w), T_+(w))$ for w, with carets numbered in infix order. First, we say that caret number p in a tree T has *type N* if caret $p + 1$ is an interior caret which lies in the right subtree of p.

Definition 3.1 Caret pair p in the reduced tree pair diagram $(T_-(w), T_+(w))$ is a *penalty caret pair* if either

(1)   Caret p has type N in either $T_-(w)$ or $T_+(w)$, and is not a left caret in either tree, or
(2)   Caret p is a right caret in both $T_-(w)$ and $T_+(w)$ and caret p is neither the final caret in the tree pair diagram, nor a left caret in either tree.

Using this notation, the following proposition is proved in [11].

Proposition 3.2 *For $w = (T_-(w), T_+(w))$, the length $l_A(w) = l_\infty(w) + 2p(w)$, where $l_\infty(w)$ is the total number of carets in both trees of the reduced tree pair diagram which are not right carets, and $p(w)$ is the number of penalty caret pairs.*

It then follows that $N(w)$ is a good estimate for the $l_A(w)$. Lemma 3.3 makes this relationship precise and is used in the proof that the tameness function of the combing we construct below is linear.

Lemma 3.3 *For $w \in F$, $N(w) - 2 \le l_A(w) \le 4N(w)$.*

Proof Proposition 3.2 shows that each caret pair in the reduced tree pair diagram for w contributes 0, 1, 2, 3 or 4 to $l_A(w)$, and the upper bound on $l_A(w)$ follows immediately. The caret pair can contribute either 0, 1, or 2 to $l_\infty(w)$, and can contribute another 2 if it is a penalty pair. In order for a caret pair to contribute zero to the word length of the element, both carets must be on the right side of the tree in order to not contribute to $l_\infty(w)$, and either one is the root (in which case the pair is not a penalty pair because the root is also a left caret), or the pair is the last caret pair. So at most two caret pairs do not contribute anything to $l_A(w)$, which yields the lower bound on $l_A(w)$.   □

Finally, we include here a lemma which will be used in Sect. 4 and describes a family of words in F which are always nontrivial.

Lemma 3.4 *Let $w \in F$, and suppose $w = a_1 a_2 \quad a_k$ where for each i, either a_i or a_i^{-1} is in $X_\infty = \{x_0, x_1, x_2, \ldots\}$. In addition, suppose that for each i, if $a_i = x_r^{\pm 1}$, then $a_{i+1} = x_{r+1}^{\pm 1}$ or $x_{r-1}^{\pm 1}$. Then $w \neq 1$ in F.*

Proof We prove the lemma by induction on k. The base case $k = 1$ is trivial. Suppose $w = a_1 a_2 \quad a_k$, and the indices of the generators satisfy the hypothesis of the lemma, and let i be the smallest index appearing in w. We will show that if $w = 1$, then we can obtain a shorter word satisfying the conditions on indices which is also 1 in F, contradicting the inductive hypothesis.

Utilizing the representation of elements of F as piecewise linear homeomorphisms of the unit interval (see [3] for details), $x_i^{\pm 1}$ has a breakpoint at $1 - (1/2)^i$, and the right derivative

is $2^{\mp 1}$ at that breakpoint, but $x_j^{\pm 1}$ has support in $[1 - (1/2)^j, 1] \subset [1 - (1/2)^i, 1]$ for $j \geq i$. It follows that the net exponent of all generators $x_i^{\pm 1}$ occurring in w must be zero. We can write $w = x_i^{\epsilon_1} w_1 x_i^{\epsilon_2} \quad w_m x_i^{\epsilon_{m+1}}$, where for each j, w_j is a nontrivial word in generators of the form x_l for $l > i$ and $\epsilon_j \in \{1, -1\}$, except possibly ϵ_1 and ϵ_{m+1} which may be zero. Note that if either ϵ_1 or ϵ_{m+1} are zero, then necessarily $m \geq 2$, and if both are zero, then $m \geq 3$. In any case, $m \geq 1$, and for some pair of indices r and s, $\epsilon_r = 1$ and $\epsilon_s = -1$.

Case 1: If for some j, $\epsilon_j = -1$ and $\epsilon_{j+1} = 1$, then let w_j' be the word obtained from w_j by increasing the index of each generator by 1. Then $w_j' = x_i^{-1} w_j x_i = x_i^{\epsilon_j} w_j x_i^{\epsilon_{j+1}}$ in F. Furthermore, as i is the minimal index in the word w, we know that w_j' begins and ends with $x_{i+2}^{\pm 1}$, w_{j-1} ends in $x_{i+1}^{\pm 1}$ (or does not exist if $j = 1$), and w_{j+1} begins in $x_{i+1}^{\pm 1}$ (or does not exist if $j = m$), and so replacing $x_i^{\epsilon_j} w_j x_i^{\epsilon_{j+1}}$ by w_j' produces a word of length $k - 2$ satisfying the hypotheses of the lemma.

Case 2: If no such index j exists, and neither ϵ_1 nor ϵ_{m+1} is zero, then w begins with x_i and ends with x_i^{-1}. Therefore, since $w = 1$ in F, $x_i^{-1} w x_i = 1$ as well, and if w' is the word of length $k - 2$ obtained from w by deleting the first and last letters, then $w' = x_i^{-1} w x_i = 1$ in F, and w' satisfies the hypotheses of the lemma.

Case 3: If no such index j exists, and $\epsilon_1 = 0$, then $m \geq 2$ and $\epsilon_2 = 1$. Let w_1' be the word w_1 with the index of each generator increased by one. Then since w_1' ends in $x_{i+2}^{\pm 1}$ and w_2 begins in $x_{i+1}^{\pm 1}$, then replacing $w_1 x_i$ by $x_i w_1'$ results in a word of the same length which still satisfies the hypotheses. Either this new word satisfies the conditions of Case 2, or else it does not end in x_i^{-1}. But if not, then $\epsilon_{m+1} = 0$, and one can do a similar substitution at that end to obtain a new word ending in x_i^{-1} and beginning in x_i which satisfies the conditions of Case 2. Applying the argument in Case 2 to this new word yields a word of length $k - 2$ satisfying the hypotheses of the lemma. $\qquad\square$

3.1 Nested traversal normal forms

In general, there are many minimal length representatives of elements of F with respect to the standard finite generating set, and Fordham [8] described effective methods for finding all such minimal length representatives. Cleary and Taback [5] described a straightforward procedure which canonically produces a minimal length element (with respect to the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$) for a purely positive or purely negative element in F; that is, an element w whose infinite normal form $w_p w_n$ satisfies $w = w_p$ (hence contains only terms with positive exponents) or $w = w_n$ (hence contains only terms with negative exponents). They call these paths *nested traversal paths* due to their construction. The combing paths used below will be built from concatenating and then freely reducing these nested traversal paths.

Let $w \in F$ be a strictly negative element; that is, $w = w_n$, and w is represented by a reduced tree pair diagram of the form $(T_-(w), R_{N(w)})$, where $R_{N(w)}$ is a tree consisting only of $N(T_-(w))$ right carets. To construct the nested traversal path corresponding to w, we proceed as follows. We number the carets of the tree $T_-(w)$ in infix order, beginning with 1. We proceed through the carets in infix order, adding generators x_0, x_0^{-1}, and x_1^{-1} to the right end of the nested traversal path at each step according to the following rules.

(1) If the infix number of the caret is 1, add nothing to the nested traversal path.
(2) If the caret is a left caret with infix number greater than 1, add x_0^{-1} to the nested traversal path.

(3) If the caret is an interior caret, let T be the right subtree of the caret. If T is nonempty, add $x_0^{-1} \gamma_T x_0 x_1^{-1}$ to the nested traversal path, where γ_T is the nested traversal path obtained by following these rules for the carets of T.

(4) If the caret is an interior caret and the right subtree of T is empty, then add x_1^{-1} to the nested traversal path.

(5) If the caret is a right non-root caret, and its right subtree T contains an interior caret, add $x_0^{-1} \gamma_T x_0$ to the nested traversal path, where γ_T is as above.

(6) If the caret is a right non-root caret, and its right subtree T contains no interior carets, then add nothing to the nested traversal path.

It is proved in [5] that this method produces a minimal length word representing a negative element w_n of F, with respect to the generating set $\{x_0, x_1\}$. We denote this nested traversal path for w_n by $\eta(w_n)$. For a reduced tree pair diagram, the situation in rule (6) above never arises. Including it allows one to extend the algorithm to tree pair diagrams obtained by appending only right carets to both the last leaf in the tree $T_-(w)$ and to the last leaf of $R_{N(w)}$ without changing the word produced by the algorithm.

We define the *nested traversal normal form* $\eta(w)$ of an element $w \in F$ as follows. Let $w = w_p w_n$ be the infinite normal form for w. Then the element w_p^{-1}, represented by the (not necessarily reduced) tree pair diagram $(T_+(w), R_{N(w)})$, is strictly negative, and so has a nested traversal path formed according to the above rules, which is not affected by the possible reduction of the diagram, according to rule (6) of the procedure above. Hence we can define the nested traversal normal form for w_p to be $\eta(w_p) := (\eta(w_p^{-1}))^{-1}$. It follows from the nested traversal construction that the words $\eta(w_p)$ and $\eta(w_n)$ are freely reduced, considered separately. However, their concatenation $\eta(w_p)\eta(w_n)$ may not be, so we define $\eta(w)$, the nested traversal normal form for w, to be the result of freely reducing the word $\eta(w_p)\eta(w_n)$. Note that $\eta(w)$ is not necessarily a minimal length word representing the element w.

Cleary and Taback show in the proof of Theorem 6.1 of [5] that along a strictly negative nested traversal normal form $\eta(w_n) = a_1 a_2 \ldots a_n$, the number of carets in the tree pair diagrams corresponding to the prefixes $a_1 a_2 \ldots a_i$ for $i \in \{1, 2, \ldots, n\}$ never decreases, that is, $N(a_1 a_2 \quad a_i) \le N(a_1 a_2 \quad a_{i+1})$. This follows from the construction of the paths: the multiplication $(a_1 a_2 \ldots a_i) \, a_{i+1}$ never causes a reduction of carets. We prove below that the same holds for the general nested traversal normal form $\eta(w)$. Our proof of the tameness of the 0-combing given by the nested traversal normal forms uses this property combined with the relationship between $l_A(w)$ and $N(w)$ described in Lemma 3.3.

Theorem 3.5 *For $w \in F$, if $\eta(w) = a_1 a_2 \ldots a_p$, then $N(a_1 a_2 \quad a_{i-1}) \le N(a_1 a_2 \quad a_i)$ for all $1 \le i \le p$.*

Proof We first claim that for any $u \in F$ and generator $a \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$, carets cannot be both added and removed in the process of multiplying ua. We check one case of this for the reader, in which $a = x_1$ and the right child of the root caret in $T_-(u)$ exists and has an exposed left leaf labeled n. In this case it is necessary to add a single caret to leaf n of both $T_-(u)$ and $T_+(u)$, which has exposed leaves numbered n and $n + 1$. Before reduction of carets, we obtain a possibly unreduced tree pair diagram (T'_-, T'_+) in which leaves n and $n + 1$ of T'_- no longer form a caret. Any exposed caret with leaves numbered greater than $n + 1$ in $(T_-(u), T_+(u))$ has its leaf labels increased by 1 in (T'_-, T'_+). Thus if a caret pair is exposed in (T'_-, T'_+), it would have been exposed in $(T_-(u), T_+(u))$. However, $(T_-(u), T_+(u))$ was reduced, and hence (T'_-, T'_+) is reduced as well, and so equals $(T_-(ux_1), T_+(ux_1))$. Other cases are checked similarly.

Note that $N(ua)$ and $N(u)$ can be related in one of the following three ways. Either

- $N(ua) > N(u)$ if carets must be added in order to perform the multiplication, or
- $N(ua) = N(u)$ if no carets must be added in order to perform the multiplication and no carets must be removed in order to reduce the resulting tree pair diagram, or
- $N(ua) < N(u)$ if no carets must be added in order to perform the multiplication, but carets must be removed to reduce the resulting tree pair diagram.

We remark that in any case, $N(ua)$ and $N(u)$ differ by at most 1 when $a = x_0^{\pm 1}$ and by at most 2 when $a = x_1^{\pm 1}$. In addition to the possible change in the number of carets, the resulting trees are rearranged slightly, in very constrained ways.

We will first prove the theorem in the case where $a_i = x_1^{\pm 1}$. Observe that for any $u \in F$, either

- $T_+(ux_1)$ has one more interior caret than $T_+(u)$, both $T_-(ux_1)$ and $T_-(u)$ have the same number of interior carets, and $N(ux_1) > N(u)$, or
- $T_-(ux_1)$ has one fewer interior caret than $T_-(u)$, both $T_+(ux_1)$ and $T_+(u)$ have the same number of interior carets, and $N(ux_1) \leq N(u)$.

To see this, note that if carets must be added in order to perform the multiplication ux_1; that is, if either the root of $T_-(u)$ does not have a right child, or this root does have a right child but this child does not have a left child, then $T_+(ux_1)$ has one more interior caret than does $T_+(u)$. In that case, $T_-(ux_1)$ is the tree $T_-(u)$ with one or two right carets added to the last leaf, and the number of interior carets in the negative tree is preserved. Also as noted above, no carets can be removed, and hence $N(ux_1) > N(u)$. On the other hand, if no carets are added in performing the multiplication ux_1, then $N(ux_1) \leq N(u)$. Moreover, during the multiplication process, the left child of the right child of the root of $T_-(u)$ is an interior caret, but in the tree $T_-(ux_1)$, either this caret has been removed in the multiplication process, or the caret with this same number is a right caret, and hence either way, the number of interior carets in $T_-(ux_1)$ is strictly less than that of $T_-(u)$. If carets are removed, they must be the final one or two carets of the tree pair, and these must be right carets in the tree $T_+(u)$, so the number of interior carets in the positive tree is left unchanged. Similarly, one can verify that either

- $T_-(ux_1^{-1})$ has one more interior caret than $T_-(u)$, both $T_+(ux_1^{-1})$ and $T_+(u)$ have the same number of interior carets, and $N(ux_1^{-1}) \geq N(u)$, or
- $T_+(ux_1^{-1})$ has one fewer interior caret than $T_+(u)$, both $T_-(ux_1^{-1})$ and $T_-(u)$ have the same number of interior carets, and $N(ux_1^{-1}) < N(u)$.

Furthermore, if $a = x_0^{\pm 1}$, then $T_-(ua)$ and $T_-(u)$ have the same number of interior carets, as do $T_+(ua)$ and $T_+(u)$.

Now in the (possibly not freely reduced) word $\eta(w_p)\eta(w_n)$, all occurrences of x_1 precede all occurrences of x_1^{-1}. Furthermore, if $T_-(w)$ has r interior carets and $T_+(w)$ has s interior carets, then by construction, $\eta(w_p)\eta(w_n)$ has precisely s occurrences of x_1 followed by r occurrences of x_1^{-1}. Thus in any prefix of $\eta(w)$, if $a_i = x_1^{\pm 1}$ it is always true that the number of interior carets in the tree pair diagram corresponding to $a_1 a_2 \quad a_i$ is one more than the number of interior carets in the tree pair diagram corresponding to $a_1 a_2 \quad a_{i-1}$.

Hence, it follows from the previous observations that if ux_1 is a prefix of $\eta(w_p)$, then $T_+(ux_1)$ has one more interior caret than $T_+(u)$, and $N(ux_1) > N(u)$. Similarly, if ux_1^{-1} is a prefix of $\eta(w_n)$, then $T_-(\eta(w_p)ux_1^{-1})$ has one more interior caret than $T_-(\eta(w_p)u)$, and $N(\eta(w_p)ux_1^{-1}) \geq N(\eta(w_p)u)$, which proves the theorem in the case $a_i = x_1^{\pm 1}$.

In addition, we can conclude from this analysis a few more facts about the relationship between the reduced word $\eta(w)$ and the potentially longer word $\eta(w_p)\eta(w_n)$, which

we note here for use again later. In particular, $\eta(w) = w_1 w_2$, where $\eta(w_p) = w_1 x_0^n$ and $\eta(w_n) = x_0^{-n} w_2$ for some $n \geq 0$, and if w_1 and w_2 are both nonempty words, then either w_1 ends in x_1 or w_2 begins with x_1^{-1}, but not both. Moreover, for any prefix $a_1 \quad a_i$ of w_1, $T_-(a_1 \quad a_i)$ contains no interior carets.

