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ORIGINAL PAPER

Free limits of Thompson’s group

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Abstract We produce a sequence of markings S_k of Thompson s group F within the space $\mathcal{G}_n$ of all marked n-generator groups so that the sequence (F, S_k) converges to the free group on n generators, for $n \geq 3$. In addition, we give presentations for the limits of some other natural (convergent) sequences of markings to consider on F within $\mathcal{G}_3$, including $(F, \{x_0, x_1, x_n\})$ and $(F, \{x_0, x_1, x_0^n\})$.

Keywords Thompson s group · Limit group · Free group · Girth

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1 Introduction

Sela defined the notion of a limit group in conjunction with his solution to the problem of Tarski which asks whether all free groups of rank at least 2 have the same elementary theory [17,18]. Limit groups arise in Sela s analysis of equations in free groups, and he shows that they coincide with the class of finitely generated, fully residually free groups [Se1]. Work of Sela, along with Kharlampovich and Myasnikov [14,15,17] shows that limit groups can be constructed recursively from building blocks consisting of free, surface and free abelian groups by taking a finite sequence of free products and amalgamations over $\mathbb{Z}$. Alternately, a group is a limit group in the sense of Sela if and only if it is an iterated generalized double, defined below in Sect. 4.

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Define a marked group (G, S) to be a group G with a fixed and ordered set of generators $S = \{g_1, g_2, \ldots, g_n\}$, and let $\mathcal{G}_n$ be the set of all groups marked by n elements up to isomorphism of marked groups. A marked group (G, S) is equipped with a canonical epimorphism from the free group on $|S|$ letters to G. The space $\mathcal{G}_n$ admits a topology in which two marked groups (G_1, S_1) and (G_2, S_2) are at distance at most e^{-R} if they have same relations of length at most R. With respect to this topology, the limit groups of Sela, equivalently the class of all finitely generated fully residually free groups, arise naturally as limits of marked free groups.

This topological approach towards marked groups opens the notion of limit groups to include limits of other, non free, groups within $\mathcal{G}_n$, for a fixed n. This topology was defined in [12] by Grigorchuk, and an earlier equivalent construction was presented in [9]; Champetier and Guirardel study this topology in [10], and Guyot and Stalder in [19,13] investigate limits of marked copies of Baumslag-Solitar groups.

Just as the class of finitely generated fully residually free groups arises naturally as limits of marked free groups, one can extend the definition of "fully residually" to non-free classes of groups in such a way that these groups arise naturally as limits of marked sequences of other, non-free, groups. More specifically, a finitely generated group G is defined to be *fully residually* $\mathcal{P}$ (where we view $\mathcal{P}$ as a property of groups, e.g. free or finite) if for any finite collection $\{w_1, w_2, \ldots, w_k\}$ of elements of G, there is a surjective homomorphism ϕ from G to a $\mathcal{P}$ group so that the images $\{\phi(w_1), \phi(w_2), \ldots, \phi(w_k)\}$ of the original set of elements are all nontrivial. In the case $\mathcal{P}$=free, we can omit the requirement that ϕ be surjective, as any subgroup of a free group is itself free. Just as for the special case of fully residually free groups, if a group G is fully residually $\mathcal{P}$, there is a sequence of markings of the $\mathcal{P}$ group so that the sequence of marked groups converges to G. With this definition, any group which is fully residually Thompson arises as a limit of marked copies of Thompson s group F. One aim of this paper is to show that the free group F_k for $k \geq 3$ is fully residually Thompson.

The goal of this paper is to analyze several sequences of markings of Thompson s group F. We begin in Sect. 3 by partially answering a question of Sapir, and show in Corollary 3.7 that there exist sequences of marked copies of Thompson s group F which converge to the free group F_k within $\mathcal{G}_k$ for any $k \geq 3$, that is, we show that the free group is fully residually Thompson. Brin [4] has recently shown that there is a sequence of markings of F in $\mathcal{G}_2$ which converges to the free group F_2. This is related to the notion of a group exhibiting *k-free-like behavior*, defined by Olshanskii and Sapir in [16]. A group G is said to be k-free-like for $k \geq 2$ if there exists a sequence of generating sets Z_i for $i \geq 1$, each with k elements, such that the Cayley graph $\Gamma(G, Z_i)$ satisfies no relation of length less than i, and the Cheeger constant of this graph is uniformly (in i) bounded away from zero. The Cheeger constant is the infimum over all subsets A of the group of the ratio of the size of the boundary of A to the size of A, with respect to a fixed generating set. An answer to the question of whether F is amenable will also decide whether F exhibits k-free-like behavior.

In Sect. 4 we investigate the limits of several sequences of marked copies of F within $\mathcal{G}_3$ where the markings are chosen to be very "natural", for example the sequence $\{x_0, x_1, x_n\}$, where the generators are taken from the standard infinite presentation for F. We generalize our concrete examples to sequences of markings where the third element in the triple is simply subject to certain conditions on its support. We note that in each case, there is an amalgamated product (or HNN extension) of copies of Thompson s group F, or subgroups of F, over a maximal abelian subgroup which is a generalized double over the limit group obtained.

The problem of determining all sequences of markings of the form $\{x_0, x_1, g_n\}$ of F, where $g_n \in F$ which converge in $\mathcal{G}_3$, and the presentation of any resulting limit groups, is extremely interesting to the authors. In his thesis, Zarzycki considers this problem as well;

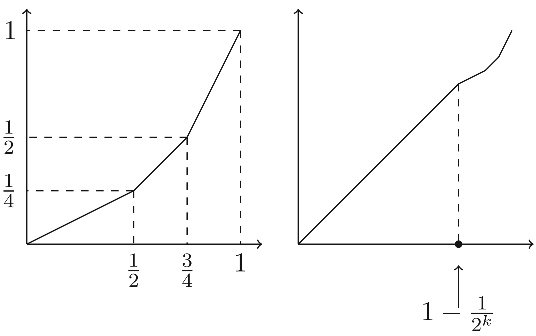

Fig. 1 The generators x_0 and x_n of F as homeomorphisms of [0, 1]

he has obtained preliminary results which state that limit of a sequence of marked copies of F using markings of the form $\{x_0, x_1, g_n\}$ can never be a central HNN-extension [20,21]. His methods are significantly different from the techniques presented here.

2 Preliminaries

In this section we present brief background material on several topics used in this paper.