To prove the theorem for the cases where $a_i = x_0^{\pm 1}$, we repeatedly refer to the following six facts, each of which can be deduced by carefully following the process of multiplying by a generator.

(1) $N(u x_0^{-1}) < N(u)$ if and only if the first caret of $T_+(u)$ is exposed, and the first two right carets of $T_-(u)$ have no left children.
(2) $N(u x_0) < N(u)$ if and only if the last caret of $T_+(u)$ is exposed, and the last two left carets of $T_-(u)$ have no right children.
(3) If $N(u x_0^{-1}) \geq N(u)$, then the root caret of $T_-(u x_0^{-1})$ has a left child.
(4) If $N(u x_0) \geq N(u)$, then the root caret of $T_-(u x_0)$ has a right child.
(5) If $N(u x_1^{-1}) \geq N(u)$, then the second right caret of $T_-(u x_1^{-1})$ has a left child, so in particular the root caret has a right child.
(6) If $N(u x_1) \geq N(u)$, then the root caret of $T_-(u x_1)$ has a right child.

We proceed by induction to prove the theorem. Clearly $N(a_1) > N(\epsilon)$, so now assume that $N(a_1 \quad a_k) \geq N(a_1 \quad a_{k-1})$ for all $k < i$.
Case 1: $a_i = x_0$.

Either $a_{i-1} = x_0$, $a_{i-1} = x_1^{-1}$, or $a_{i-1} = x_1$. But then, since by the inductive hypothesis $N(a_1 \quad a_{i-1}) \geq N(a_1 \quad a_{i-2})$, facts 4, 5, and 6 show that the root caret of $T_-(a_1 \quad a_{i-1})$ always has a right child. Hence, fact 2 above implies that $N(a_1 \quad a_i) \geq N(a_1 \quad a_{i-1})$.
Case 2: $a_i = x_0^{-1}$.

If $a_{i-1} = x_0^{-1}$ or $a_{i-1} = x_1^{-1}$, then since by the inductive hypothesis $N(a_1 \quad a_{i-1}) \geq N(a_1 \quad a_{i-2})$, facts 3 and 5 above show that either the root caret or the second right caret of $T_-(a_1 \quad a_{i-1})$ has a left child. Hence, in these cases fact 1 above implies that $N(a_1 \quad a_i) \geq N(a_1 \quad a_{i-1})$. So we must check the one remaining possibility, that $a_{i-1} = x_1$. We claim that for $a_{i-1} = x_1$, either the root caret of $T_-(a_1 \quad a_{i-1})$ has a left child, or the first caret of $T_+(a_1 \quad a_{i-1})$ is not exposed, which will again imply by fact 1 above that $N(a_1 \quad a_i) \geq N(a_1 \quad a_{i-1})$. To verify the claim, first note that since the letter x_1 cannot occur in $\eta(w_n)$, the word $a_1 \quad a_i$ is a prefix of w_1, and so neither $T_-(a_1 \quad a_{i-2})$ nor $T_-(a_1 \quad a_{i-1})$ contain any interior carets. Now if the root caret of $T_-(a_1 \quad a_{i-2})$ has a left child, then so does the root caret of $T_-(a_1 \quad a_{i-1})$. On the other hand, if $T_-(a_1 \quad a_{i-2})$ consists only of right carets, then $T_+(a_1 \quad a_{i-1})$ is obtained by hanging a caret from the second leaf of $T_+(a_1 \quad a_{i-2})$, so that in this case, the first caret of $T_+(a_1 \quad a_{i-1})$ is not exposed.   □

In addition, we remark that the nested traversal forms, which are certainly not in general geodesics, are at least quasigeodesics. To see this, it is helpful to first make some preliminary observations about nested traversal paths for strictly negative words, each of which can be deduced from the algorithm for the construction of nested traversal paths.

Observation 3.6 (1) *A word $a_1 \quad a_n$, where $a_i \in \{x_0, x_0^{-1}, x_1^{-1}\}$ for each i, is a nested traversal path if and only if the word satisfies the following three conditions:*

(a) *The word is freely reduced.*
(b) *The exponent sum of x_0 in any prefix $a_1 \quad a_k$ for $1 \leq k \leq n$ is not positive.*
(c) *If $a_k = a_{k+1} = x_0$ for some k, then $a_j = x_0$ for $k \leq j \leq n$.*

Hence, any prefix of a nested traversal path is again a nested traversal path.

(2) *Let $a_1 \quad a_n$ be a nested traversal path with reduced tree pair diagram (T_-, R_l) for some l. Then the first caret of T_- is exposed if and only if $a_1 = x_0^{-1}$ and the exponent sum of x_0 in every nonempty prefix $a_1 \quad a_k$ is strictly negative. Note that the final caret of T_-, always a right caret, is never exposed.*

(3) *If $a_1 \quad a_n$ is a nested traversal path with reduced tree pair diagram (T_-, R_l) for some l, the numbers labeling the exposed carets of T_- can be algorithmically determined as follows. Caret 2 is exposed if $a_1 = x_1^{-1}$, and not if $a_1 = x_0^{-1}$. So by reading through the prefix a_1, it can be determined whether or not caret 2 is exposed. Inductively, suppose that by reading through a prefix $a_1 \quad a_k$, you have decided whether or not carets 2 through j are exposed. Then for caret $j + 1$, if*

$$
a_{k+1} = \begin{cases}
x_0 & \text{then move on to } a_{k+2} \text{ to make a decision about caret } j + 1. \\
x_1^{-1} & \text{and } a_k = x_0, \text{ then move on to } a_{k+2} \text{ to make a decision about caret } j+1. \\
x_0^{-1} & \text{then caret } j + 1 \text{ is not exposed.} \\
x_1^{-1} & \text{and } a_k = x_0^{-1}, \text{ then caret } j + 1 \text{ is exposed.} \\
x_1^{-1} & \text{and } a_k = x_1^{-1}, \text{ then caret } j + 1 \text{ is not exposed.}
\end{cases}
$$

In the latter three cases, then move on to a_{k+2} to make a decision about whether caret $j + 2$ is exposed.

With this observation in hand, we are ready to prove that nested traversal normal forms are quasigeodesics. Recall that for a group G with finite generating set A, a word y is a (λ, c)-quasigeodesic for constants $\lambda \geq 1$ and $c \geq 0$ if the unit speed path $p : [0, l(y)] \to \Gamma(G, A)$ labeled by y satisfies $\frac{1}{\lambda}|s - t| - c \leq d_A(p(s), p(t)) \leq \lambda|s - t| + c$ for all $s, t \in [0, l(y)]$.

Theorem 3.7 *For every $w \in F$ the nested traversal normal form $\eta(w)$ is a (λ, c)-quasigeodesic with $\lambda = 6$ and $c = 0$.*

Proof Let $w \in F$. From facts noted in the proof of Theorem 3.5, we can write $\eta(w_p)\eta(w_n) = w_1 x_0^n x_0^{-n} w_2$ and $\eta(w) = w_1 w_2 = a_1 \quad a_p$ for some $n \geq 0$ and each $a_i \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$. It follows from the formula for length in Proposition 3.2 that each pair of carets in $T_-(w)$ and $T_+(w)$, other than the first pair and the last pair, contribute some nonzero number between one and 4 to $l_A(w)$. However, each such caret pair contributes at most 6 to $l(\eta(w))$, so the length contribution to the normal form is at most 6 times the contribution to length in the Cayley graph. The first and last caret pairs are a slightly special case, since together they may contribute 0, 1, or 2 to both $l_A(w)$ and $l(\eta(w))$. However, one checks that if there is no contribution to $l_A(w)$ from these carets, then the first caret in both trees is the root caret, and so the contribution to $l(\eta(w))$ from these carets will be zero as well. Thus we obtain

$$l_A(w) \leq l(\eta(w)) \leq 6\, l_A(w)$$

for any nested traversal normal form $\eta(w)$.

We now show that these inequalities hold for an arbitrary subword of $\eta(w)$. So suppose $u = a_i \quad a_{i+k}$, with $1 \leq i < i + k \leq r$. If a_{i+k} is a letter of w_1, then since $w_1 x_0^n = \eta(w_p)$ is a geodesic, then $l_A(u) \leq l(\eta(u)) \leq 6 l_A(u)$. The same argument, using the fact that $\eta(w_n)$ is a geodesic, holds in the case where a_i is a letter in w_2.

Now assume $a_1 \quad a_i$ is a prefix of w_1 and $a_{i+k} \quad a_p$ is a suffix of w_2. Let u_1 be the suffix of w_1 starting with a_i, and let u_2 be the prefix of w_2 ending with a_{i+k}; then $u = u_1 u_2$. We prove below that for the subword u, $\eta(u_p) = u_1 x_0^s$ and $\eta(u_n) = x_0^{-s} u_2$ for some $0 \leq s \leq n$. This then implies that $\eta(u) = u_1 u_2$, and therefore u is a nested traversal normal form, and $l_A(u) \leq l(\eta(u)) \leq 6 l_A(u)$.

Since $x_0^{-n}u_2$ and $x_0^{-n}u_1^{-1}$ are prefixes of $\eta(w_n)$ and $\eta(w_p^{-1})$ respectively, they are also nested traversal paths according to the first part of Observation 3.6. Let (T'_-, R_k) denote the reduced tree pair diagram for $x_0^{-n}u_2$ and let (T''_-, R_l) be the reduced tree pair for $x_0^{-n}u_1^{-1}$. There is a two step process to transform the pair of trees (T'_-, T''_-) into the reduced tree pair diagram for $u = u_1u_2$. In the first step, if the numbers k and l of carets in each of these trees are not equal, we add a string of $|k - l|$ right carets to the final leaf of the smaller tree. The second step consists of reducing the resulting tree pair. Note that by construction, the final caret of both T'_- and T''_- cannot be exposed, and so any carets added in the first step will not be removed in the second step. Each time a caret is removed, there is a corresponding change to the pair of nested traversal paths, which we track below; at the end of this process, we obtain the nested traversal paths for u_n and u_p.

If caret 1 is exposed in both trees, it must be that $n > 0$, so removing this caret pair corresponds algebraically to canceling the central $x_0x_0^{-1}$ pair to obtain the word $u_1x_0^{n-1}x_0^{-(n-1)}u_2$. Note that both $x_0^{-(n-1)}u_2$ and $x_0^{-(n-1)}u_1^{-1}$ are both again nested traversal paths. Now repeat, and eventually reach a point where caret 1 is not exposed in one of $T_-(x_0^{-s}u_2)$ and $T_-(x_0^{-s}u_1^{-1})$, for some $0 \le s \le n$. Note that if $s = 0$, caret 1 cannot be exposed in both trees. Hence this process must successfully terminate.

Next we check for possible reduction of caret pairs numbered greater than 1. Part 3 of Observation 3.6 shows that caret 2 can only be exposed in both trees $T_-(x_0^{-s}u_2)$ and $T_-(x_0^{-s}u_1^{-1})$ if $s = 0$ and both u_1^{-1} and u_2 start with the letter x_1^{-1}; however, u_2 is a prefix of the word w_2 and u_1 is a suffix of w_1, and the word w_1w_2 is freely reduced. For caret pairs numbered between 3 and $\min\{k, l\} - (n - s)$, the algebraic criteria from part (3) of Observation 3.6 by which we check for exposure of these carets only depends upon letters in the words u_2 and u_1^{-1} which, as noted above, are prefixes of the words w_2 and w_1^{-1}, respectively. Since the tree pair diagram for w_1w_2 is reduced, then none of these carets can be removed. Carets numbered above $\min\{k, l\} - (n - s)$ were added in the first step, and hence also cannot be removed.

Hence, $\eta(u_p)\eta(u_n) = u_1x_0^sx_0^{-s}u_2$, so $\eta(u) = u_1u_2$, and the inequalities follow. Thus nested traversal normal forms are (6,0)-quasigeodesics. $\square$

4 Constructing the combing of F

In this section, we construct a 1-combing of the group F with respect to the presentation

$$\langle x_0, x_1 | [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2] \rangle;$$

in Sect. 5 we will show that this combing satisfies a linear radial tameness function. Let X be the Cayley complex for this presentation.

We first construct a 0-combing of F with respect to $A = \{x_0, x_1\}$ by defining a continuous function $\Psi : X^0 \times [0, 1] \to X^1$ where, for any $w \in F$, the restriction $\Psi : \{w\} \times [0, 1] \to X^1$ is labeled by the nested traversal normal form $\eta(w)$ for w. We call this 0-combing Ψ the *nested traversal 0-combing*.

Now we must extend this 0-combing to a 1-combing. All edges in the Cayley graph fall into one of two categories, "good" and "bad". The good edges consist of those edges where the combing path to one endpoint contains the other endpoint, and thus points along that edge are combed through the 1-skeleton. The bad edges include all of those edges where this is not the case, and thus the points along the edge in question must be combed through the 2-skeleton. To make this more formal, we introduce some notation. For each $w \in F$, let

Ψ_w be the 0-combing path in X^1 from the identity to w (labeled by $\eta(w)$), and let Ψ_w^{-1} be the inverse path from w to the identity. Recall that each directed edge in the Cayley graph $X^1 = \Gamma(F, \{x_0, x_1\})$ is labeled either by the generator x_0 or the generator x_1. We formalize the notion of good and bad edges in the following definition.

Definition 4.1 If the set of endpoints of an edge e is of the form $\{w, wx_0^{-1}\}$, we denote the edge as $e_0(w)$, and if the set of endpoints of an edge e is of the form $\{w, wx_1^{-1}\}$, we denote the edge as $e_1(w)$.

Moreover, for the edge $e_a(w)$, where $a \in \{0, 1\}$, if the loop $\gamma_e := \Psi_w e_a(w) \Psi_{wx_a^{-1}}^{-1}$ is homotopic to the trivial loop in the Cayley graph, we call the edge $e_a(w)$ a *good edge*, and if not, we call the edge $e_a(w)$ a *bad edge*.

Theorem 4.2 describes conditions on the Cayley complex which allow us to extend the nested traversal 0-combing to a 1-combing.

Theorem 4.2 *Let $\mathcal{B}$ be the set of bad edges in the Cayley complex X with respect to the nested traversal 0-combing. Suppose that*

(1) *there is a partial ordering of $\mathcal{B}$ with the property that for any edge $e \in \mathcal{B}$, the set of edges $\{f \in \mathcal{B} \mid f < e\}$ is finite, and*
(2) *there is a function c from $\mathcal{B}$ to the set of 2-cells of X, so that for every $e \in \mathcal{B}$ the edge e is on the boundary of $c(e)$, and whenever $f \in \mathcal{B}$ is another edge on the boundary of $c(e)$, then $f < e$.*

Then the nested traversal 0-combing $\Psi : X^0 \times [0, 1] \to X^1$ can be extended to a 1-combing $\Psi : X^1 \times [0, 1] \to X^2$.

Proof We remark that the hypotheses of the theorem imply that the mapping from bad edges to 2-cells is injective. Let $\mathcal{G}$ be the set of good edges. We extend the 0-combing in two stages. First, extend $\Psi : X^0 \times [0, 1] \to X^1$ to $\Psi : (X^0 \cup \mathcal{G}) \times [0, 1] \to X^1$ using the homotopies for the good edges. Next, note that the partial ordering on $\mathcal{B}$ is well-founded, and so we may apply Noetherian induction to define Ψ on bad edges as follows. Suppose we have already extended the combing to $\Psi : (X^0 \cup \mathcal{G} \cup S) \times [0, 1] \to X^2$, where $S := \{e' \in \mathcal{B} \mid e' < e\}$ for a particular edge e. Since $c(e)$ is a 2-cell, there is a homotopy from e, through $c(e)$, to the remainder of the boundary $\partial c(e)$ excluding the interior $Int(e)$ of e, which fixes the endpoints of e. More specifically, let $\Theta : e \times [0, 1] \to c(e)$ satisfy: for each point p in the edge e, $\Theta(p, 0) \in \partial c(e) \setminus Int(e)$ and $\Theta(p, 1) = p$; the image $\Theta(e \times \{0\}) = \partial c(e) \setminus Int(e)$; and for the endpoints g and h of e and for all $t \in [0, 1]$, $\Theta(g, t) = g$ and $\Theta(h, t) = h$. Since all edges in $\partial c(e) \setminus Int(e)$ are in $\mathcal{G} \cup S$, the combing $\Psi : (X^0 \cup \mathcal{G} \cup S) \times [0, 1] \to X^2$ provides combing paths from the identity to each of the points of $\partial c(e) \setminus Int(e)$. Reparametrize these paths and concatenate them with the paths from the homotopy Θ to define the homotopy $\Psi : e \times [0, 1] \to X$. This yields a homotopy $\Psi : (X^0 \cup \mathcal{G} \cup S \cup \{e\}) \times [0, 1] \to X^2$. Then, by induction, the 0-combing extends to a 1-combing $\Psi : X^1 \times [0, 1] \to X^2$. $\square$

The remainder of this section is devoted to establishing the hypotheses of Theorem 4.2.

4.1 Identifying the good edges

The goal of this section is to identify the good edges in $\Gamma(F, A = \{x_0, x_1\})$. This is accomplished in the following theorem.

Theorem 4.3 *Let $w \in F$ have reduced tree pair diagram $(T_-(w), T_+(w))$. If any one of the following four conditions holds, then the edge $e_a(w)$, with $a \in \{0, 1\}$, is a good edge:*

(1) *The index $a = 0$.*
(2) *The tree $T_-(w)$ has at most two right carets.*
(3) *The tree $T_-(w)$ has at least three right carets, no carets need be removed to reduce the tree pair diagram for wx_1^{-1}, and all carets following the third right caret in $T_-(w)$, if any, are right carets.*
(4) *The tree $T_-(w)$ has at least three right carets but no interior carets, caret n must be removed to reduce the tree pair diagram for wx_1^{-1}, and caret n is the first exposed caret in $T_+(w)$.*

We prove this theorem in two lemmas, considering separately the cases $e_0(w)$ and $e_1(w)$. To prove each lemma, we simply compare the nested traversal forms for the two endpoints of the edge in each situation in the hypotheses of Theorem 4.3.

Lemma 4.4 *Let $w \in F$. Then $e_0(w)$ is a good edge.*

Proof Let $z = wx_0^{-1}$. We compare $\eta(w)$ and $\eta(z)$, and show that either $\eta(z) = \eta(w)x_0^{-1}$ or $\eta(w) = \eta(z)x_0$, and so it follows immediately that $e_0(w)$ is a good edge.

As usual, let $w = (T_-(w), T_+(w))$ and $z = (T_-(z), T_+(z))$ denote reduced tree pair diagrams. The tree pair diagram (S_-, S_+) for x_0^{-1} is given in Fig. 2.

Suppose first that $T_-(w)$ has only one right caret, the root caret. The left subtree of the root caret must then be nonempty; let $A(w)$ be this subtree, and let γ_A be the substring of $\eta(w_n)$ consisting of all generators corresponding to carets in $A(w)$. Then $\eta(w_n) = \gamma_A x_0^{-1}$. In multiplying $z = wx_0^{-1}$, a caret is appended to the rightmost leaf of each of the trees for w, and the tree $A(w)$ is appended to the leftmost leaf of the trees S_- and S_+ for x_0^{-1}. Then $z = (T_-(z), T_+(z))$ where $T_-(z)$ consists of a root caret with a left child whose left subtree is $A(w)$, and $T_+(z)$ is $T_+(w)$ with a single caret appended to its rightmost leaf. This pair is reduced, so no caret is removed in performing this product. The tree $T_-(z)$ has a left caret between the subtree $A(w)$ and the root, so $\eta(z_n) = \gamma_A x_0^{-2}$. Since $T_+(z)$ is just $T_+(w)$ with a single caret appended to its rightmost leaf, $\eta(w_p^{-1}) = \eta(z_p^{-1})$. Hence in this case, $\eta(z_p)\eta(z_n) = \eta(w_p)\gamma_A x_0^{-2} = \eta(w_p)\eta(w_n)x_0^{-1}$. Therefore, if $\eta(w)$ does not end in x_0, then $\eta(z) = \eta(w)x_0^{-1}$. However, if $\eta(w)$ does end in x_0, then $\eta(z)x_0 = \eta(w)$.

For the remainder of this proof suppose that $T_-(w)$ has at least two right carets. Let $A(w)$ be the left subtree of the root caret, let $B(w)$ denote the left subtree of the right child of the root caret, and let $E(w)$ denote the right subtree of the right child of the root. Let γ_A, γ_B, and γ_E denote the subwords of $\eta(w_n)$ corresponding to the carets of these subtrees; note that any of these trees can be the empty tree, and if so, the corresponding subword will be empty. Define $\gamma_r := 1$ if the tree $A(w)$ is the empty tree $\emptyset$ with no carets, and $\gamma_r := x_0^{-1}$ if $A(w) \neq \emptyset$, so that γ_r is the contribution of the root caret of $T_-(w)$ to the nested traversal normal form $\eta(w_n)$. The nested traversal normal form for w is then

$$\eta(w_p)\eta(w_n) = \begin{cases} \eta(w_p)\gamma_A\gamma_r\gamma_B & \text{if } E(w) = R_k \text{ for some } k \geq 0 \\ \eta(w_p)\gamma_A\gamma_r\gamma_B x_0^{-1}\gamma_E x_0 & \text{if } E(w) \neq R_k \text{ for all } k. \end{cases}$$

In this case no carets need to be added to the tree pair for w in order to perform the multiplication wx_0^{-1}; the trees $A(w)$, $B(w)$, and $E(w)$ must be appended to leaves 0, 1, and 2, respectively of the trees S_- and S_+.

A caret must be removed from the product wx_0^{-1} to obtain the reduced tree pair diagram if and only if the trees $A(w)$ and $B(w)$ are both empty, and caret 1 in the tree $T_+(w)$ is exposed. In this case, $T_+(z)$ is the tree $T_+(w)$ with the first caret removed. Note that caret 1 of $T_+(w)$ contributed nothing to the nested traversal normal form $\eta(w_p^{-1})$, and that caret 2 of $T_+(w)$ must also be a left caret, and so contributed x_0^{-1}. The latter caret is caret 1 of $T_+(z)$. Hence $\eta(w_p^{-1}) = x_0^{-1}\eta(z_p^{-1})$. Analyzing the negative trees, we note that the tree $T_-(z)$ is the tree with a single left caret, namely the root caret, having a right subtree given by $E(w)$, and so $\eta(z_n) = \gamma_E$. If $E(w) = R_k$ for some $k \geq 0$, then $\eta(w) = \eta(z)x_0$. On the other hand, if $E(w) \neq R_k$ for any k, then since $E(w)$ is the right subtree of the root of $T_-(z)$, this subtree gives a nonempty contribution to the nested traversal path, and hence $\eta(z_n)$ cannot end with the letter x_0^{-1}. Thus when freely reducing the word $\eta(w_p)\eta(w_n) = \eta(z_p)x_0x_0^{-1}\eta(z_n)x_0$, only one $x_0x_0^{-1}$ is removed, and $\eta(w) = \eta(z)x_0$.

Finally, suppose that no carets need to be removed in the multiplication wx_0^{-1}. Then $T_+(z) = T_+(w)$ and $z_p = w_p$. Therefore $\eta(z_p)\eta(z_n) = \eta(w_p)\gamma_A\gamma_r\gamma_Bx_0^{-1}\gamma_E$. If $E(w) = R_k$, then $\eta(z_p)\eta(z_n) = \eta(w_p)\eta(w_n)x_0^{-1}$, and so either $\eta(z) = \eta(w)x_0^{-1}$, when $\eta(w)$ does not end in x_0, or else $\eta(z)x_0 = \eta(w)$. On the other hand if $E(w) \neq R_k$, then $\eta(w_p)\eta(w_n) = \eta(z_p)\eta(z_n)x_0$. Since $\eta(z_n)$ cannot end in x_0^{-1}, $\eta(w) = \eta(z)x_0$. □

Next, we turn to edges of the form $e_1(w)$. For such an edge, the case where $T_-(w)$ has at least three right carets is by far the most complicated, so before stating the desired lemma, we establish some useful notation for that case.

Notation 4.5 For any $w = (T_-(w), T_+(w)) \in F$ such that $T_-(w)$ has at least three right carets:

- Let $A(w)$ denote the left subtree of the root caret, $B(w)$ denote the left subtree of the right child of the root, and $C(w)$ and $D(w)$ denote the left and right subtrees, respectively, of the third right caret of $T_-(w)$.
- Define $N(w) := N(T_-(w)) = N(T_+(w))$ (as above), and $N_D(w) := N(D(w))$ and $N_A(w) := N(A(w))$.
- Define $j(w)$ to be the number of the first exposed caret of $T_+(w)$.

To understand the good edges, and later the definition of the partial order on the edges, one must first understand explicitly how the tree pair diagram for w may change when w is multiplied by x_1^{-1}. To form the product wx_1^{-1}, if $T_-(w)$ contains at least 3 right carets, then no carets must be added to the trees of the reduced pair diagram for w, but the subtrees $A(w), B(w), C(w)$, and $D(w)$ are appended to the leaves numbered 0, 1, 2, and 3, respectively, of the trees in the reduced tree pair diagram for x_1^{-1}, which is given in Fig. 2.

Continuing the case that $T_-(w)$ contains at least three right carets, let T'_- be the negative tree of the intermediate step in the multiplication, wx_1^{-1} before any carets are removed to reduce the tree pair diagram. The carets of $T_-(w)$ and T'_- with the same number have the same type (left, right or interior) in both trees, with the exception of the caret numbered $N_A(w) + 1 + N_B(w) + 1$, which is a right caret in $T_-(w)$ and an interior caret in T'_-. This is the only caret that can be exposed in T'_- but not in $T_-(w)$, and hence the only caret that might be removed if T'_- is not reduced. As a consequence, a caret must be removed in the multiplication wx_1^{-1} if and only if the following property holds:

($\ddagger$) : $B(w) = \emptyset, C(w) = \emptyset$, and $T_+(w)$ has an exposed caret at caret number $N_A(w)+2$.

We are now ready to describe edges of the form $e_1(w)$ that are good edges.

Lemma 4.6 *Let $w \in F$. Then $e_1(w)$ is a good edge if any of the following are satisfied:*

(1) $T_-(w)$ *has at most two right carets.*
(2) $T_-(w)$ *has three or more right carets, property (‡) is not satisfied and $D(w) = R_k$ for some $k \geq 0$.*
(3) $T_-(w)$ *has three or more right carets but no interior carets, property (‡) holds and the number n of the caret that cancels satisfies $n = N_A(w) + 2 = j(w)$, so caret n is the first exposed caret in $T_+(w)$.*

Proof Let $u = wx_1^{-1}$. Again we compare $\eta(w)$ with $\eta(u)$, and we claim that when w satisfies either of the first two conditions of the hypothesis, then $\eta(u) = \eta(w)x_1^{-1}$. If w satisfies the third condition, then $\eta(w) = \eta(u)x_1$. So in all cases it follows immediately that the edge $e_1(w)$ is good. We proceed by cases according to which hypothesis is satisfied. The first two are straightforward, but for the third we separate into subcases.

Case 1. $T_-(w)$ has fewer than three right carets. Suppose first that $T_-(w)$ has only one right caret, and $A(w)$ is the left subtree of this root caret. In multiplying $u = wx_1^{-1}$, two right carets are appended to the rightmost leaf of each of the trees for w, and $A(w)$ is appended to the leftmost leaf of the trees for x_1^{-1}, but no carets are removed. The two appended right carets in $T_+(u)$ contribute nothing to $\eta(u_p^{-1})$, so $\eta(u_p) = \eta(w_p)$. The root caret of $T_-(u)$ has left subtree $A(w)$, and the right child of the root has left subtree consisting of a single interior caret which contributes x_1^{-1} to the nested traversal normal form $\eta(u_n)$. Then $\eta(u_p)\eta(u_n) = \eta(w_p)(\eta(w_n)x_1^{-1})$, so it follows that $\eta(u) = \eta(w)x_1^{-1}$. The proof in the case that $T_-(w)$ has two right carets is similar.

For the remainder of the proof, assume that $T_-(w)$ has at least three right carets. Let $\gamma_A, \gamma_r, \gamma_B, \gamma_C$, and γ_D be the subwords of the nested traversal normal form $\eta(w_n)$ corresponding to the carets of $A(w)$, the root, $B(w)$, $C(w)$, and $D(w)$, respectively. In this case the nested traversal normal form for w is then

$$\eta(w_p)\eta(w_n) = \begin{cases} \eta(w_p)\gamma_A\gamma_r\gamma_B & \text{if } C(w)=\emptyset \text{ and } D(w)=R_k \text{ for some } k \geq 0 \\ \eta(w_p)\gamma_A\gamma_r\gamma_B x_0^{-1}\gamma_C x_0 & \text{if } C(w)\neq\emptyset \text{ and } D(w)=R_k \text{ for some } k \geq 0 \\ \eta(w_p)\gamma_A\gamma_r\gamma_B x_0^{-2}\gamma_D x_0^2 & \text{if } C(w)=\emptyset \text{ and } D(w)\neq R_k \text{ for all } k \geq 0 \\ \eta(w_p)\gamma_A\gamma_r\gamma_B x_0^{-1}\gamma_C x_0^{-1}\gamma_D x_0^2 & \text{if } C(w)\neq\emptyset \text{ and } D(w)\neq R_k \text{ for all } k \geq 0. \end{cases}$$

Case 2. No carets must be removed to create the reduced tree pair diagram for $u = wx_1^{-1}$, and $D(w) = R_k$ for some $k \geq 0$. It follows immediately that $T_+(w) = T_+(u)$ and hence $\eta(w_p) = \eta(u_p)$. From the discussion of $T_-(w)$ and $T'_- = T_-(u)$ above, only caret number $N_A(w) + 2 + N_B(w)$ makes a different contribution to the respective nested traversal normal forms, yielding:

$$\eta(u_p)\eta(u_n) = \begin{cases} \eta(w_p)\gamma_A\gamma_r\gamma_B x_1^{-1} & \text{if } C(w)=\emptyset \text{ and } D(w)=R_k \text{ for some } k \geq 0 \\ \eta(w_p)\gamma_A\gamma_r\gamma_B x_0^{-1}\gamma_C x_0 x_1^{-1} & \text{if } C(w) \neq \emptyset \text{ and } D(w)=R_k \text{ for some } k \geq 0. \end{cases}$$

Comparing these words with the corresponding words $\eta(w_p)\eta(w_n)$ given above yields $\eta(u) = \eta(w)x_1^{-1}$.

Case 3. Caret n is removed when we form the product wx_1^{-1} (equivalently, property (‡) holds), $j(w) = n$, and $T_-(w)$ has no interior carets. From (‡), this caret necessarily has caret number $n = N_A(w) + 2$. As $T_-(w)$ has no interior carets in Case 3, we must have $A(w) = L_{n-2}$, the tree with $n - 2$ left carets, where $n - 2 \geq 0$ and $D(w) = R_k$ for some $k \geq 0$. Then $\eta(w_n) = x_0^{-(n-2)}$. When caret n is removed to form the tree pair diagram for u, we see that $T_-(u)$ then has $n - 1$ left carets including the root and $k + 1$ right non-root carets, and so $\eta(u_n) = x_0^{-(n-2)}$ as well.

Note that $N(w) \geq N_A(w) + 3 = n + 1$, and so caret n of $T_+(w)$ is neither the first nor the last caret of this tree. Then this is an interior caret of $T_+(w)$ which is an exposed caret, in particular it has an empty right subtree. This caret will contribute x_1^{-1} to the nested traversal normal form $\eta(w_p^{-1})$. The tree $T_+(u)$ is the tree $T_+(w)$ with caret n removed.

For $1 \leq j \leq N(w)$, let C_j denote caret j of the tree $T_+(w)$. Caret C_1 contributes nothing to the nested traversal normal form $\eta(w_p^{-1})$. Whenever $2 \leq j < n - 1$, the unexposed caret C_j is either an interior caret or a right caret, and in both cases C_j has a nonempty right subtree containing the interior caret C_n. Hence each of these carets C_j adds x_0^{-1} to $\eta(w_p^{-1})$ before the subword x_1^{-1} corresponding to caret C_n, and also adds either $x_0 x_1^{-1}$ or x_0 to $\eta(w_p^{-1})$ after this subword. Then $\eta(w_p^{-1}) = x_0^{-(n-2)} x_1^{-1} \beta$ for some word β, and hence $\eta(w_p)\eta(w_n) = (\beta^{-1} x_1 x_0^{n-2})(x_0^{-(n-2)})$, so $\eta(w) = \beta^{-1} x_1$.