2.1 A brief introduction to Thompson s group F

Thompson s group F can be viewed as the group of piecewise-linear orientation-preserving homeomorphisms of the unit interval, subject to two conditions:

(1) the coordinates of all breakpoints lie in the set of dyadic rational numbers, and
(2) the slopes of all linear pieces are powers of 2.

Group elements can be viewed uniquely in this way if we require that the slope change at all given breakpoints. Group multiplication then simply corresponds to function composition.

This group is commonly studied via a standard infinite presentation $\mathcal{P}$:

$$\mathcal{P} = \langle x_0, x_1, x_2, \ldots | x_i^{-1} x_j x_i = x_{j+1} \text{ for } i < j \rangle.$$

It is clear from the above presentation that x_0 and x_1 are sufficient to generate the group, and we thus obtain the standard finite presentation   :

$$= \langle x_0, x_1 | [x_1 x_0^{-1}, x_0^{-1} x_1 x_0],\ [x_1 x_0^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

The generators x_0 and x_n are depicted as homeomorphisms of the interval in Fig. 1. Notice that the support of x_n is exactly the interval $[1 - \frac{1}{2^n}, 1]$. As a consequence, any element with support contained in $[0, 1 - \frac{1}{2^n}]$ will commute with x_n, as elements of F with disjoint support always commute. Analyzing the relators in the finite presentation   , it is not hard to see that the support of $x_1 x_0^{-1}$ is $[0, \frac{3}{4}]$, while the supports of $x_2 = x_0^{-1} x_1 x_0$ and $x_3 = x_0^{-2} x_1 x_0^2$ are contained in $[\frac{3}{4}, 1]$, hence these elements commute.

With respect to the infinite presentation $\mathcal{P}$, elements of F have a standard (infinite) normal form, given by

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

where $r_i, s_i > 0$, $0 \leq i_1 < i_2 \quad < i_k$ and $0 \leq j_1 < j_2 \quad < j_l$, and if both x_i and x_i^{-1} occur, so does x_{i+1} or x_{i+1}^{-1}, as discussed by Brown and Geoghegan in [7]. A *positive* word (resp. *negative word*) has normal form containing only positive (resp. negative) exponents.

For a thorough introduction to Thompson s group F we refer the reader to [8].

2.2 Girth and limit groups

The girth of a group G with respect to a finite generating set S is defined to be the length of the shortest relator satisfied in (G, S). The girth of a finitely generated group G is the supremum of $girth(G, S)$ over all finite generating sets S for G. It is proven in [2] that a non-cyclic, finitely generated hyperbolic group (or one-relator or linear group) has infinite girth if and only if it is not virtually solvable.

We will use the notion of girth together with the following proposition of Stalder, which provides an algebraic formulation of the convergence of a series of marked groups in the topology defined in [9] and [12], to exhibit free limits of marked copies of Thompson s group F. If (G, S) is a marked group and $|S| = k$, we say that S is a marking of G of *length* k.

Proposition 2.1 [19] *Let (Γ_n, S_n) be a sequence of marked groups on k generators. The following are equivalent*:

(1) *(Γ_n, S_n) is convergent in $\mathcal{G}_k$*;
(2) *for all $w \in F_k$ we have either $w = 1$ in Γ_n for n large enough, or $w \neq 1$ in Γ_n for n large enough, where F_k is the free group on k letters.*

In the sections below, we will be interested not only in whether particular sequences of marked groups converge, but in the presentation of the limit group of such a convergent sequence. The following proposition, while a restatement of the definition of convergence of a sequence of marked groups, states explicitly how we characterize this limit group.

Proposition 2.2 *Let (Γ_n, S_n) be a sequence of marked groups on k generators. Then (Γ_n, S_n) converges to (Γ, S) in $\mathcal{G}_k$ if for all words $w \in F_k$*

(1) *if $w = 1$ in Γ, then $w = 1$ in Γ_n for sufficiently large n.*
(2) *if $w = 1$ in Γ_n for infinitely many n, then $w = 1$ in Γ.*

In the case that the limit group is a free group, the first property holds trivially, so we need only check the second. Combining the notion of girth with the above proposition yields a straightforward characterization of when a sequence of markings of a fixed group converges to a free group. Namely:

Proposition 2.3 *If a group G has a sequence of markings S_n of length k so that the girth of (G, S_n) goes to infinity with n, then the sequence (G, S_n) converges to F_k in $\mathcal{G}_k$.*

3 Free limits of Thompson's group

The goal of this section is to prove that there is a sequence $\{S_n\}$ of markings of F so that (F, S_n) converges to the free group of rank k in $\mathcal{G}_k$, for all $k \geq 3$. This is proven by exhibiting a sequence of markings of F of a fixed length so that the girth of the group with respect to these markings approaches infinity. As a corollary we obtain that the same sequence of markings on F converges to the free group of the appropriate rank. As the finiteness of the girth of a group is closely related to whether the group satisfies a law, we first recall the result of Brin and Squier [5] (and reproven by Abért [1] and later by Esyp [11]) that Thompson s group F satisfies no law.

First define $\mathcal{W}_{m,k}$ to be the set of all nontrivial reduced words in the free group $F(a, b_1, b_2, \ldots, b_{k-1})$ of rank k of length at most m. The reason for using both a and b_j to denote the generators of the free group will be clear below.

Proposition 3.1 *Fix $m \in$ and let $k \geq 2$. There exist $u_1, u_2, \ldots u_k \in F$ so that for any $w \in \mathcal{W}_{m,k}$ the word $w(u_1, u_2, \ldots u_k) \in F$ is nontrivial.*

Proof Suppose that there are l words $w_1, \ldots, w_l$ in $\mathcal{W}_{m,k}$. As F satisfies no group law, for each i with $1 \leq i \leq l$ we can find $u'_{i,1}, u'_{i,2}, \ldots u'_{i,k} \in F$ so that $w_i(u'_{i,1}, u'_{i,2}, \ldots u'_{i,k})$ is nontrivial. For any two dyadic rationals a and b with $a < b$, there is an isomorphism $\phi_{[a,b]}$ from F to the isomorphic copy of F supported on the interval $[a, b]$, which we denote $F_{[a,b]}$, as detailed in [3]. Choosing dyadic rationals $0 = a_0 < a_1 < a_2 \ldots < a_{l-1} < a_l = 1$, let $I_j = [a_{j-1}, a_j]$ for $1 \leq j \leq l$ and define $\phi_{I_j} : F \to F_{I_j}$ to be the corresponding isomorphisms. Let $u_{j,r} = \phi_{I_j}(u'_{j,r})$ for $1 \leq r \leq k$ and $1 \leq j \leq l$. It is clear that the support of $u_{j,r}$ lies in the interval I_j.