To analyze the nested traversal normal forms $\eta(u_p^{-1})$ and $\eta(u)$ further, we now divide into four subcases, as follows.

Case 3a. Suppose $A(w) = \emptyset$. Then it follows that $n = 2$, and the tree $T_+(u)$ is $T_+(w)$ with caret $n = 2$ removed. Hence $\eta(u) = \beta^{-1}$, and so $\eta(w) = \eta(u)x_1$.

Case 3b. Suppose $A(w) \neq \emptyset$, caret n is the left child of its parent caret in $T_+(w)$, and $N(w) = n + 1$. Then it follows that all other carets of $T_+(w)$ are right carets, or else a caret with infix number less than n would be the first exposed caret in $T_+(w)$. Thus $\eta(w_p)\eta(w_n) = x_0^{-(n-2)} x_1 x_0^{n-2} x_0^{-(n-2)}$. The tree $T_+(u)$ contains only right carets, and so $\eta(u_p)$ is trivial, and hence $\eta(u_p)\eta(u_n) = x_0^{-(n-2)}$. Therefore the nested traversal normal form for w is $\eta(w) = x_0^{-(n-2)} x_1 = \eta(u)x_1$ and so $\eta(w) = \eta(u)x_1$.

Case 3c. Suppose that $A(w) \neq \emptyset$, caret n is the left child of its parent caret in $T_+(w)$, and $N(w) > n + 1$. Caret C_{n+1} is the parent of caret C_n in this case. If C_{n+1} is an interior caret of $T_+(w)$, then C_{n+1} is an interior caret contained in the right subtree of carets C_j for all $2 \leq j \leq n - 1$, and so in the tree $T_+(u)$, these carets C_j also contain an interior caret in their right subtrees. If instead C_{n+1} is a right caret, then C_j is a right caret for all $1 \leq j \leq n - 1$. Note that the final caret $N(T_-(w))$ of $T_-(w)$ is exposed, and the tree pair $(T_-(w), T_+(w))$ is reduced, so caret number $N(T_+(w)) > n + 1$ of $T_+(w)$ is not exposed. Then the left subtree of the latter caret contains an interior caret C_i of $T_+(w)$ with $i > n$, and hence this interior caret is contained in the right subtrees of all of the carets C_j with $2 \leq j \leq n - 1$. Then for both types of parent caret the nested traversal path for u_p^{-1} is the same as that for w_p^{-1} except that the x_1^{-1} subword corresponding to caret n is removed. Then $\eta(u_p)\eta(u_n) = \beta^{-1} x_0^{n-2} x_0^{-(n-2)}$, and so once again $\eta(w) = \eta(u)x_1$.

Case 3d. Suppose that $A(w) \neq \emptyset$, and caret n is the right child of its parent in $T_+(w)$. Since $n \geq 3$, $N(T_+(w)) \geq n + 1$, and caret C_n is the first exposed caret in $T_+(w)$, then caret C_{n-1} must be an interior caret in $T_+(w)$, which is contained in the right subtree of each C_j with $2 \leq j \leq n - 2$. Then the nested traversal path for u_p^{-1} is the same as that for w_p^{-1} except that the $x_0^{-1} x_1^{-1} x_0 x_1^{-1}$ subword of $\eta(w_p^{-1})$ corresponding to carets C_{n-1} and C_n is replaced with the word x_1^{-1} corresponding to the caret C_{n-1} of $T_+(u)$. In this case $\eta(w_p)\eta(w_n) = (^{-1} x_1 x_0^{-1} x_1 x_0^{n-2})(x_0^{-(n-2)})$, (where $\beta = x_0 x_1^{-1}$), and $\eta(u_p)\eta(u_n) = (^{-1} x_1 x_0^{n-3})(x_0^{-(n-2)})$. Therefore, $\eta(w) = ^{-1} x_1 x_0^{-1} x_1 = \eta(u)x_1$. □

Lemmas 4.4 and 4.6 complete the proof of Theorem 4.3. In the next sections, we will most frequently apply the contrapositive of Theorem 4.3, rewritten below following Notation 4.5.

Corollary 4.7 *Let $w \in F$. If $e_a(w)$ is a bad edge, then $a = 1$, the tree $T_-(w)$ has at least three right carets, and either*

(1) $D(w) \neq R_{N_D(w)}$,
(2) $D(w) = R_{N_D(w)}$, $A(w) = L_{N_A(w)}$, *property* (‡) *holds, and* $2 \leq j(w) \leq N_A(w)$, *or*
(3) $D(w) = R_{N_D(w)}$, $A(w) \neq L_{N_A(w)}$, *and property* (‡) *holds.*

Proof From Theorem 4.3 parts (1) and (2) we know that $a = 1$ and $T_-(w)$ contains at least 3 right carets.

If no caret is removed in the multiplication wx_1^{-1} (that is, if property (‡) fails), then Part (3) of Theorem 4.3 shows that we must have $D(w) \neq R_{N_D(w)}$.

If a caret is removed in the multiplication wx_1^{-1}, then property (‡) holds, Additionally, if we are not in either of the cases (1) or (3) of this corollary, then we have $D(w) = R_{N_D(w)}$ and $A(w) = L_{N_A(w)}$. In this case, $T_-(w)$ has no interior carets, so $B(w) = \emptyset$ and the caret that is canceled in the multiplication is caret number $N_A(w) + 2$, which must be exposed in $T_+(w)$. It follows from part (4) of Theorem 4.3 that this caret is not the first exposed caret in $T_+(w)$, and so $j(w) < N_A(w) + 2$. However, two consecutive carets cannot be exposed, and we conclude that $j(w) \leq N_A(w)$. Furthermore, if $j(w) = 1$, then caret 1 would be exposed in both $T_-(w)$ and $T_+(w)$ and the tree pair diagram would not be reduced. Hence, $2 \leq j(w) \leq N_A(w)$, and case (2) of the corollary holds.   □

4.2 Defining a partial order on the bad edges

We now define a partial order on the set of all bad edges $e_1(w)$ as required for Theorem 4.2. This partial order is based on numerical measures related to the tree pair diagram for w. These include $N(w)$, as well as $N_A(w)$ and $N_D(w)$, the number of carets in the subtrees $A(w)$ and $D(w)$ defined in Notation 4.5 above. To order the edges $e_1(w)$ and $e_1(w')$ where the values N_A and N_D are the same for both elements, we first need to construct, for each fixed number k, several different partial orderings of the set of all rooted binary trees with k carets. Before explaining these posets, we first need some additional combinatorial information associated to a rooted binary tree.

Definition 4.8 Let T be a rooted, binary tree.

- We order the right carets of T in infix order, and call them $r_1, r_2, r_3, \ldots, r_k$, where r_1 is the root caret of T. Let T_i be the (possibly empty) left subtree of caret r_i. Let $s_r(T) := i$, where i is the smallest index with $0 \leq i \leq k$ satisfying the property that for every $i < t \leq k$, the subtree T_t is empty.
- Similarly, we call the left carets of T, in infix order, $l_m, l_{m-1}, \ldots, l_1$, where l_1 is the root caret of T, and let S_i be the (possibly empty) right subtree of caret l_i. Then let $s_l(T) := i$, where i is the smallest index , $0 \leq i \leq m$, such that S_t is empty for every $i < t \leq m$.
- Let $C_r(T) := N(T) - (k - s_r(T))$ where k is the number of right carets in T; that is, $C_r(T)$ is the number of carets in T up to and including caret $r_{s_r(T)}$.
- Let $C_l(T) := N(T) - (m - s_l(T))$ where m is the number of left carets in T; that is, $C_l(T)$ is the number of carets in T after, and including, caret $l_{s_l(T)}$.

We remark that the simple condition of whether a tree consists either only of right carets or only of left carets, which was critical in recognizing bad edges in Corollary 4.7, simply translates into whether s_r or s_l equals zero. More precisely, the condition $s_r(T) = 0$ (respectively $s_r(T) > 0$) is equivalent to $T = R_{N(T)}$ (respectively $T \neq R_{N(T)}$). Similarly, the condition $s_l(T) = 0$ (respectively $s_l(T) > 0$) is equivalent to $T = L_{N(T)}$ (respectively $T \neq L_{N(T)}$).

In order to sort, rather than simply recognize, the bad edges, however, we need to keep track of the numerical values s_r and s_l.

Consider the set of rooted binary trees with k carets. We define the *right poset of rooted binary trees with k carets* which will be used to order edges $e_1(w)$ where $N_D(w) = k$. For each tree D with k carets with $s_r(D) > 0$, we define the tree $f(D)$ as follows:

- If $s_r(D)$ is odd, and T_1, the left subtree of the root caret of D, is empty, $f(D)$ is the tree formed by rotating D to the left at caret r_1. That is, if g is the element of F with tree pair diagram (D, R_k) where R_k is the tree consisting of k right carets, then gx_0^{-1} has (possibly unreduced) tree pair diagram $(f(D), R_k)$.
- If $s_r(D)$ is odd, and T_1 is not empty, $f(D)$ is the tree formed by rotating D to the right at caret r_1. That is, if g is the element of F with tree pair diagram (D, R_k), then gx_0 has tree pair diagram $(f(D), R_k)$.
- If $s_r(D)$ is even, and T_2, the left subtree of the right child of the root caret of D, is empty, $f(D)$ is the tree formed by rotating D to the left at caret r_2. If g is the element of F with tree pair diagram (D, R_k), then gx_1^{-1} has tree pair diagram $(f(D), R_k)$.
- If $s_r(D)$ is even, and T_2 is not empty, $f(D)$ is the tree formed by rotating D to the right at caret r_2. If g is the element of F with tree pair diagram (D, R_k), then gx_1 has tree pair diagram $(f(D), R_k)$.

Now declare $f(D) <_r D$ for every D. We claim that the transitive closure of this order is a well-founded partial order, with unique minimal element R_k, the tree with k right carets. To see this, notice that $C_r(D) = 0$ if and only if $D = R_k$. Now $C_r(f(D)) \leq C_r(D)$, and if $C_r(f(D)) = C_r(D)$, then $s_r(D)$ and $s_r(f(D))$ have different parities. So if $f^n(D) = D$ for some positive integer n, this implies that there is a word $x_0^{\pm 1} x_1^{\pm 1} \quad x_0^{\pm 1} x_1^{\pm 1}$ (where possibly the first and/or last generators are absent) which is trivial in F, contradicting Lemma 3.4.

Since there are only a finite number of trees with k carets, $C_r(f^m(D)) < C_r(D)$ for some m, and hence $C_r(f^n(D)) = 0$ for some n. Hence, we see that this is a partial order with a unique minimal tree R_k, which is less than all other trees in the poset. We denote the order in this poset by $<_r$.

We now define the *left posets of rooted binary trees with k carets*, which will be used to sort bad edges $e_1(w)$ for which $N_A(w) = k$. Using the method given above, we could have constructed a poset using s_l, S_i, and C_l instead of s_r, T_i, and C_r, replacing the words "rotate left " by "rotate right" and vice-versa. This yields a dual poset, where the minimal element is the tree L_k consisting of only left carets. We denote relationships in this order by $A_1 <_l A_2$.

However, in some cases we will need a modification of this left poset in order to sort our edges, depending on an index $1 \leq j \leq k$. For any natural numbers $k \geq 3$ and $2 \leq j \leq k-1$, let $B_j(k)$ be a tree consisting of k carets, none of which are interior, so that the root caret has infix number $j + 1$. Note that $B_{k-1}(k) = L_k$, the tree consisting of k left carets. In the left poset with order relation $<_l$, there is a unique path from each tree to the minimal element $B_{k-1}(k) = L_k$, and hence there also is a unique (undirected) path from each tree to $B_j(k)$. For each $2 \leq j \leq k-2$, we form a new poset, reordering the trees by declaring $A_1 <_l^j A_2$ if A_1 is on the unique path from A_2 to $B_j(k)$. For each such j, the new poset now has least element $B_j(k)$, and whereas $L_k = B_{k-1}(k) <_l B_{k-2}(k) <_l \quad <_l B_{j+1}(k) <_l B_j(k)$, exactly the reverse holds in $<_l^j$, namely $B_j(k) <_l^j B_{j+1}(k) <_l^j \quad <_l^j B_{k-2}(k) <_l^j B_{k-1}(k) = L_k$. If $j = 1, k-1$ or k, we use the original poset, and declare $<_l^j = <_l$. Thus we have constructed only $k - 2$ distinct posets in all, for each $k \geq 3$. In the trivial cases $k = 1$ and $k = 2$, simply declare $<_l^j = <_l$ for any $1 \leq j \leq k$.

To summarize: for each natural number $k \geq 3$, we have defined $k - 1$ distinct partial orderings of the set of rooted binary trees with k carets. There is a unique right poset which has as minimal element R_k, which will be used to sort bad edges $e_1(w)$ with $N_D(w) = k$; there is a family of $k - 2$ distinct left posets which have, respectively, the trees $B_j(k)$ for $2 \leq j \leq k - 1$ as unique minimal elements, which will be used to sort edges with $N_A(w) = k$ and $j(w) = j$.

The following notation, based on the quantities introduced in Notation 4.5 and Definition 4.8, will simplify the description of the ordering.

**Notation 4.9** Let $e_1(w)$ be a bad edge, for an element $w = (T_-(w), T_+(w)) \in F$.

- Let $s_r(w) := s_r(D(w))$.
- Let $s_l(w) := s_l(A(w))$.
- Let $C_r(w) := C_r(D(w))$.
- Let $C_l(w) := C_l(A(w))$.
- Let $n(w)$ be the infix number of the right caret of $T_-(w)$ whose left subtree is not empty, but whose right subtree is either empty or consists only of right carets. If no such caret exists, $T_-(w)$ consists only of right carets, and we set $n(w) = 0$ (see Fig. 3).

In the following definition, we define a set of comparisons between certain pairs of bad edges. We then prove that the transitive closure of this set of order relationships is a partial order. Some details of this partial order (particularly the fourth set of comparisons) may seem mysterious at this point, but they are exactly the relationships needed for the cell map from the set of bad edges into the 2-cells which is defined in the next section to satisfy the hypotheses of Theorem 4.2.

Definition 4.10 Let $e_1(w)$ and $e_1(z)$ be bad edges. We say $e_1(z) < e_1(w)$ in the following situations:

(1) $N(z) < N(w)$.
(2) $N(z) = N(w)$, $T_+(z) = T_+(w)$, both $s_r(w) > 0$ and $s_r(z) > 0$, and either:

    (a) $N_D(z) < N_D(w)$ and $n(z) = n(w)$, or
    (b) $N_D(z) = N_D(w)$, $n(z) \leq n(w)$, and $D(z) <_r D(w)$.

(3) $N(z) = N(w)$, $T_+(z) = T_+(w)$, $s_r(z) = s_r(w) = 0$, and either:

    (a) $N_A(z) < N_A(w)$ and $n(z) \leq n(w)$ or
    (b) $N_A(w) = N_A(z)$, $n(z) = n(w)$, and $A(z) <_l^j A(w)$ for $j = j(w) = j(z) \leq N_A(w)$.

Fig. 3 In the example of the pair of trees $(T_-(w), T_+(w))$ given in this figure, the subtree $D(w)$ (resp. $A(w)$) has four right (resp. *left*) carets. We compute the following quantities: $N(w) = 15$, $N_D(w) = 7$, $N_A(w) = 5$, $s_r(w) = 2$, $s_l(w) = 1$, $C_r(w) = 5$, $C_l(w) = 2$, $n(w) = 13$ and $j(w) = 3$
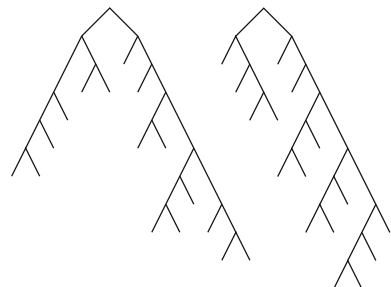
(4) $N(z) = N(w)$, $T_+(z) = T_+(w)$, exactly one of the pair $\{s_r(w), s_r(z)\}$ is zero, and either:

(a) $s_r(z) = 0$, $s_r(w) = 1$ or 2, and $n(z) < n(w)$, or
(b) $s_r(z) = 1$, $s_r(w) = 0$, $n(z) = n(w)$, and $N_A(z) < N_A(w)$.