For $1 \leq r \leq k$, define $u_r(x) = u_{j,r}(x)$ for $x \in I_j$. If $w \in \mathcal{W}_{m,k}$, then $w(u_1, u_2, \ldots u_k)$ must be nontrivial on at least one interval I_j by construction. □

The next proposition shows that for any word $w = w(a, b_1, b_2, \ldots b_{k-1}) \in \mathcal{W}_{m,k}$, we can always find $u_1, u_2, \ldots u_{k-1} \in F$ so that $w(x_0, u_1, u_2, \ldots, u_{k-1})$ is nontrivial. We first state a basic lemma regarding the construction of elements of F with certain proscribed values. A proof of this lemma can be found in [8].

Lemma 3.2 ([8], Lemma 4.2) *If $0 = x_0 < x_1 < x_2 < \quad < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \quad < y_n = 1$ are partitions of $[0, 1]$ consisting of dyadic rational numbers, then there exists $f \in F$ so that $f(x_i) = y_i$ for $i = 0, 1, 2, \ldots, n$. Furthermore, if $x_{i-1} = y_{i-1}$ and $x_i = y_i$ for some i with $1 \leq i \leq n$, then f can be taken to be the identity on the interval $[x_{i-1}, x_i]$.*

Proposition 3.3 *Fix $m \in \quad, 0 < \epsilon < \frac{1}{2}$, and $k \geq 2$. For any $w = w(a, b_1, b_2, \ldots b_{k-1}) \in \mathcal{W}_{m,k}$, there exist elements $u_1, u_2, \ldots u_{k-1} \in F$ so that for $1 \leq i \leq k-1$, we have $Support(u_i) \subseteq [0, \epsilon)$ and $w(x_0, u_1, u_2, \ldots, u_{k-1})|_{[0,\epsilon)} \neq 1$.*

Proof As m is fixed, define $\mathcal{N}_m = \{-m, -(m-1), \ldots, 0, 1, 2, \ldots, m\}$. Choose $I_0 \subset (0, \epsilon)$ to be a small closed interval whose endpoints lie in $\mathbb{Z}[\frac{1}{2}]$ with the property that the collection of intervals $\{I_i = x_0^i(I_0)\}$ is pairwise disjoint for $i \in \mathcal{N}_m$, and all $I_i \subset (0, \epsilon)$. Given $w \in \mathcal{W}_{m,k}$ we will construct $u_1, u_2, \ldots u_{k-1} \in F$ supported in $\cup_{i \in \quad m} I_i$ so that $w(x_0, u_1, u_2, \ldots u_{k-1})$ is nontrivial.

More specifically, choosing x in the interior of I_0, we will define a sequence of elements $u_1, u_2, \ldots u_{k-1}$ in F so that $w(x_0, u_1, u_2, \ldots, u_{k-1})(x) \neq x$. Each element u_j will be constructed so that

(1) u_j preserves each interval, though not point-wise, that is, $u_j(I_i) = I_i$ for $i \in \mathcal{N}_m$, and
(2) u_j is the identity off the union of these intervals, that is, for any $y \notin \cup_{i \in \quad m} I_i$, we have $u_j(y) = y$.

The construction will be accomplished by selecting, for each j and each i, a (possibly empty) pair of sequences of points $\ _1 < \ _2 < \quad \ _r$ and $\beta_1 < \beta_2 < \quad < \beta_r$ in the interior of I_i, and applying Lemma 3.2 to define $u_j|_{I_i}$.

To begin the construction, choose $x \in \mathbb{Z}[\frac{1}{2}]$ in the interior of I_0. Let

$$w = a^{e_1} B_1 a^{e_2} B_2 \dots a^{e_r} B_r$$

be a word in $F(a, b_1, b_2, \dots, b_{k-1})$ where each B_j is a word in $b_1^{\pm 1}, b_2^{\pm 1}, \dots, b_{k-1}^{\pm 1}$. We allow $e_1 = 0$ and $B_r = 1$, but $e_l \neq 0$ for $1 < l \leq r$ and B_l is not the trivial word for $1 \leq l < r$.

For $1 \leq s \leq r$, let $B_s = b_{s,1} b_{s,2} \dots b_{s,q_s}$ for $b_{s,j} \in \{b_1^{\pm 1}, \dots, b_{k-1}^{\pm 1}\}$. All points that are chosen in the construction below are assumed to lie both in the interior of an interval I_j and in $\mathbb{Z}[\frac{1}{2}]$. First choose an increasing sequence of dyadic rationals in the interior of I_0 beginning with x (which we relabel in keeping with our indexing scheme),

$$x = y_r = y_r^{q_r+1} < y_r^{q_r} < y_r^{q_r-1} < \quad < y_r^1 = x_r.$$

Note that the length of this sequence is one more than the length of B_r. Moving through the word w from right to left, we now consider a^{e_r}, which indicates which interval we use to choose the next sequence of points. More precisely, recalling that x_0 will be substituted for a, this next sequence will be chosen in the interior of $x_0^{e_r}(I_0)$, beginning with the image of the final point in the previous sequence under $x_0^{e_r}$ and with length $|B_{r-1}| + 1$. Namely, let $y_{r-1} = x_0^{e_r}(x_r) \in x_0^{e_r}(I_0) = I_j$ for $I_j \neq I_0$, and then choose an increasing sequence of dyadic rationals in the interior of I_j,

$$y_{r-1} = y_{r-1}^{q_{r-1}+1} < y_{r-1}^{q_{r-1}} < y_{r-1}^{q_{r-1}-1} < \quad < y_{r-1}^1 = x_{r-1}.$$

Continue in this way through the word w, constructing increasing sequences of points in the various intervals I_j. We remark that for each index s the sequence constructed consists of more than one point, with the possible exception of $s = r$, that is, the initial sequence constructed. When $s \neq r$, we know that $B_s \neq 1$, hence $y_s < x_s$. Notice that it is possible for more than one such sequence to be chosen within a single interval I_j. If it is the case that two sequences of the form $\{y_l^i\}$ and $\{y_h^i\}$ for fixed l and h where $l < h$ both are chosen in I_j, then by construction, $y_h^s \leq y_l^p$ for any superscripts s and p. In fact, we claim this inequality is strict. To see this, note that since $h > 1$, $e_h \neq 0$, and hence $h - 1 \neq l$. In particular, $l + 1 \neq r$, and hence $y_{l+1} < x_{l+1}$. Therefore, $y_h^s \leq y_{l+1} < x_{l+1} \leq y_l^p$, and so $y_h^s < y_l^p$, as desired.

Now for each pair of points $y_s^{i+1} < y_s^i$, if $b_{s,i} = b_n^{\pm 1}$, we define an ordered pair of points $(z, u_n(z))$ as follows. If $b_{s,i} = b_n$, choose $(z, u_n(z)) = (y_s^{i+1}, y_s^i)$, and if $b_{s,i} = b_n^{-1}$, choose $(z, u_n(z)) = (y_s^i, y_s^{i+1})$. We claim that for any n, the collection of pairs $\{(z \ , u_n(z \ ))\}$ defined above satisfy the hypotheses of Lemma 3.2. Namely, the domain points $\{z \ \}$ are all distinct, and if $z_{\ 1} < z_{\ 2}$ then $u_n(z_{\ 1}) < u_n(z_{\ 2})$.

To verify this claim, we must show that if $(z_1, u_n(z_1))$ and $(z_2, u_n(z_2))$ are two such pairs, corresponding to two letters $b_n^{\epsilon_1}$ and $b_n^{\epsilon_2}$ in the word w, for some $n \in \{1, 2, \dots, k-1\}$ and $\epsilon_1, \epsilon_2 \in \{+1, -1\}$ where $z_1 \leq z_2$, then in fact $z_1 \neq z_2$ and $u_n(z_1) < u_n(z_2)$.

To see this, first consider the case that both $b_n^{\epsilon_1}$ and $b_n^{\epsilon_2}$ occur within the same subword B_j. If $\epsilon_1 = \epsilon_2$, the claim is clear. If $\epsilon_1 \neq \epsilon_2$, since B_j is freely reduced, $b_n^{\epsilon_1}$ and $b_n^{\epsilon_2}$ are not adjacent in the word. Hence, $b_n^{\epsilon_1} = b_{j,i}$ and $b_n^{\epsilon_2} = b_{j,l}$ with $l + 1 < i$, and $y_j^{i+1} < y_j^i < y_j^{l+1} < y_j^l$. Hence, regardless of the values of ϵ_1 and ϵ_2, since we have the setwise equalities $\{z_1, u_n(z_1)\} = \{y_j^{i+1}, y_j^i\}$ and $\{z_2, u_n(z_2)\} = \{y_j^{l+1}, y_j^l\}$, we see that $z_1 \neq z_2$ and $u_n(z_1) < u_n(z_2)$.

On the other hand, suppose $b_n^{\epsilon_1}$ is in the subword B_j and $b_n^{\epsilon_2}$ is in the subword B_k with $k \neq j$. Then if $z_1 \in I_i$ and $z_2 \in I_l$ with $i \neq l$, since $I_l \cap I_i = \emptyset$, the claim is clearly true. So suppose both z_1 and z_2 are in I_i. Then since $z_1 \leq z_2$, $k < j$. Then $\{z_1, u_n(z_1)\} = \{y_j^r, y_j^{r+1}\}$

and $\{z_2, u_n(z_2)\} = \{y_k^s, y_k^{s+1}\}$ for some superscripts r and s. Then as previously established, $z_1 < z_2$ and $u_n(z_1) < u_n(z_2)$.

We have shown that for a given $n \in \{1, 2, \ldots, k-1\}$, the collection of all of the pairs $\{(z\ , u_n(z\ ))\}$ defined above satisfy the hypothesis Lemma 3.2. Therefore, we can apply Lemma 3.2 for any pair j and n to define $u_n|_{I_j}$. If, for a given pair, no points have been chosen in I_j to be domain and range points for u_n, then simply define $u_n|_{I_j}$ to be the identity. Also, for points y outside all intervals I_j, define $u_n(y) = y$ for any n. If it is the case that $exp_a(w)$, the sum of the exponents of all instances of a in the word w, is zero then $w(x_0, u_1, u_2, \ldots, u_{k-1})(x) \in I_0$, and hence by construction, $w(x_0, u_1, u_2, \ldots, u_{k-1})(x) > x$. If $exp_a(w) \neq 0$, then $w(x_0, u_1, u_2, \ldots, u_{k-1})(x) \notin I_0$, and hence $w(x_0, u_1, u_2, \ldots, u_{k-1})(x) \neq x$. In either case, $w(x_0, u_1, u_2, \ldots, u_{k-1})$ is nontrivial. □

We next extend Proposition 3.3 to a collection of words in $F(a, b_1, b_2, \ldots, b_{k-1})$ of length at most m. The proof of this next proposition is analogous to that of Proposition 3.1.

Proposition 3.4 *Fix $m \in\ $, $k \geq 2$, $0 < \epsilon < \frac{1}{2}$ and let $w_1, w_2, \ldots, w_q \in \mathcal{W}_{m,k}$. There exists $u_1, u_2, \ldots, u_{k-1} \in F$ with $Support(u_j) \subseteq [0, \epsilon)$ for $1 \leq j \leq k-1$ and $w_i(x_0, u_1, u_2, \ldots, u_{k-1})|_{[0,\epsilon)} \neq 1$ for all $1 \leq i \leq q$.*

Proof Choose the interval I_0 as in the proof of Proposition 3.3 so that it is contained in $(0, \epsilon)$ and its translates under x_0^i are pairwise disjoint for $i \in \mathcal{N}_m = \{-m, \ldots, m\}$ and all contained in $(0, \epsilon)$. Choose q pairwise disjoint subintervals $J_1, J_2, \ldots, J_q$ of I_0, all having endpoints which are dyadic rationals, with the same properties as I_0, that is, the translates of each J_l are pairwise disjoint under the above list of powers of x_0. Following the proof of Proposition 3.3, for each i with $1 \leq i \leq q$ define elements $u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k-1}$ supported in $\cup_{s\in\ _m} x_0^s(J_i)$ so that $w_i(x_0, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k-1})$ is nontrivial.