Lemma 4.11 *The transitive closure of the set of order relationships defined above is a partial order satisfying the property that for all bad edges e, the set of bad edges less than e with respect to this partial order is finite.*

Proof In order to show this is a partial order, we must show that for every set of bad edges satisfying $e_1(w_1) > e_1(w_2) > \quad > e_1(w_n)$, $w_1 \neq w_n$. Suppose $e_1(w_1) > e_1(w_2) > \quad > e_1(w_n)$. If $N(w_i)$ is not constant for all i, then $N(w_n) < N(w_1)$, and so $w_1 \neq w_n$. So we may assume $N = N(w_i)$ for $1 \leq i \leq n$. Next we observe that if $s_r(w_i) > 0$ for every $1 \leq i \leq n$, then for each i either $N_D(w_{i+1}) < N_D(w_i)$, or else $N_D(w_{i+1}) = N_D(w_i)$ and $D(w_{i+1}) <_l D(w_i)$, so $w_1 \neq w_n$. Similarly, if $s_r(w_i) = 0$ for every i, either $N_A(w_{i+1}) < N_A(w_i)$, or else $N_A(w_{i+1}) = N_A(w_i)$ and $A(w_{i+1}) <_l^j A(w_i)$ for $j = j(w_i) = j(w_{i+1})$, so $w_1 \neq w_n$. Therefore, if $w_1 = w_n$ the value of the variable s_r must change twice in the sequence of edges between a strictly positive value and 0. Thus there must be indices for which the value of s_r increases from 0 to a (strictly) positive number and for which the value decreases from positive to 0. In particular, there must be some index i for which $s_r(D(w_i)) = 1$ or 2, $s_r(D(w_{i+1})) = 0$, and $n(w_{i+1}) < n(w_i)$. But since for every index j we have $n(w_{j+1}) \leq n(w_j)$, then $n(w_n) < n(w_1)$, and hence $w_1 \neq w_n$. Finally, since the subset of all edges $e_1(w)$ with a fixed value of $N(w)$ is finite, the finiteness condition is satisfied and this partial order is well-founded.   □

4.3 The mapping from the set of bad edges to the set of 2-cells in the Cayley complex

In this section we define a mapping c from the set of bad edges to the set of 2-cells in the Cayley complex. We will set up the map c so that the bad edge $e_1(w)$ is on the boundary of the cell $c(e_1(w))$.

In order to specify this mapping, we will first define notation for 2-cells in the Cayley complex with a specified basepoint and orientation. For each vertex w and edge $e_1(w)$ in the Cayley complex, there are eight 2-cells containing this edge in their boundaries. For four of these 2-cells, there are 10 edges on the boundary; these are the 2-cells labeled $Rr_1^{\pm 1}(w)$ and $Rl_1^{\pm 1}(w)$ in Fig. 4. For the other four 2-cells whose boundaries contain $e_1(w)$, there are 14 boundary edges; these are the 2-cells labeled $Rr_2^{\pm 1}(w)$ and $Rl_2^{\pm 1}(w)$ in Fig. 5.

In each of these 2-cells, in addition to $e_1(w)$ the boundary contains three other edges of the form $e_1(v)$ for some $v \in F$, and none of the e_1 edges in the boundary of a particular 2-cell are adjacent. The edge $e_1(w)$ will be referred to as the top e_1 edge in these eight 2-cells. The e_1 edges closest to w and wx_1^{-1} are the left and right side edges $e_1(z_l)$ and $e_1(z_r)$, respectively, and the last e_1 edge is the bottom edge $e_1(z_b)$.

For a bad edge $e_1(w)$, the 2-cell $c(e_1(w))$ must be chosen from among these eight cells. The map will be defined so that z_b can be represented by a (not necessarily reduced) tree pair diagram $(T'_-(z_b), T'_+(z_b))$, where the negative trees $T_-(w)$ and $T'_+(z_b)$ differ by a single rotation at a particular caret, and the positive trees satisfy $T_+(w) = T'_+(z_b)$. The notation $Ra_n^{\pm 1}(w)$ (where R stands for relator,) has been motivated by this. The letter $a = l$ or $a = r$ depends on whether the rotation needed to transform $T_-(w)$ to $T'_-(z_b)$ takes place at a left or right caret of $T_-(w)$. The superscript ± 1 takes into account the direction of this rotation, and the subscript n specifies at which caret the rotation takes place. More specifically, in the
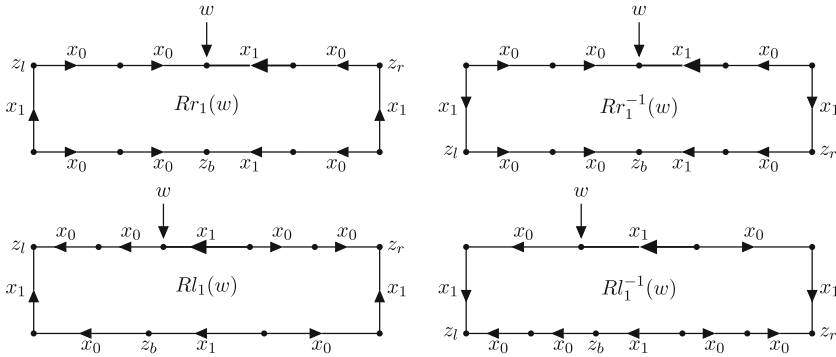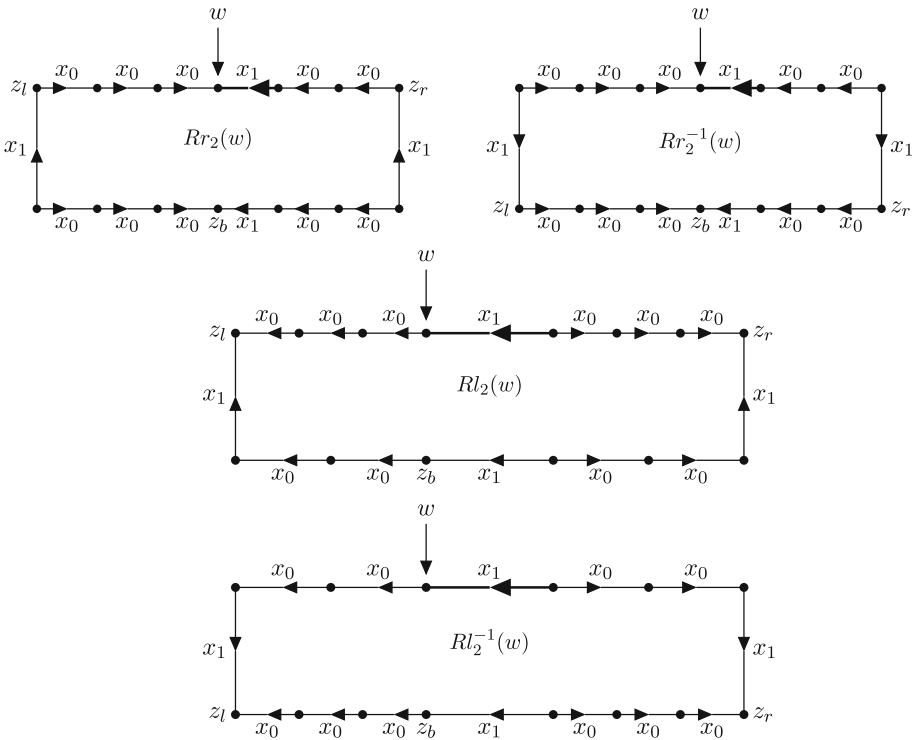
Fig. 4 The four 2-cells $Rr_1^{\pm 1}(w)$ and $Rl_1^{\pm 1}(w)$ with boundary consisting of 10 edges including $e_1(w)$. In each rectangle, the vertices w, z_l, z_r, and z_b are labeled

Fig. 5 The four 2-cells $Rr_2^{\pm 1}(w)$ and $Rl_2^{\pm 1}(w)$ with boundary consisting of 14 edges including $e_1(w)$. In each rectangle, the vertices w, z_l, z_r, and z_b are labeled

case of a rotation at a left caret, $n = 1$ means this caret is the left child of the root of $T_-(w)$, while $n = 2$ means rotation is at the left child of the left child of the root. In the case of a rotation at a right caret, if caret m is the right child of the right child of the root of $T_-(w)$, then $n = 1$ means rotating at the right child of caret m, and $n = 2$ means rotating at the right child of the right child of caret m.

Rewriting the result of Corollary 4.7 using the quantities in Notation 4.9, we have that the bad edge $e_1(w)$ satisfies either $s_r(w) > 0$ or else property ($\ddagger$) holds and either $s_l(w) > 0$ or

$2 \leq j(w) \leq N_A(w)$. It will be useful to re-organize these cases for the definition of the map c, as follows.

Corollary 4.12 *Let $w \in F$. If $e_a(w)$ is a bad edge, then $a = 1$, the tree $T_-(w)$ has at least 3 right carets, and either*

(1) $s_r(w) > 0$,
(2) $s_r(w) = 0$, $s_l(w) \in \{0, 1\}$, *property* (‡) *holds, $N_A(w) \geq 2$, and either*

    (a) $2 \leq j(w) \leq N_A(w) - 1$ *and* $A(w) = B_{j(w)}(N_A(w))$,
    (b) $j(w) = N_A(w)$ *and* $A(w) = B_{N_A(w)-1}(N_A(w))$, *or*
    (c) $2 \leq j(w) \leq N_A(w) - 2$ *and* $A(w) = B_i(N_A(w))$ *with* $j(w)+1 \leq i \leq N_A(w)-1$,

    or

(3) $s_r(w) = 0$, $s_l(w) > 0$, *property* (‡) *holds, and the conditions of case* (2) *are not satisfied.*

The proof of this corollary follows directly from Corollary 4.7, using the fact that when $s_l(w) = 0$ then $A = L_{N_A(w)} = B_{N_A(w)-1}(N_A(w))$, and is left to the reader.

Using these cases, we will choose $c(e_1(w))$ to accomplish the following:

- If $s_r(w) > 0$, then $D(w)$ is not the minimal element $R_{N_D(w)}$ relative to $<_r$; in this case $c(e_1(w))$ is chosen so that either $N(z_b) < N(w)$, or $N(z_b) = N(w)$, $N_D(z_b) = N_D(w)$ and $D(z_b) <_r D(w)$ (see part (1) of the definition below).
- If $s_r(w) = 0$, but $A(w)$ is not the minimal tree relative to $<_l^{j(w)}$, $c(e_1(w))$ is chosen (in parts (2c) and (3) of the definition below) so that either $N(z_b) < N(w)$, or $N(z_b) = N(w)$, $N_A(z_b) = N_A(w)$ and $A(z_b) <_l^{j(w)} A(w)$.
- Finally, if both $A(w)$ and $D(w)$ are minimal, then $c(e_1(w))$ is chosen (in parts (2a) and (2b) of the definition below) so that caret $j(w)$ is removed in moving around the 2-cell from w to z_b, so $N(z_b) < N(w)$.

Definition 4.13 We define a map c from the set of bad edges to the set of 2-cells in several cases. Consider a bad edge $e_1(w)$, and let $k = N_A(w)$. Let T_1 be the left subtree of the root of $D(w)$, and let T_2 be the left subtree of the right child of the root of $D(w)$. Similarly, let S_1 be the right subtree of the root caret of $A(w)$, and let S_2 be the right subtree of the left child of the root caret of $A(w)$.

(1)   If $s_r(w) > 0$ and:

    - If $s_r(w)$ is odd, and T_1 is empty, let $c(e_1(w)) := Rr_1(w)$.
    - If $s_r(w)$ is odd, and T_1 is not empty, let $c(e_1(w)) := Rr_1^{-1}(w)$.
    - If $s_r(w)$ is even, and T_2 is empty, let $c(e_1(w)) := Rr_2(w)$.
    - If $s_r(w)$ is even, and T_2 is not empty, let $c(e_1(w)) := Rr_2^{-1}(w)$.

(2)   If $s_r(w) = 0$, $s_l(w) \in \{0, 1\}$, property (‡) holds, $k \geq 2$, and:

    (a) If $2 \leq j(w) \leq k - 1$ and $A(w) = B_{j(w)}(k)$, then let $c(e_1(w)) := Rl_2(w)$.
    (b) If $j(w) = k$ and $A(w) = B_{k-1}(k)$, then let $c(e_1(w)) := Rl_1(w)$.
    (c) If $2 \leq j(w) \leq k - 2$ and $A(w) = B_i(k)$ for $j(w) + 1 \leq i \leq k - 1$, let $c(e_1(w)) := Rl_1(w)$.

(3)   If $s_r(w) = 0$, $s_l(A) > 0$, property (‡) holds, the conditions of case (2) are not satisfied, and:

    - If $s_l(w)$ is odd, and S_1 is empty, then let $c(e_1(w)) := Rl_1(w)$.

- If $s_l(w)$ is odd, and S_1 is not empty, let $c(e_1(w)) := Rl_1^{-1}(w)$.
- If $s_l(w)$ is even, and S_2 is empty, let $c(e_1(w)) := Rl_2(w)$.
- If $s_l(w)$ is even, and S_2 is not empty, let $c(e_1(w)) := Rl_2^{-1}(w)$.

See Figs. 6 and 8 for examples of bad edges and their corresponding two cells. Figures 7 and 9 show the tree pair diagrams corresponding to the elements w and z_b, where $e_1(z_b)$ is the edge across the two-cell from the bad edge $e_1(w)$.

In the following theorem, we verify that the map defined above and the partial order on the set of bad edges satisfy the hypothesis of Theorem 4.2. In addition, we prove another

Fig. 6 The 2-cell corresponding to the bad edge $e_1(x_5^{-1}x_3^{-2})$, where $z_b = x_5^{-1}x_3^{-1}$
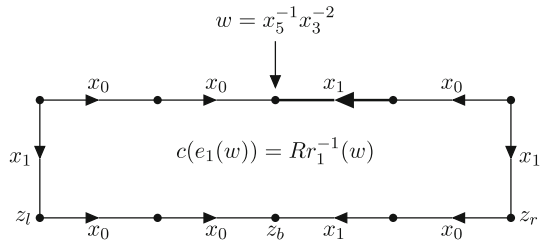

Fig. 7 The left (negative) trees from the pair diagrams corresponding to $w = x_5^{-1}x_3^{-2}$ and $z_b = x_5^{-1}x_3^{-1}$. Notice that these two trees differ by a rotation at the root caret of the subtree $D(w)$
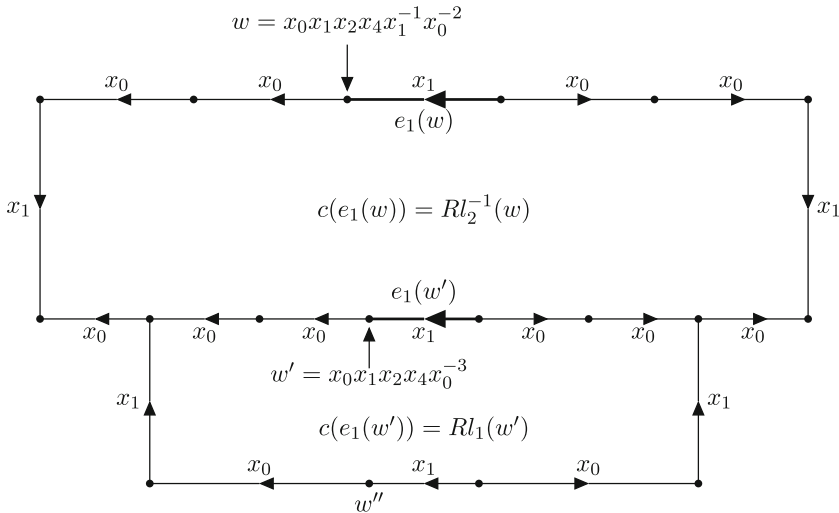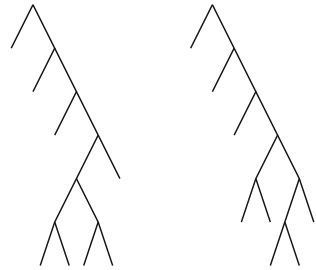

Fig. 8 The 2-cells corresponding to the bad edges $e_1(w)$ and $e_1(w')$, for $w = x_0x_1x_2x_4x_1^{-1}x_0^{-2}$ and $w' = x_0x_1x_2x_4x_0^{-3}$. The edge across the bottom 2-cell from $e_1(w')$ is $e_1(w'')$ where $w'' = x_0x_1x_3x_0^{-2}$
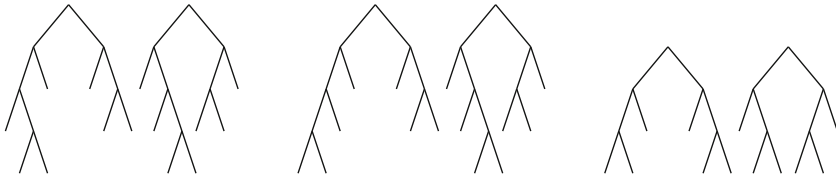
Fig. 9 The tree pair diagrams corresponding to $w = x_0x_1x_2x_4x_1^{-1}x_0^{-2}$, $w' = x_0x_1x_2x_4x_0^{-3}$, and $w'' = x_0x_1x_3x_0^{-2}$ which are labeled in Fig. 8

fact which will be used later in showing that the combing satisfies a linear radial tameness function.