For $1 \leq j \leq k-1$, define $u_j(x) = u_{l,j}(x)$ for $x \in \cup_{s\in\ _m} x_0^s(J_l)$ over all $1 \leq l \leq q$, and $u_j(x) = x$ for all other $x \in [0, 1]$. By construction, each u_j is supported in $[0, \epsilon)$ for $1 \leq j \leq k-1$. Moreover, as homeomorphisms of $[0, 1]$ we have $w_r(x_0, u_1, u_2, \ldots, u_{k-1})|_{[0,\epsilon)} \neq 1$ for $1 \leq r \leq q$. □

Theorem 3.5 *Thompson's group F has infinite girth. Moreover, for any $l \geq 3$ there is a sequence of generating sets $S_{l,m}$ of length l for F so that the girth of $(F, S_{l,m})$ approaches infinity as m approaches infinity.*

Proof Fix $l \in\ $ with $l \geq 3$. To prove the theorem we exhibit a family of generating sets of length l for F so that the girth with respect to these generating sets approaches infinity. This proves the second statement in the theorem, which includes the first statement in the theorem.

Choose $m \in\ $, and fix $\epsilon > 0$ so that $2^{m^2}\epsilon < \frac{1}{2}$. This choice is made initially so that later in the argument, the supports of certain elements are disjoint from $supp(x_1) = [\frac{1}{2}, 1]$.

Using Proposition 3.4 with $k = l - 1$ we can find $u_{1,m}, u_{2,m}, \ldots, u_{l-2,m} \in F$ so that for all $w(a, b_1, b_2, \ldots, b_{l-2}) \in \mathcal{W}_{2m^2,l-1}$ we know that as a homeomorphism $w(x_0, u_{1,m}, u_{2,m}, \ldots, u_{l-2,m})$ is nontrivial on the interval $(0, \epsilon)$.

Consider the following generating set of length l for F:

$$S_{l,m} = \{\ \ = x_0, \beta = x_0^m u_{1,m}^m x_1, \gamma_1 = u_{1,m}, \ldots, \gamma_{l-2} = u_{l-2,m}\}.$$

Any freely reduced word w of length at most m in the free group on generators $\{\ , \beta, \gamma_1, \ldots, \gamma_{l-2}\}$ can be rewritten as a word w_1 in the free group on the generators

$\{x_0, x_1, u_{1,m}, \ldots u_{l-2,m}\}$ simply replacing by x_0, β by $x_0^m u_{1,m}^m x_1$, and γ_i by $u_{i,m}$, and then freely reducing, at the cost of increasing word length by a factor of at most $2m + 1$. Namely,

$$w(\ , \beta, \gamma_1, \gamma_2, \ldots, \gamma_{l-2}) = w_1(x_0, x_1, u_{1,m}, u_{2,m}, \ldots, u_{l-2,m})$$

and since the length of w is at most m, then the length of w_1 is at most $2m^2 + m$.

Similarly, if we remove all instances of the letter x_1 from w_1 and reduce the resulting word, we obtain a new word $w_2(x_0, u_{1,m}, u_{2,m}, \ldots, u_{l-2,m})$. Since w_2 can be obtained from w by replacing by x_0, β by $x_0^m u_{1,m}^m$, and γ_i by $u_{i,m}$, and then freely reducing, w_2 has length at most $2m^2$.

Now observe the following fact:

Fact 3.6 *Let $H =< x, y, d >$ be a free group, and $G =< x, y, d | d = x^m y^m >$ for some positive integer m. Then if $w \in H$ with $w = 1$ in G, then the length of w in the free group H is greater than m.*

Fact 3.6 implies that w_2 has length strictly greater than 1. We note that w_2, when viewed as an element of F, is nontrivial, as from Proposition 3.4 we know that all nontrivial freely reduced words $w(x_0, u_{1,m}, u_{2,m}, \ldots, u_{l-2,m})$ of length at most $2m^2$ are nontrivial in F. Moreover, there is some $x \in (0, \epsilon)$ such that $w_2(x) \neq x$. As $supp(u_{i,m}) \subset [0, \epsilon)$ by construction and $supp(x_1) \subset [\frac{1}{2}, 1]$, these supports are disjoint. Since there are at most m^2 occurrences of the generator x_0 in w_1 and $2^{m^2}\epsilon < \frac{1}{2}$, we see that $w_2(x) = w_1(x)$, and hence $w_1(x) \neq x$. As $w(x) = w_1(x)$, we have shown that $w(\ , \beta, \gamma_1, \gamma_2, \ldots, \gamma_{l-2})$ is nontrivial. Hence, the girth of $(F, S_{l,m})$ is at least m. Therefore, the girth of $(F, S_{l,m})$ approaches infinity as m approaches infinity. □

As a direct consequence of Proposition 2.3, we obtain the following corollary.

Corollary 3.7 *For each $l \in$, $l \geq 3$, the sequence of marked groups*

$$G_m = (F, \{x_0, x_0^m u_{1,m}^m x_1, u_{1,m}, \ldots, u_{l-2,m}\})$$

converges to the free group $(F_l, \{a_1, a_2, \ldots, a_l\})$, where $u_{1,m}, \ldots, u_{l-2,m}$ are the elements constructed above in the proof of Proposition 3.4 with $k = l - 1$.

4 Non-free limits of within $_3$

The results in this section are motivated by considering several natural sequences of markings of F within $\mathcal{G}_3$ of the form $\{x_0, x_1, g_n\}$ for some $g_n \in F$. In particular, we consider the cases $g_n = x_n$, the $(n+1)$-st generator in the infinite presentation for F, and $g_n = x_0^n$. However, the convergence of the resulting sequence of marked groups relies less on the actual elements chosen and more on their supports. Hence we are able to state convergence results for more general sequences of markings of F. As a corollary of Theorem 4.3 we see that $(F, \{x_0, x_1, x_n\})$ is convergent in $\mathcal{G}_3$ and obtain a presentation for the resulting limit group; in Theorem 4.5 we prove that $(F, \{x_0, x_1, x_0^n\})$ is convergent in $\mathcal{G}_3$ as well and give a presentation of the limit group. For consistency in the notation of the marking, we identify $a = x_0$, $b = x_1$ and $c = g_n$, the additional generator in the marking.

When limits of free groups are studied, a standard construction used to create new examples of limit groups from existing examples is the generalized double. Namely, if a group

G is a generalized double over a limit group L (in the sense of Sela), then it is itself a limit group in this sense. This construction is sufficient to create the entire class of limit groups (as limits of free groups), as proven by Champetier and Guirardel in the following theorem, derived from work of Sela.

Theorem 4.1 ([10], Theorem 4.6) *A group is a limit group if and only if it is an iterated generalized double.*

To make this precise, we define the notion of a generalized double over a limit group L which is the limit of a convergent sequence of marked copies of free groups.

Definition 4.2 A generalized double over a limit group L is a group $G = A *_C B$ (or $G = A*_C$) such that both vertex groups A and B are finitely generated and

(1) C is a nontrivial abelian group whose images under both embeddings are maximal abelian in the vertex groups, and
(2) there is an epimorphism $\varphi : G \quad L$ which is one-to-one in restriction to each vertex group.

While there is no analogous characterization for limits of non-free groups, we note that in each of our theorems below there is an amalgamated product (or HNN extension) of copies of Thompson s group F, or subgroups of F, over a maximal abelian subgroup which is a generalized double over the limit group obtained. Thus the same structure emerges in our limit groups as appears in the case of limits of a convergent sequence of marked free groups.

The first example, given in Theorem 4.3, is a generalization of the natural sequence of markings $\{x_0, x_1, x_n\}$ of F in $\mathcal{G}_3$, that is, $a = x_0$, $b = x_1$ and $c = x_n$.

Theorem 4.3 *Let*

$$G_n = \langle a, b, c | R_1 = [ba^{-1}, a^{-1}ba],\ R_2 = [ba^{-1}, a^{-2}ba^2],\ c^{-1}w_n\rangle,$$

where w_n is a word in a and b, such that viewed as a element of F where $a = x_0$ and $b = x_1$, w_n has support in $[t_n, 1]$, where $\lim_{n\to\infty} t_n = 1$, and w_n maps $[\frac{3+t_n}{4}, 1]$ linearly to $[\frac{1+t_n}{2}, 1]$. Then the sequence $(G_n, \{a, b, c\})$ converges to $(G, \{a, b, c\})$, where

$$G = \langle a, b, c | R_1, R_2, R_3 = [ca^{-1}, a^{-1}ca],\ R_4 = [ca^{-1}, a^{-2}ca^2], \text{ and } [ba^{-1}, a^i ca^{-i}] \text{ for all } i \in \mathbb{Z}\rangle.$$

Proof First, choose N_0 so that for $n \geq N_0$, $t_n \geq \frac{1}{2}$. Then for any $n \geq N_0$, R_3 and R_4 are true in G_n for the following reason. Since w_n maps $[\frac{3+t_n}{4}, 1]$ linearly to $[\frac{1+t_n}{2}, 1]$, the support of ca^{-1} is contained in $[0, \frac{1+t_n}{2}]$. Now since $t_n \geq \frac{1}{2}$, a^{-1} takes $[t_n, 1]$ linearly to $[\frac{1+t_n}{2}, 1]$. As the support of $a^{-i}ca^i$ is $a^{-i}(Supp(c))$ and the support of c is contained in $[t_n, 1]$, we see that $Supp(a^{-i}ca^i) \subseteq a^{-i}[t_n, 1] \subseteq [\frac{1+t_n}{2}, 1]$ for $i \geq 1$. As the supports of these two elements are disjoint, they must commute and we obtain the relations R_3 and R_4.

Next, for a relator of the form $[ba^{-1}, a^i ca^{-i}]$, choose N_i (and there are infinitely many such choices) so that $a^i[t_{N_i}, 1] \subseteq [3/4, 1]$. Then for any $n \geq N_i$, we see that $a^i ca^{-i}$, as a homeomorphism in G_n, has support in $[3/4, 1]$ and hence commutes in G_n with ba^{-1}, whose support lies in $[0, 3/4]$. Therefore the relation $[ba^{-1}, a^i ca^{-i}]$ holds in G_n for all $n \geq N_i$.

We follow Proposition 2.2, and first prove that if w is trivial in G then w is trivial in G_n for sufficiently large n. Let $w = w(a, b, c)$ be a word in a, b, c and their inverses which is the identity in G. Then w may be expressed as a product of conjugates of finitely many relators of G. But, for any finite set of relators, there is some N so that the relators of G all hold in G_n for $n \geq N$, and thus w is trivial in G_n for all $n \geq N$.

Now suppose w is a word in the letters a, b, c and their inverses which is the identity in G_n for infinitely many n. We must show that w must be trivial in G as well.

We first show that the word w can be rewritten in a certain constrained form in both the group G and the groups G_n for n sufficiently large. First, in G, using relations of the form $[ba^{-1}, a^i ca^{-i}]$ for at most finitely many i, we may move all occurrences of $c^{\pm 1}$ to the left of all occurrences of $b^{\pm 1}$ with the penalty of increasing the number of occurrences of the generator a in the word.

For example, suppose w has a suffix of the form $w' = ba^i ca^j$. Then $w' = (ba^{-1})(a^{i+1}ca^{-(i+1)})a^{j+i+1}$. Since $[ba^{-1}, a^{i+1}ca^{-(i+1)}]$ is a relator in G, then in G, $w' = (a^{i+1}ca^{-(i+1)})(ba^{-1})a^{j+i+1}$. Moving through the word to the left in this manner, using a finite number of the commutator relators, we can rewrite w in this way as $w_1 w_2$, where w_1 is a word in the generators a and c and w_2 is a word in the generators a and b. Choose M large enough so that the finite collection of relators used in this process all hold in G_n for $n > M$, and thus we can rewrite w in G and in G_n for $n > M$ in the desired form.