Theorem 4.14 *If $e_1(w)$ is a bad edge, then all other vertices z on the boundary of $c(e_1(w))$ have $N(z) \leq N(w)$. Furthermore, every edge of the form $e_1(z)$ along the boundary is either a good edge, or precedes $e_1(w)$ in the ordering of the bad edges.*

Proof Let $e_1(z_b)$ be the bottom e_1 edge in the 2-cell $c(e_1(w))$, and $e_1(z_l)$ (respectively $e_1(z_r)$) be the left (respectively right) side e_1 edges. The first statement in the theorem is a consequence of the following observation. The tree $T_-(w)$ has enough carets in the left subtree of the root caret, and in both subtrees of the right child of the root caret to ensure that as we read around $c(e_1(w))$ to the left, starting from w, terminating at z_b, and form the successive products, no carets ever need to be added to the tree pair diagrams in order to perform these multiplications. The same holds for the path from wx_1^{-1}, around to the right ending at $z_bx_1^{-1}$. Since $N(wx_1^{-1}) \leq N(w)$, it follows that for each vertex z of $c(e_1(w))$, $N(z) \leq N(w)$. In addition, if $N(z) = N(w)$, then $T_+(z) = T_+(w)$.

To prove the second statement of the theorem, we proceed by cases according to the size of $s_r(w)$. In each case we show that $e_1(z) < e_1(w)$, or else $e_1(z)$ is a good edge. We consider separately the three subcases of $e_1(z)$ for $z \in \{z_b, z_l, z_r\}$.

Case 1: $s_r(w) > 0$. In this case $c(e_1(w)) = Rr_n^{\pm 1}$ for $n \in \{1, 2\}$. Also, note that $n(w) = N(w) - N_D(w) + C_r(w)$.

(a) $z = z_l$. In this case either:

- $N(z_l) < N(w)$ (and $e_1(z_l) < e_1(w)$ by (1) of Definition 4.10 if $e_1(z_l)$ is a bad edge), or
- $N(z_l) = N(w)$, $s_r(z_l) > 0$, $N_D(z_l) < N_D(w)$ and $n(z_l) = n(w)$ (and $e_1(z_l) < e_1(w)$ by (2a) of Definition 4.10 if $e_1(z_l)$ is a bad edge), or
- $N(z_l) = N(w)$ and $s_r(z) = 0$. But one checks that if $s_r(z) = 0$, then $c(e_1(w)) = Rr_n^{-1}, n \in \{1, 2\}$, and $s_r(w) = n$. But since no carets are ever added in moving from $z_lx_1^{-1}$ to z_l, $e_1(z_l)$ is a good edge.

(b) $z = z_r$. If it is not the case that $N(z_r) < N(w)$, then it is easily checked through the definition of $Rr_n^{\pm 1}(w)$ that $T_-(z_l)$ and $T_-(z_r)$ differ only in the configuration of the carets in the left subtree of the root. Therefore, the argument for $e_1(z_l)$ goes through exactly, replacing z_l by z_r.

(c) $z = z_b$. In this case either:

- $N(z_b) < N(w)$ (and $e_1(z_b) < e_1(w)$ by (1) of Definition 4.10 if $e_1(z_b)$ is a bad edge), or

- $N(z_b) = N(w)$ and $s_r(z_b) > 0$, in which case $N_D(z_b) = N_D(w)$, and $D(z_b) <_r D(w)$. Then $C_r(z_b) \le C_r(w)$, which implies that $n(z_b) \le n(w)$, (and $e_1(z_b) < e_1(w)$ by (2b) of Definition 4.10 if $e_1(z_b)$ is a bad edge), or
- $N(z_b) = N(w)$ and $s_r(z_b) = 0$. However, this can happen only when $s_r(w) = n$ for $n \in \{1, 2\}$, $c(e_1(w)) = Rr_n^{-1}(w)$, and $n(z_b) < (n_w)$ (and $e_1(z_b) < e_1(w)$ by (4a) of Definition 4.10 if $e_1(z_b)$ is a bad edge).

Case 2: $s_r(w) = 0$. In this case, $c(e_1(w)) = Rl_n^{\pm 1}$ for $n \in \{0, 1\}$. Also, note that $n(w) = N_A(w) + 1$.

(a) $z = z_l$. In this case, $N_A(z_l) < N_A(w)$. Now either:

- $N(z_l) < N(w)$ (and $e_1(z_l) < e_1(w)$ by (1) of Definition 4.10 if $e_1(z_l)$ is a bad edge), or
- $N(z_l) = N(w)$ and $s_r(z_l) = 0$, and hence $n(z) \le n(w)$ (and $e_1(z_l) < e_1(w)$ by (3a) of Definition 4.10 if $e_1(z_l)$ is a bad edge), or
- $N(z_l) = N(w)$ and $s_r(z_l) > 0$. However, this only occurs if $c(e_1(w)) = Rl_2^{\pm 1}(w)$, and then $s_r(z_l) = 1$ and $n(z_l) = n(w)$ (and $e_1(z_l) < e_1(w)$ by (4b) of Definition 4.10 if $e_1(z_l)$ is a bad edge).

(b) $z = z_r$. If $e_1(z_r)$ is a bad edge, then $s_r(w) = 0$ implies that property ($\ddagger$) holds. In this case, $N(z_r) < N(w)$ because a caret is removed when moving from w to wx_1^{-1}.

(c) $z = z_b$. Then either:

- In cases (2a) and (2b) of Definition 4.13, $N(z_b) < N(w)$, since caret $j(w)$ is removed in moving from z_l to $z_l x_1^{-1}$ (and $e_1(z_b) < e_1(w)$ by (1) of Definition 4.10 if $e_1(z_b)$ is a bad edge).
- In cases (2c) and (3) of Definition 4.13, either $N(z_b) < N(w)$ (and $e_1(z_b) < e_1(w)$ by (1) of Definition 4.10 if $e_1(z_b)$ is a bad edge), or $N(z_b) = N(w)$, $s_r(z_b) = 0$, $N_A(z_b) = N_A(w)$, $n(z_b) = n(w)$, and $A(z_b) <_l^{j(w)} A(w)$ (and $e_1(z_b) < e_1(w)$ by (3b) of Definition 4.10 if $e_1(z_b)$ is a bad edge). □

Since all hypotheses of Theorem 4.2 have now been verified, Theorem 4.2 shows that the nested traversal 0-combing Ψ extends to a 1-combing $\Psi : X^1 \times [0, 1] \to X$.

5 The combing of F satisfies a linear radial tameness function

The fact that our combing Ψ satisfies a linear radial tameness function will follow from the fact that the number of carets in the tree pair diagrams representing the vertices along a nested traversal normal form path never decreases, and from the close relationship between word length over the alphabet $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$ and the number of carets. First, we extend the concept of the number of carets in a tree pair diagram from $F = X^0$ to all of X.

Definition 5.1 For any $x \in X$, we define $N_{\text{Max}}(x)$ and $N_{\text{Min}}(x)$ by cases.

(1) If $x \in X^0$, then $x = g \in F$, and we let $N_{\text{Max}}(x) = N_{\text{Min}}(x) = N(g)$, the number of carets in either tree of a reduced tree pair diagram for g.
(2) If $x \in X^1 - X^0$, then x is on the interior of some edge, with vertices $g, h \in X^0$. Then define $N_{\text{Max}}(x) = \text{Max}(N(g), N(h))$, and $N_{\text{Min}}(x) = \text{Min}(N(g), N(h))$.
(3) If $x \in X - X^1$, then x is in the interior of some 2-cell, with vertices $g_1, g_2, \ldots, g_n$ along the boundary. Then we define $N_{\text{Max}}(x) = \text{Max}(N(g_1), N(g_2), \ldots, N(g_n))$, and $N_{\text{Min}}(x) = \text{Min}(N(g_1), N(g_2), \ldots, N(g_n))$.

The following lemma proves that using this expanded notion of the number of carets of $x \in X$, the number of carets does not decrease along the combing paths defined by Ψ.

Lemma 5.2 *For any $x \in X^1$ and $0 \leq s < t \leq 1$, we have $N_{\mathrm{Max}}(\Psi(x, s)) \leq N_{\mathrm{Max}}(\Psi(x, t))$, where Ψ is the 1-combing defined in Sect. 4.*

Proof In the case where $x \in X^0$, from Theorem 3.5 we know that along the nested traversal normal form $\eta(x) = a_1 a_2 \ldots a_n$, we have $N(a_1 a_2 \quad a_i) \leq N(a_1 a_2 \quad a_{i+1})$. For $x \in X^1 - X^0$, if x is in the interior of a good edge the conclusion of this lemma follows from the previous sentence. If x is in the interior of a bad edge e, then the inequality follows from Noetherian induction and the fact that for y on any bad edge e and z on the complement of the edge e in the closure of the 2-cell $c(e)$, we have $N_{\mathrm{Max}}(z) \leq N_{\mathrm{Max}}(y)$ as shown in Theorem 4.14. $\qquad\square$

The next lemma relates the level of $x \in X$ to the quantities $N_{\mathrm{Min}}(x)$ and $N_{\mathrm{Max}}(x)$. Recall that when $x \in X^0$, the level of x and $l_A(x)$, the word length of x with respect to A, are identical. The lengths of the two relators in this presentation are 10 and 14, so the constant c used in defining the level of a point in the interior of a 2-cell of the Cayley complex for this presentation of F will be $c = 4(10)(14) + 1$.

Lemma 5.3 *For any $x \in X$ we have*

$$N_{\mathrm{Min}}(x) - 2 \leq \mathrm{lev}(x) < 4N_{\mathrm{Max}}(x) + 1$$

and additionally $N_{\mathrm{Max}}(x) - N_{\mathrm{Min}}(x) \leq 9$.

Proof When $x \in X^0$, Lemma 3.3 gives $N(x) - 2 \leq l_A(x) \leq 4N(x)$. Therefore

$$N_{\mathrm{Min}}(x) - 2 = N(x) - 2 \leq \mathrm{lev}(x) = l_A(x) \leq 4N(x) = 4N_{\mathrm{Max}}(x) < 4N_{\mathrm{Max}}(x) + 1$$

for $x \in X^0$. If, on the other hand, $x \in X^1 - X^0$, then x is in the interior of an edge, whose endpoints are $g, h \in F$. Now

$$\mathrm{lev}(x) = \frac{\mathrm{lev}(g) + \mathrm{lev}(h)}{2} + \frac{1}{4} \leq \frac{4N(g) + 4N(h)}{2} + \frac{1}{4} \leq 4\mathrm{Max}(N(g), N(h)) + \frac{1}{4}$$

$$= 4N_{\mathrm{Max}}(x) + \frac{1}{4} < 4N_{\mathrm{Max}}(x) + 1.$$

But, on the other hand,

$$\mathrm{lev}(x) \geq \frac{N(g) + N(h)}{2} - 2 + \frac{1}{4}$$

$$\geq \mathrm{Min}(N(g), N(h)) - 2 + \frac{1}{4} = N_{\mathrm{Min}}(x) - 2 + \frac{1}{4} \geq N_{\mathrm{Min}}(x) - 2.$$

In summary, in this case, we have

$$N_{\mathrm{Min}}(x) - 2 \leq \mathrm{lev}(x) \leq 4N_{\mathrm{Max}}(x) + \frac{1}{4} < 4N_{\mathrm{Max}}(x) + 1.$$

And finally, if $x \in X - X^1$, x is in the interior of some 2-cell, with vertices $g_1, g_2, \ldots, g_k$ on its boundary, then

$$\mathrm{lev}(x) = \frac{\mathrm{lev}(g_1) + \quad + \mathrm{lev}(h)}{k} + \frac{1}{4} + \frac{1}{c} \leq \frac{4(N(g_1) + \quad + N(g_k))}{k} + \frac{1}{4} + \frac{1}{c}$$

$$\leq 4\mathrm{Max}(N(g_1), \ldots, N(g_k)) + \frac{1}{4} + \frac{1}{c} = 4N_{\mathrm{Max}}(x) + \frac{1}{4} + \frac{1}{c} < 4N_{\mathrm{Max}}(x) + 1.$$

On the other hand,

$$\text{lev}(x) \geq \frac{N(g_1) + \cdots + N(g_k)}{k} - 2 + \frac{1}{4} \geq \text{Min}(N(g_1), \ldots, N(g_k)) - 2 + \frac{1}{4} + \frac{1}{c}$$

$$= N_{\text{Min}}(x) - 2 + \frac{1}{4} + \frac{1}{c} \geq N_{\text{Min}}(x) - 2.$$

And so, in this case, we have

$$N_{\text{Min}}(x) - 2 \leq \text{lev}(x) \leq 4N_{\text{Max}}(x) + \frac{1}{4} + \frac{1}{c} < 4N_{\text{Max}}(x) + 1.$$

This establishes the first statement of the lemma. Now if $x \in X^0$, $N_{\text{Max}}(x) = N_{\text{Min}}(x)$. For $x \in X^1 - X^0$, $N_{\text{Max}}(x) - N_{\text{Min}}(x) \leq 2$, since one either needs to add at most 1 caret (or can cancel at most one caret) when multiplying by $x_0^{\pm 1}$, and one needs to add at most two carets (or can cancel at most two carets) when multiplying by $x_1^{\pm 1}$. Now the relators in our presentation of F have length either 10 or 14, and two vertices v and w on the boundary of a relator can be at most seven edges apart. Furthermore, examining the relators, we see that at most two of these seven edges correspond to multiplication by $x_1^{\pm 1}$. Therefore, for $x \in X - X^1$, $N_{\text{Max}}(x) - N_{\text{Min}}(x) \leq 2(2) + 5 = 9$. □

We are now able to prove that the combing Ψ defined in Sect. 4 satisfies a linear radial tameness function.

Theorem 5.4 *Thompson's group F is in TC_ρ with ρ linear. More specifically, the Cayley complex of the presentation $F = \langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$ has a 1-combing admitting a radial tameness function of $\rho(q) = 4q + 45$.*

Proof Let $\Psi : X^1 \times [0, 1] \to X$ be the 1-combing of F constructed in the previous section. Suppose that $x \in X^1, 0 \leq s < t \leq 1$, and $\text{lev}(\Psi(x, s)) > 4q + 45$. In Lemma 5.3 we have shown that $\text{lev}(\Psi(x, s)) < 4N_{\text{Max}}(\Psi(x, s)) + 1$, which implies that $4N_{\text{Max}}(\Psi(x, s)) > 4q + 44$, or $N_{\text{Max}}(\Psi(x, s)) > q + 11$. From Lemma 5.2 we have $N_{\text{Max}}(\Psi(x, t)) \geq N_{\text{Max}}(\Psi(x, s))$, and so $N_{\text{Max}}(\Psi(x, t)) > q + 11$. The last statement in Lemma 5.3 also shows that $N_{\text{Max}}(\Psi(x, t)) - N_{\text{Min}}(\Psi(x, t)) \leq 9$, and so $N_{\text{Min}}(\Psi(x, t)) > q + 2$. Using Lemma 5.3 once more, we obtain $\text{lev}(\Psi(x, t)) > q$. □

6 Linear tame combing for $BS(1, p)$

In this section we prove the following.