In addition, as long as we choose $M \geq N_0$ as well, then in both G_n and in G, the elements a and c satisfy the relators R_3 and R_4. The subgroup $\langle a, c | R_3, R_4 \rangle \cong F$ and so we can rewrite w_1 in an infinite normal form using $c_0 = a, c_1 = c, \ldots, c_{1+i} = a^{-i}ca^i, \ldots$ for $i \geq 1$. Similarly, we can rewrite w_2 using the relators R_1 and R_2 in the standard infinite normal form in the letters $x_0 = a, x_1 = b, x_2, \ldots, x_{1+i} = a^{-i}ba^i, \ldots$ for $i \geq 1$, in both G and G_n. Suppose that $w_1 = c_0^{\epsilon} w_1' c_0^{-\delta}$, where w_1' has infinite normal form in $c_i^{\pm 1}$ for $i \geq 1$, and $w_2 = x_0 \, w_2' x_0^{-\beta}$, where w_2' has infinite normal form in $x_i^{\pm 1}$ for $i \geq 1$. Combine the terms $c_0^{-\delta}$ and x_0, and if $= \delta$, these terms cancel. Otherwise, we move the combination to either the right or the left as follows. If $- \delta < 0$, rewrite $c_0^{-\delta} x_0 \, w_2' x_0^{-\beta} = x_0 \, ^{-\delta} w_2' x_0^{-\beta}$ in infinite normal form using R_1 and R_2. If $- \delta > 0$, rewrite $c_0^{\epsilon} w_1' c_0^{-\delta} x_0 = c_0^{\epsilon} w_1' c_0 \, ^{-\delta}$ in infinite normal form in the generators $\{c_0, c_1, c_2, \ldots\}$ and their inverses, using R_3 and R_4. As a result of the application of these relators, subscripts of letters in w_1' and w_2' may be increased, and the initial and final exponents ϵ and β may change as well. Without loss of generality we then assume that there is an $M \in$ so that for $n > M$ we can write $w = w_1 w_2$ in both G and G_n, where $w_1 = c_0^{\epsilon} w_1'$, $w_2 = w_2' x_0^{-\beta}$, and w_1' has infinite normal form in $c_i^{\pm 1}$, for $i \geq 1$, and w_2' has infinite normal form in $x_i^{\pm 1}$, for $i \geq 1$.

Now recall that w is the identity in G_n for infinitely many n. Then w is certainly the identity for infinitely many $n > M$, where we can write $w = w_1 w_2 = c_0^{\epsilon} w_1' w_2' x_0^{-\beta}$ as described above. For any such n, in G_n the word w represents a particular homeomorphism. As a homeomorphism, c_1 has support in $[t_n, 1]$, so c_k has support contained in $[t_n, 1]$ as well for all $k \geq 2$. Thus w_1' must also have support in $[t_n, 1]$. Similarly, as the support of x_1 is $[\frac{1}{2}, 1]$ and the support of x_k is contained in $[\frac{1}{2}, 1]$ for all $k \geq 2$, we see that w_2' has support in $[\frac{1}{2}, 1]$. But then the slope of $w_1 w_2$ near zero will be $2^{\beta - \epsilon}$; as this homeomorphism is the identity in G_n, we must have $\epsilon = \beta$.

For each of the infinitely many $n > M$ for which w is the identity in G_n, recalling that x_0 and c_0 are both equal to the generator a, we may conjugate w by a^{ϵ} to obtain the word $w_1' w_2'$, which must also be the identity in G_n. We claim that w_2' must in fact be the empty word. For if not, then thinking of w_2' as a homeomorphism in G_n, there is some $x \in (0, 1)$ which is not fixed by w_2'. However, for sufficiently large n, $w_2'(x)$ will be outside of the support of w_1', and hence $w_1' w_2'$ will not fix x in G_n for such large n. But if w_2' is the empty word, then w_1' must be the empty word as well. But this means that in fact, for all $n > M$, the original

w can be written as $c_0^{\epsilon} x_0^{-\epsilon}$ in G, where $c_0 = x_0 = a$, and hence can be transformed to the identity in G. □

Using the notation in the definition of the generalized double over a limit group, we remark that when $A = B = F$ and $C = \mathbb{Z}$, where both inclusions of C in A and B map C to the subgroup generated by x_0, then $A *_C B$ is a generalized double over the limit group G obtained in Theorem 4.3. We note that as the support of x_0 is the entire interval [0, 1], the subgroup generated by x_0 is a maximal abelian subgroup of F [6].

In the previous example, the additional generator c in G_n had support in a small neighborhood of 1 for large n. Alternatively, if we choose a sequence of additional generators to have supports in arbitrarily small intervals close to zero but not including zero, we obtain the following convergent sequence of marked copies of F.

Theorem 4.4 *Choose $g_n \in F$ to have support in $[r_n, s_n] \subseteq [0, 1]$, where $r_n < s_n < 2r_n$ and $\lim_{n\to\infty} r_n = 0$, and choose a word w_n in $a^{\pm 1}, b^{\pm 1}$ so that $w_n(x_0, x_1) = g_n$. Let*

$$G_n = \langle a, b, c \mid R_1 = [ba^{-1}, a^{-1}ba], R_2 = [ba^{-1}, a^{-2}ba^2], c^{-1}w_n(a, b)\rangle.$$

Then the sequence $(G_n, \{a, b, c\})$ converges to $(G, \{a, b, c\})$, where

$$G = \langle a, b, c \mid R_1, R_2, [a^i ca^{-i}, c], [a^i ba^{-i}, c] \text{ for every } i \in \mathbb{Z}\rangle.$$

Proof First, suppose $w = 1$ in G. Then w can be transformed to the empty word using only a finite number of relations of G. There exists some M so that $s_n < 1/2$ for all $n \geq M$, and then it follows that $x_0([r_n, s_n])$ is disjoint from $[r_n, s_n]$. Therefore, for any $n \geq M$, the relations of the form $[a^i ca^{-i}, c]$ hold in G_n. On the other hand, we may choose N_i so that $[a^i ba^{-i}, c] = 1$ holds in G_n for $n > N_i$; it follows that there is an $N = \max\{M, N_i\}$ so that the finite set of relations used to transform w to the empty word hold in G_n for $n \geq N$, and thus w is trivial in G_n for $n \geq N$.

Now suppose $w = 1$ in G_n for infinitely many n; once again we show that w can be rewritten in a particular form in both G_n and G for sufficiently large n. By inserting pairs of the form $a^{-i}a^i$ adjacent to certain instances of the generator $c^{\pm 1}$ in the word w as in the proof of Theorem 4.3 and using a finite number of the fourth type of relation given in the presentation for G, the word w can be rewritten in the form $w_1 w_2$ (in both G and G_n for sufficiently large n) where w_1 is a word in the generators $a^{\pm 1}$ and $c^{\pm 1}$, and w_2 is a word in the generators $a^{\pm 1}$ and $b^{\pm 1}$.