Theorem 6.1 *For every natural number $p \geq 3$, the group $BS(1, p)$ is in the class TC_ρ with ρ linear. Moreover, the Cayley complex for the presentation $BS(1, p) = \langle a, t | tat^{-1} = a^p \rangle$ has a 1-combing admitting a radial tameness function of $\rho(q) = 4(h + 2)q + [4(h + 2) + 1](h + 4)$ where $h = \lfloor \frac{p}{2} \rfloor$.*

Throughout this section, let $p \geq 3$, $G = BS(1, p)$, $h = \lfloor \frac{p}{2} \rfloor$, and $A = \{a^{\pm 1}, t^{\pm 1}\}$. Let X denote the Cayley 2-complex associated with the presentation $\langle a, t | tat^{-1} = a^p \rangle$ of G. Give the Cayley graph $X^1 = \Gamma(G, A)$ the path metric. For a word v over the alphabet A, let $l(v)$ denote the length of the word v, and let $l_\Gamma(v)$ denote the length of a geodesic (with respect to the path metric on Γ) representative of the element of G represented by v. Each element of G has a particularly simple, not necessarily geodesic, normal form $t^{-m} a^j t^s$ with

$m \geq 0$ and $s \geq 0$. The combing we construct will be based on this set of normal forms. The following lemma, which characterizes a set of geodesics for elements of G and relates them to the normal forms above, is a direct consequence of Elder and Hermiller [6, Prop. 2.3].

Lemma 6.2 *Let g be an arbitrary element of G. Then g is represented by a geodesic word w satisfying one of the following.*

(1) $w = t^k a^{i_k} t^{-1} a^{i_{k-1}} t^{-1} \quad t^{-1} a^{i_{-m}} t^s$ *with* $0 \leq s \leq m, 0 < m, -m \leq k, |i_l| \leq h$ *for* $-m \leq l \leq k - 1, |i_k| \leq h + 1,$ *and either* $1 \leq |i_k|$ *or* $(k = -m$ *and* $s = 0)$.

(2) $w = t^{-m} a^{i_{-m}} t a^{i_{-m+1}} t \quad t a^{i_k} t^{s-m-k}$ *with* $0 \leq m \leq s, -m \leq k, |i_l| \leq h$ *for* $-m \leq l \leq k - 1, 1 \leq |i_k| \leq h + 1,$ *and either* $1 \leq |i_k|$ *or* $k = -m = 0$.

Moreover, in each case, g also has a (not necessarily geodesic) representative of the form $t^{-m} a^j t^s$ with $m \geq 0, s \geq 0,$ and $j = i_{-m} + i_{-m+1} p + \quad + i_k p^{k+m} \in \mathbb{Z}$.

The next two lemmas show that lower and upper bounds on the length of a geodesic representative of $t^{-m} a^j t^s$ in the Cayley graph imply lower and upper bounds, respectively, on the value of $|j|$.

Lemma 6.3 *If $0 \leq m < n, 0 \leq s < n, h + 2 < B,$ and $l_\Gamma(t^{-m} a^j t^s) > Bn,$ then $|j| > p^{(\frac{1}{h+2} B - 2)n}$.*

Proof We will prove the contrapositive; suppose that $0 \leq m < n, 0 \leq s < n, h + 2 < B,$ and $|j| \leq p^{(\frac{1}{h+2} B - 2)n}$. Let w be a word in one of the forms (1)–(2) from Lemma 6.2 that is a geodesic representative of the element of G that is also represented by $t^{-m} a^j t^s$. As w is a geodesic, it follows that $l_\Gamma(t^{-m} a^j t^s)$ is simply the length $l(w)$ of the word w.

First note that if $i_k = 0$, then $j = 0$ and either $s = 0$ or $m = 0$. In both of these instances, we have $l_\Gamma(t^{-m} a^j t^s) < n < Bn$.

For the rest of the proof we suppose that $|i_k| \geq 1$. In both cases (1)–(2), we have $|j| = |i_{-m} + i_{-m+1} p + \quad + i_k p^{k+m}| \leq p^{(\frac{1}{h+2} B - 2)n}$, and hence $|i_k| p^{k+m} - | \sum_{l=-m}^{k-1} i_l p^{l+m}| \leq p^{(\frac{1}{h+2} B - 2)n}$. Since each $|i_l| \leq h$ for $-m \leq l \leq k - 1$, then $|\sum_{l=-m}^{k-1} i_l p^{l+m}| \leq \sum_{l=-m}^{k-1} h p^{l+m} = h \frac{p^{k+m}-1}{p-1} < \frac{2}{3} p^{k+m}$, where the last inequality uses the hypothesis that $p \geq 3$. Plugging this into the previous inequality, and using the fact that $|i_k| \geq 1$, gives $\frac{1}{3} p^{k+m} \leq |i_k| p^{k+m} - \frac{2}{3} p^{k+m} < p^{(\frac{1}{h+2} B - 2)n}$. Then $p^{k+m-(\frac{1}{h+2} B - 2)n} < 3$, and so $k + m - (\frac{1}{h+2} B - 2)n \leq 0$. Since $0 \leq m$, this gives $k \leq (\frac{1}{h+2} B - 2)n$.

If w is of the form in (1) with $k > 0$, then

$$l(w) = 2k + m + s + |i_{-m}| + \quad + |i_k| < 2\left(\frac{1}{h+2} B - 2\right) n$$
$$+ n + n + h(k - 1 + m) + h + 1$$
$$< 2\left(\frac{1}{h+2} B\right) n + h\left(\left(\frac{1}{h+2} B - 2\right) n + n\right) < Bn.$$

If w is of the form in (1) with $k \leq 0$, then

$$l(w) = m + s + |i_{-m}| + \quad + |i_k| < n + n + h(m - 1) + h + 1 < (h + 2)n + 1.$$

For w of the form in (2) with $k > s - m$, we have

$$l(w) = 2m + k + (k - s + m) + |i_{-m}| + \quad + |i_k|$$
$$< 2n + 2\left(\frac{1}{h+2}B - 2\right)n + h(m + k - 1) + h + 1$$
$$\leq 2\left(\frac{1}{h+2}B\right)n + h\left(n + \left(\frac{1}{h+2}B - 2\right)n\right) < Bn.$$

And finally, for w of the form in (2) with $k \leq s - m$, we have

$$l(w) = m + s + |i_{-m}| + \quad + |i_k| < n + n + h(k - 1 + m) + h + 1 < 2n + hs + 1$$
$$< 2n + hn + 1 = (h + 2)n + 1.$$

Hence, in all possible cases, $l_\Gamma(t^{-m}a^j t^s) = l(w) < Bn + 1$, and so this nonnegative integer satisfies $l_\Gamma(t^{-m}a^j t^s) \leq Bn$. ☐

Lemma 6.4 *If* $0 \leq m < n, 0 \leq s < n, 1 < E$, *and* $|j| > p^{En}$, *then* $l_\Gamma(t^{-m}a^j t^s) > (E - 1)n$.

Proof Suppose that $0 \leq m < n, 0 \leq s < n, 1 < E$, and $|j| > p^{En}$. Let w be a word in one of the forms (1)–(2) from Lemma 6.2 that is a geodesic representative of the element of G also represented by $t^{-m}a^j t^s$.

In both cases, we have $p^{En} < |j| = |i_{-m} + i_{-m+1}p + \quad + i_k p^{k+m}|$, and in particular we must have $|i_k| \geq 1$. Using the fact that $|\sum_{l=-m}^{k-1} i_l p^{l+m}| < \frac{2}{3}p^{k+m}$ (see the proof of Lemma 6.3) and the inequality $|i_k| \leq h+1$ yields $p^{En} < \frac{2}{3}p^{k+m} + |i_k|p^{k+m} < (h+2)p^{k+m}$. Since $p \geq 3$, this gives $p^{En-k-m} < h + 2 \leq p$, and so $En - k - m \leq 0$. Then $(E - 1)n < En - m \leq k$.

Note that the inequality $(E - 1)n < k$ implies that $0 < k$. We again consider the length of w in each case.

If w is of the form in (1), then

$$l(w) = 2k + m + s + |i_{-m}| + \quad + |i_k| > 2(E - 1)n + 0 + 0 + 1.$$

For w of the form (2) with $k > s - m$, we have

$$l(w) = 2m + k + (k - s + m) + |i_{-m}| + \quad + |i_k| > 0 + (E - 1)n + 0 + 1.$$

And finally, for w in form (2) with $k \leq s - m$, we have $k \leq s$ and hence

$$l(w) = m + s + |i_{-m}| + \quad + |i_k| > 0 + k + (s - k) + 1 > (E - 1)n + 0 + 1.$$

Thus in all possible cases we have $l_\Gamma(t^{-m}a^j t^s) = l(w) > (E - 1)n$. ☐

The Cayley complex X can be constructed using rectangles homeomorphic to $[0, 1] \times [0, 1]$, with the top labeled a and oriented to the right, the bottom labeled a^p and also oriented to the right, and the left and right sides labeled t and oriented upward. Gluing these rectangles along commonly labeled and oriented sides, the Cayley complex X is homeomorphic to the product $\mathbb{R} \times T$ of the real line with a tree T. The projection maps $\pi : X \to \mathbb{R}$ and $\pi_T : X \to T$ are continuous, and we can write a point $x \in X$ uniquely as $[\pi(x), \pi_T(x)]$.

The vertices of T are the projections via π_T of the vertices of X. Two vertices of X project to the same vertex of T if and only if there is a path in X^1 labeled by a power of a between the two vertices. Each edge of T can be considered as oriented upward with a label t, the

projection under π_T of edges labeled by t in the Cayley complex. Each vertex of T is the initial vertex for p edges and the terminal vertex for one edge.

The projection $\pi_{\mathbb{R}}$ maps a vertex $t^{-m}a^j t^s$ to the real number jp^{-m}. The points on a vertical edge between vertices $t^{-m}a^j t^s$ and $t^{-m}a^j t^{s+1}$ also all map under $\pi_{\mathbb{R}}$ to jp^{-m}, and the projection $\pi_{\mathbb{R}}$ maps the horizontal edge from $t^{-m}a^j t^s$ to $t^{-m}a^j t^s a$ homeomorphically to the interval from jp^{-m} to $jp^{-m} + p^{-m+s}$.

On a rectangular ($[0, 1] \times [0, 1]$) 2-cell, the top left and top right vertices have the form $[jp^{-m}, z]$ and $[jp^{-m} + p^{-m+s}, z]$, respectively. Two points $x = [jp^{-m}, \pi_T(x)]$ and $y = [jp^{-m} + p^{-m+s}, \pi_T(y)]$ on the left and right sides of this 2-cell, respectively, determine a horizontal line segment if $\pi_T(x) = \pi_T(y)$ is a point on the unique edge in the tree T oriented toward the vertex z. The projection $\pi_{\mathbb{R}}$ maps this horizontal line segment homeomorphically to the interval from jp^{-m} to $jp^{-m} + p^{-m+s}$ in $\mathbb{R}$, and the projection π_T is constant on this segment. Let z' be the initial vertex of the edge in T whose terminus is z, and let r be any real number in the interval from jp^{-m} to $jp^{-m} + p^{-m+s}$. The two points $[r, z']$ and $[r, z]$ are on the bottom and top sides of this 2-cell, respectively, and they determine a vertical line segment in the 2-cell which maps via $\pi_{\mathbb{R}}$ to r, and which maps via π_T homeomorphically to the edge from z' to z.

It will frequently be useful to move from points in the interiors of 1-cells or 2-cells to vertices in the Cayley complex. If y is a vertex in X, let $\tilde{y} := y$. If y is in the interior of a 1-cell in X labeled t directed upward, then let $\tilde{y}$ be the initial vertex of that edge. If y is in the interior of a 1-cell in X labeled a directed rightward, then let $\tilde{y}$ be the endpoint of that edge whose image under $\pi_{\mathbb{R}}$ has the maximum absolute value. Finally if y is in the interior of a 2-cell, let $\tilde{y}$ be the bottom (left or right) corner of that rectangular 2-cell whose image under $\pi_{\mathbb{R}}$ has the maximum absolute value. In a 2-cell there are $p + 3$ vertices, so the difference in levels of vertices in that cell is at most $h + 2$, resulting in a bound on the difference between the level of the 2-cell and the level of any vertex in that cell, as well. Then in all cases, we have $\tilde{y} \in X^0$ and $|\text{lev}(\tilde{y}) - \text{lev}(y)| < h + 3$. We will call $\tilde{y}$ the *vertex associated to* y.

More information on Cayley complexes for Baumslag–Solitar groups can be found in the references by Epstein et al. [7] or Elder and Hermiller [6].

Next, we apply the lemmas above to prove Theorem 6.1.

Proof We first define a 1-combing $\Psi : X^1 \times [0, 1] \to X$ as follows. Let ϵ denote the identity vertex in X.

Let x be an arbitrary point in X^1. Since T is a tree, there is a unique geodesic in T from $\pi_T(\epsilon)$ to $\pi_T(x)$. This geodesic first follows a (possibly empty) edge path in the direction opposite to each edge orientation from $\pi_T(\epsilon)$ down to a point $z(x)$ (which we will call the *nadir* of x), and then follows a (possibly empty) edge path in the same direction as each edge orientation up to $\pi_T(x)$. If $z(x) \neq \pi_T(x)$ so that the upward portion is nonconstant, then the nadir $z(x)$ must be a vertex of T. Let the path $p_x : [0, \frac{1}{3}] \to T$ follow the geodesic from $\pi_T(\epsilon)$ to $z(x)$ with constant speed, and let the path $q_x : [\frac{2}{3}, 1] \to T$ follow the geodesic from $z(x)$ to $\pi_T(x)$ with constant speed (with respect to the path metric on T).

Define the path $\Psi : \{x\} \times [0, 1] \to X$ by $\Psi(x, u) = [0, p_x(u)]$ for $u \in [0, \frac{1}{3}]$, $\Psi(x, u) = [3(u - \frac{1}{3})\pi_{\mathbb{R}}(x), z(x)]$ for $u \in [\frac{1}{3}, \frac{2}{3}]$, and $\Psi(x, u) = [\pi_{\mathbb{R}}(x), q_x(u)]$ for $u \in [\frac{2}{3}, 1]$. In the first third of the interval this path goes directly downward, in the second third it travels horizontally, and in the last third it goes directly upward in the Cayley complex X. Note that some of these three component paths may be constant. We will refer to this path as the *DHU-path* for x.

Continuity of the function $\Psi : X^1 \times [0, 1] \to X$ defined by these DHU-paths follows from the continuity of the two projection functions $\pi_{\mathbb{R}}$ and π_T. For a vertex $x \in X^0$ regarded as an

element of G, the representative $t^{-m}a^j t^s$ of x from Lemma 6.2 satisfies $z(x) = \pi_T(t^{-m})$, and so the path $\Psi : \{x\} \times [0, 1] \to X^1$ follows the edge path labeled by the word $t^{-m}a^j t^s$ and remains in the 1-skeleton of X. Hence Ψ is a 1-combing, which we will refer to as the *DHU-combing*.

In order to show that the DHU-combing satisfies a linear radial tameness function, we will show that for the constants $B := 4(h + 2)$ and $C := (h + 4)(B + 1)$, whenever $x \in X^1, 0 \le b < c \le 1, 0 \le q \in \mathbb{Q}$, $\mathrm{lev}(\Psi(x, b)) > Bq + C$, and $\mathrm{lev}(\Psi(x, c)) \le q$, we have a contradiction.

To that end, fix a point x in $X^1, 0 \le b < c \le 1$, and $0 \le q \in \mathbb{Q}$. Let $v := \Psi(x, b)$, $w := \Psi(x, c)$, $\sigma := \Psi(x, \frac{1}{3})$, and $\tau := \Psi(x, \frac{2}{3})$, and assume that $\mathrm{lev}(v) > Bq + C$ and $\mathrm{lev}(w) \le q$.
Case I. Suppose that $w \in \Psi(\{x\} \times [0, \frac{1}{3}])$. Then both v and w are points on the downward portion of the DHU-path for x, on the infinite ray labeled $t^{-\infty}$ going down from ϵ in X. Now t^{-m} is a geodesic in the Cayley graph for all $m \ge 0$, and so traveling along a downward path, the level is a nondecreasing function. Then we have $Bq + C < \mathrm{lev}(v) \le \mathrm{lev}(w) \le q$. Hence we obtain the required contradiction in this case.
Case II. Suppose that $w \in \Psi(\{x\} \times (\frac{1}{3}, \frac{2}{3}]) \setminus \{\sigma\}$. In this case the DHU-combing path for x has a nontrivial horizontal component and w is in its image.

If $v \in \Psi(\{x\} \times [0, \frac{1}{3}))$, then let $v' := \sigma$ and $b' := \frac{1}{3}$; otherwise $v \in \Psi(\{x\} \times [\frac{1}{3}, c))$ and we let $v' = v$ and $b' = b$. Again applying the fact that the level is a nondecreasing function on a downward path from the identity ϵ, we also have that $v' = \Psi(x, b')$ with $\frac{1}{3} \le b' < c$ and $Bq + C < \mathrm{lev}(v) \le \mathrm{lev}(v')$. Now the points v' and w are both on the horizontal portion of the DHU-path for x, satisfying $\pi_T(v') = \pi_T(w) = z(x)$ and $|\pi(v')| < |\pi(w)|$.

The geodesic in T from $\pi_T(\epsilon)$ to $\pi_T(x)$ may not have an upward component in Case II, and so the nadir $z(x)$ may not be a vertex of T. This implies that the points v' and w may not be in X^1. Let $\tilde{v}'$ and $\tilde{w}$ be the vertices associated to v' and w, respectively. Since $\pi_T(v') = \pi_T(w)$ is on the $t^{-\infty}$ ray in T, the associated vertices project to a vertex $\pi_T(\tilde{v}') = \pi_T(\tilde{w}) = \pi_T(t^{-m})$ for some integer $m \ge 0$ on this ray. The construction of the associated vertices implies that $\tilde{v}'$ is represented by a word $t^{-m}a^i$ and $\tilde{w}$ is represented by a word $t^{-m}a^j$ with $0 \le m$ and $|i|p^{-m} = |\pi(\tilde{v}')| \le |\pi(\tilde{w})| = |j|p^{-m}$.

We also have $|\mathrm{lev}(\tilde{w}) - \mathrm{lev}(w)| < h + 3$, so $l_\Gamma(t^{-m}a^j) = l_\Gamma(\tilde{w}) = \mathrm{lev}(\tilde{w}) < q + h + 3 \le \lfloor q \rfloor + h + 4$. Define $n := \lfloor q \rfloor + h + 4$. Then $l_\Gamma(t^{-m}a^j) < n$ and as a consequence $0 \le m < n$ as well. Similarly since $|\mathrm{lev}(\tilde{v}') - \mathrm{lev}(v')| < h + 3$, we have

$$l_\Gamma(t^{-m}a^i) = l_\Gamma(\tilde{v}') = \mathrm{lev}(\tilde{v}') > Bq + C - (h + 3)$$
$$= B(\lfloor q \rfloor + h + 4) + B(q - \lfloor q \rfloor) - (h + 4)B + C - (h + 3) > Bn.$$

Now $B > h + 2$, and so we may apply Lemma 6.3 to $t^{-m}a^i$, yielding the inequality $|i| > p^{(\frac{1}{h+2}B - 2)n}$.

Combining the inequalities at the ends of the last two paragraphs together with the value $B = 4(h + 2)$ gives $|j| > p^{2n}$. Lemma 6.4 applied to the word $t^{-m}a^j$ with $E = 2$ says that $l_\Gamma(t^{-m}a^j) > (E - 1)n = n$. However, from the previous paragraph we have $l_\Gamma(t^{-m}a^j) < n$, a contradiction.
Case III. Suppose that $w \in \Psi(\{x\} \times (\frac{2}{3}, 1]) \setminus \{\tau\}$. In this case the DHU-combing path for x has a nontrivial upward component, and w is in its image.

As in Case II, define $v' := \sigma$ and $b' := \frac{1}{3}$ if $v \in \Psi(\{x\} \times [0, \frac{1}{3}))$, and define $v' := v$ and $b' := b$ if $v \in \Psi(\{x\} \times [\frac{1}{3}, c))$. Then $v' = \Psi(x, b')$ with $\frac{1}{3} \le b' < c$, $Bq + C < \mathrm{lev}(v) \le \mathrm{lev}(v')$, and v' is either on the horizontal or the upward portion of the DHU-path for x.

The geodesic in T from $\pi_T(1)$ to $\pi_T(x)$ must have an upward component in case III, and hence the nadir $z(x)$ is a vertex of T. Then $z(x) = \pi_T(t^{-m})$ for some integer $0 \le m$.

The DHU-path for x travels from $v' = \Psi(x, b')$ to $w = \Psi(x, c)$ either via a nontrivial upward path, or else through a horizontal and then nonconstant upward path. The DHU-paths for v' and w are reparameterizations of the portion of the DHU-path for x traveling from ϵ to each endpoint, and so they have the same nadir $z(x) = z(v') = z(w) = \pi_T(t^{-m})$. Moreover, we have $|\pi(v')| \leq |\pi(w)|$, and there is an upward path in T from $\pi_T(v')$ to $\pi_T(w)$.

Although the horizontal portion of the DHU-path for x must stay in the 1-skeleton of X (since it projects to $\pi_T(t^{-m})$), the upward portion of the DHU-path for x may leave X^1, and so v' and w may not be in X^1. Let $\tilde{v}'$ and $\tilde{w}$ be the vertices associated to v' and w, respectively. It follows from the definition of associated vertices that these vertices satisfy $z(\tilde{v}') = z(\tilde{w}) = \pi_T(t^{-m})$, $|\pi(\tilde{v}')| \leq |\pi(\tilde{w})|$, and there is there is a (possibly empty) upward path in T from $\pi_T(\tilde{v}')$ to $\pi_T(\tilde{w})$.

Using Lemma 6.2, the vertex $\tilde{w}$ is represented by a word $t^{-m}a^j t^s$ and the vertex $\tilde{v}'$ is represented by a word $t^{-m}a^i t^r$. The relations between these associated vertices above imply that $0 \leq |i| \leq |j|$ and $0 \leq r \leq s$.

The definition of associated vertices implies that $|\text{lev}(\tilde{w}) - \text{lev}(w)| < h + 3$, and hence $l_\Gamma(t^{-m}a^j t^s) = l_\Gamma(\tilde{w}) = \text{lev}(\tilde{w}) < q + h + 3 \leq \lfloor q \rfloor + h + 4 =: n$ as in case II. As a consequence we have both $0 \leq m < n$ and $0 < s < n$ as well.

Also, as in case II, the inequality $|\text{lev}(\tilde{v}') - \text{lev}(v')| < h + 3$ implies that $l_\Gamma(t^{-m}a^i t^r) = l_\Gamma(\tilde{v}') = \text{lev}(\tilde{v}') > Bn$. Combining inequalities from above, we also have $r < n$.

The rest of the proof in this case is similar to that in Case II. In particular, Lemma 6.3 applied to $t^{-m}a^i t^r$ yields the inequality $|i| > p^{(\frac{1}{h+2}B-2)n} = p^{2n}$. Combining this with the inequality $|i| \leq |j|$ from above yields $|j| > p^{2n}$. In turn, using Lemma 6.4 with the word $t^{-m}a^j t^s$ and $E = 2$ shows that $l_\Gamma(t^{-m}a^j t^s) > n$, contradicting the inequality $l_\Gamma(t^{-m}a^j t^s) < n$ found above.

Having achieved a contradiction in each case, this shows that the DHU-combing for the group $BS(1, p)$ and generating set $\{a, t\}^{\pm 1}$ satisfies a radial tameness function $\rho : \mathbb{Q} \to \mathbb{R}_+$ for the linear function $\rho(q) = Bq + C$ with the constants $B = 4(h + 2)$ and $C = (h + 4)(B + 1)$. $\qquad \square$

7 Coefficients in linear tame combings

In this section we show that the linear coefficient for a linear tame combing can be bounded away from 1 for a specific generating set.

Theorem 7.1 *For every natural number $p \geq 8$, the group $G = BS(1, p) = \langle a, t \mid tat^{-1} = a^p \rangle$ with the generating set $A = \{a^{\pm 1}, t^{\pm 1}\}$ does not admit a 1-combing with radial tameness function of the form $\rho(q) = q + C$ for any constant C.*

Proof Let $p \geq 8$ and let X be the Cayley complex of the presentation $\langle a, t \mid tat^{-1} = a^p \rangle$, described in Sect. 6. Suppose to the contrary that $\Psi : X^1 \times [0, 1] \to X$ is a 1-combing with radial tameness function $\rho(q) = q + C$. Replacing C by any larger constant results in another radial tameness function satisfied by the 1-combing Ψ, so we may assume that C is a natural number larger than four.

Consider the word $t^C at^{-C} at^C a^{-1} t^{-C} a^{-1}$. Since $t^C at^{-C} = a^{p^C}$ in the group, this word labels a loop in the Cayley graph X^1 based at the vertex corresponding to the identity ϵ of G. Let $Y \subset X^1$ be the subcomplex of points on the vertices and edges along this loop. The restriction $\Psi : Y \times [0, 1] \to X$ of the 1-combing then defines a homotopy from the identity

vertex to the loop Y, and so the image $\Psi(Y \times [0, 1])$ is the image of a disk filling in the loop Y.

Since the Cayley complex X is the product of the real line $\mathbb{R}$ with the tree T (described in Sect. 6), this complex is aspherical. Then the image set $\Psi(Y \times [0, 1])$ must include all of the points in the rectangle of points $z \in X$ with projections $0 \leq \pi_R(z) \leq p^C$ and $\pi_T(z)$ on the geodesic in T from $\pi_T(1)$ to $\pi_T(t^C)$; that is, the rectangle in X bounded by the loop labeled $t^C a t^{-C} a^{-p^C}$ based at 1, including this boundary loop. (The image $\Psi(Y \times [0, 1])$ must also contain all of the points in the rectangle of X bounded by the loop $t^C a t^{-C} a^{-p^C}$ based at a.) In particular, the vertex corresponding to the element $g \in G$ represented by the word $a^{(h-2)\frac{p^C-1}{p-1}}$, where $h = \lfloor \frac{p}{2} \rfloor$ as before, is in $\Psi(Y \times [0, 1])$. We obtain two estimates for $l_\Gamma(g)$ which, taken together, contradict our assumption that $C > 4$.

First, we observe that the points in the set Y all lie on the (geodesic) paths $t^C a t^{-C} a$ or $a t^C a t^{-C}$ starting at 1, and so the levels of all of the points in Y are at most $2C + 2$. Then the level of every point in the image set $\Psi(Y \times [0, 1])$ must be at most $\rho(2C + 2) = 3C + 2$. Hence $l_\Gamma(g) = \text{lev}(g) \leq 3C + 2$.

On the other hand, note that $(h - 2)\frac{p^C-1}{p-1} = (h - 2) + (h - 2)p + \quad + (h - 2)p^{C-1}$.

Thus the element $g =_G a^{(h-2)\frac{p^C-1}{p-1}}$ of $G = BS(1, p)$ is also represented by the word $v = (a^{(h-2)}t)^{C-1}a^{(h-2)}t^{-(C-1)}$. We claim that the word v is a geodesic. First note that since g is a nontrivial power of the generator a, we have $m = s = 0$ in Lemma 6.2, and the geodesic word w representing g provided by the lemma is in the form (2), $w = a^{i_0}t a^{i_1}t \quad t a^{i_k}t^{-k}$ with $0 \leq k, |i_l| \leq h$ for $0 \leq l \leq k - 1$, and $1 \leq |i_k| \leq h + 1$. We will show that in fact the words v and w are the same. So far we have $w =_G v$; that is, $a^{i_0}t a^{i_1}t \quad t a^{i_k}t^{-k} =_G (a^{(h-2)}t)^{C-1}a^{(h-2)}t^{-(C-1)}$, and hence $i_0 + i_1 p + \quad + i_k p^k = (h - 2) + (h - 2)p + \quad (h - 2)p^{C-1}$. If $k \geq C$, then

$$|i_k|p^k \leq \left| \sum_{l=0}^{C-1}((h - 2) - i_l)p^l \right| + \left| \sum_{l=C}^{k-1} -i_l p^l \right| \leq \sum_{l=0}^{C-1}(p - 2)p^l + \sum_{l=C}^{k-1} hp^l$$

$$< (p - 2)\frac{p^k - 1}{p - 1} < p^k.$$

Since $1 \leq |i_k|$, this shows that we must have $k \leq C - 1$. If $k \leq C - 2$, then

$$(h - 2)p^{C-1} \leq \left| \sum_{l=0}^{k}(i_l - (h - 2))p^l \right| + \left| \sum_{l=k+1}^{C-2} -(h - 2)p^l \right|$$

$$\leq \sum_{l=0}^{k}(p - 2)p^l + \sum_{l=k+1}^{C-2}(h - 2)p^l$$

$$< (p - 2)\frac{p^{C-1} - 1}{p - 1} < p^{C-1},$$

again resulting in a contradiction. Hence $k = C - 1$. Subtracting once more, we get

$$|i_{C-1} - (h - 2)|p^{C-1} \leq \left| \sum_{l=0}^{C-2}((h - 2) - i_l)p^l \right| \leq \sum_{l=0}^{C-2}(p - 2)p^l < p^{C-1},$$

and so $i_{C-1} = h - 2$. Using induction, then $i_l = h - 2$ for all $0 \leq l \leq C - 1$. Hence w and v are the same word, and the word v is a geodesic.

This gives us another way to compute the word length over A of g, since v is a geodesic representative of g, and so $l_\Gamma(g) = l(v) = ((h-2)+1)(C-1) + (h-2) + C - 1 = hC - 2$. Earlier in this proof we had $l_\Gamma(g) \leq 3C + 2$, which gives $hC - 2 \leq 3C + 2$. The hypothesis that $p \geq 8$ gives $h \geq 4$, and so we have $C \leq \frac{4}{h-3} \leq 4$, contradicting our choice of C. □

References

1. Belk, J., Bux, K.-U.: Thompson's group F is maximally nonconvex. In: Geometric Methods in Group Theory. Contemp. Math., vol. 372, pp. 131–146. American Mathematical Society, Providence (2005)
2. Brown, K.S., Geoghegan, R.: An infinite-dimensional torsion-free FP_∞ group. Invent. Math. **77**(2), 367–381 (1984)
3. Cannon, J.W., Floyd, W.J., Parry, W.R.: Introductory notes on Richard Thompson's groups. Enseign. Math. (2) **42**(3–4), 215–256 (1996)
4. Cannon, J.W.: Almost convex groups. Geom. Dedicata **22**(2), 197–210 (1987)
5. Cleary, S., Taback, J.: Combinatorial properties of Thompson's group F. Trans. Am. Math. Soc. **356**(7), 2825–2849 (2004) (electronic)
6. Elder, M., Hermiller, S.: Minimal almost convexity. J. Group Theory **8**(2), 239–266 (2005)
7. Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S., Thurston, W.P.: Word Processing in Groups. Jones and Bartlett Publishers, Boston, MA (1992)
8. Fordham, S.B.: Minimal length elements of Thompson's group F. Geom. Dedicata **99**, 179–220 (2003)
9. Guba, V.S.: The Dehn function of Richard Thompson's group F is quadratic. Invent. Math. **163**(2), 313–342 (2006)
10. Hermiller, S., Meier, J.: Measuring the tameness of almost convex groups. Trans. Am. Math. Soc. **353**(3), 943–962 (2001) (electronic)
11. Horak, M., Stein, M., Taback, J.: Computing word length in alternate presentations of Thompson's group F. Int. J. Algebra Comput. **19**(8), 963–997 (2009)
12. Kapovich, I.: A note on the Poénaru condition. J. Group Theory **5**(1), 119–127 (2002)
13. Mihalik, M.L., Tschantz, S.T.: Tame combings of groups. Trans. Am. Math. Soc. **349**(10), 4251–4264 (1997)
14. Miller, C. F. III., Shapiro, M.: Solvable Baumslag–Solitar groups are not almost convex. Geom. Dedicata **72**(2), 123–127 (1998)
15. Poénaru, V.: Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. J. Differ. Geom. **35**(1), 103–130 (1992)
16. Poénaru, V.: Geometry "à la Gromov" for the fundamental group of a closed 3-manifold M^3 and the simple connectivity at infinity of $\tilde{M}^3$. Topology **33**(1), 181–196 (1994)
17. Riley, T.R.: The geometry of groups satisfying weak almost-convexity or weak geodesic-combability conditions. J. Group Theory **5**(4), 513–525 (2002)
18. Thiel, C.: Zur fast-Konvexität einiger nilpotenter Gruppen. Bonner Mathematische Schriften [Bonn Mathematical Publications], 234. Universität Bonn Mathematisches Institut, Bonn, 1992. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn (1991)