Let $c_0 = c$ and $c_i = a^i ca^{-i}$ for $i \geq 1$. Then the word w_1 can be rewritten, in any G_n and in G, as a word in the $c_i^{\pm 1}$, at the expense of a power of a at the right, which we shift into w_2. So we may assume that w_1 is a word in the c_i, and w_2 is a word in a and b. For sufficiently large n, as a homeomorphism in G_n, the element w_1 has slope 1 near $x = 0$ and $x = 1$, so w_2 must be supported in $[\epsilon, 1 - \epsilon]$ for some $\epsilon > 0$. But for sufficiently large n, we know that w_1 is supported in $[0, \epsilon]$ as a homeomorphism in G_n. Therefore, since w is the identity in G_n for infinitely many of these n, it follows that w_2 must be the identity, and thus can be reduced to the empty word using R_1 and R_2. Therefore, w_1 is the identity in G_n for infinitely many n. But in both G_n for $n > M$ and in G, the element c_i commutes with c_j for every i and j. Since the supports of these elements, viewed as homeomorphisms, are disjoint for $n > M$, $exp_i(w_1)$, the net exponent of all occurrences of c_i, must be zero for every i. Thus w_1 can be transformed to the empty word in G, and hence w is the identity in G. □

Using the notation in the definition of a generalized double over a limit group, we remark that when $A = F$, $B = \mathbb{Z} \wr \mathbb{Z} = \langle a, c | [a^{-i}ca^i, a^{-j}ca^j] \text{ for all } i, j \in \mathbb{Z}\rangle$ and $C = \mathbb{Z}$, where the inclusions of C in A maps C to the subgroup generated by x_0, and the inclusion of C in B maps C to the subgroup generated by a, then $A *_C B$ is a generalized double over the limit group G obtained in Theorem 4.4.

In the list of motivating "natural" sequences of markings, it remains to consider the case where the additional generator g_n of G_n is taken to be x_0^n.

Theorem 4.5 *Let*

$$G_n = \langle a, b, c | R_1 = [ba^{-1}, a^{-1}ba], R_2 = [ba^{-1}, a^{-2}ba^2], c^{-1}a^n\rangle.$$

Then the sequence $(G_n, \{a, b, c\})$ *converges to* $(G, \{a, b, c\})$, *where*

$$G = \langle a, b, c | R_1, R_2, [c, a], [(a^{-j}ba^j)a^{-1}, c^{-i}(a^{-k}ba^k)c^i], i \geq 1, j \geq 0, k \geq 0\rangle.$$

Proof Note first that in both G and G_n, a and b generate a subgroup isomorphic to F, and that the relator $[c, a]$ holds in G_n for all n as $c = a^n$. As above, we identify x_0 with a, x_1 with b, and recall that $x_{i+1} = a^{-i}ba^i$; making these substitutions into the final relator in the presentation of G with $c = a^n$, we see that this relator, viewed in G_n, claims that x_{ni+k+1} commutes with $x_{j+1}x_0^{-1}$. As the support of x_{ni+k+1} is easily seen to be $[1 - \frac{1}{2^{ni+k+1}}, 1]$ and the support of $x_{j+1}x_0^{-1}$ is $[0, 1 - \frac{1}{2^{j+2}}]$, a relator of this form is satisfied in G_n as long as $j + 2 \leq ni + k + 1$, that is, $n \geq \frac{j-k+1}{i}$.

We note for later use that the last two types of relators in G, when combined, yield relations of the form

$$x_{j+1}(c^{-i}x_{k+2}c^i) = (c^{-i}x_{k+1}c^i)x_{j+1},$$

for $j \geq 0, i \geq 1, k \geq 0$. Furthermore, if $n \geq (j - k + 1)/i$, then this relation holds in G_n as well.

Suppose w is a word in a, b, c and their inverses which is the identity in G. Then it may be expressed as a product of conjugates of finitely many relators of G. But, for any finite set of relators, there is some M so that the relators all hold in G_n for $n \geq M$, so w is also the identity in G_n for all $n \geq M$.

Given any word w in a, b, c and their inverses, w can be expressed, in both G and G_n for any n, as

$$c^{n_1}w_1c^{n_2}w_2 \ldots c^{n_k}w_k,$$

where for each i, w_i is a word in $a^{\pm 1}$ and $b^{\pm 1}$ and $n_j \neq 0$ for $2 \leq j \leq k$. As usual, using the notation $x_0 = a, x_1 = b, x_{1+i} = a^{-i}ba^i$, we may assume for each i that w_i is a word in the standard infinite normal form for F. Next, since x_0 commutes with c in both G and G_n, and then also using relators involving just the x_j, that is, the standard relators in F which are a consequence of R_1 and R_2, we may assume (changing the w_i s without renaming) that w is of the form

$$x_0^a c^{n_1}w_1c^{n_2}w_2 \ldots c^{n_k}w_k x_0^{-b},$$

where a and b are positive integers and w_i is a word in infinite normal form without $x_0^{\pm 1}$, in both G and in G_n for any n.

Now suppose that w is the identity in G_n for infinitely many n. Then it follows that the total exponent sum of x_0 must be zero for those indices n, in other words, $(a - b) + (n_1 + n_2 + \quad + n_k)n = 0$ for each of those indices n. Therefore $a = b$ and $n_1 + n_2 + \quad + n_k = 0$. But $w = id$ in G_n if and only if $x_0^{-a}wx_0^a = id$, so we know that

$$w' = c^{n_1}w_1c^{n_2}w_2 \ldots c^{n_k}w_k,$$

where w_i is a word in infinite normal form without $x_0^{\pm 1}$, is the identity in G_n for infinitely many n. Moreover, conjugating if necessary, we may assume that $n_i \neq 0$ (for $i \neq 1$) and w_i is not the empty word, for all i. Now since $n_1 + n_2 + \quad + n_k = 0$, we may rewrite w' as:

$$w' = (c^{n_1} w_1 c^{-n_1})(c^{n_1+n_2} w_2 c^{-(n_1+n_2)}) \ldots (c^{n_1+ \quad n_{k-1}} w_{k-1} c^{-(n_1+ \quad n_{k-1})})(w_k).$$

We claim that the word w' must also be the identity in G. For if not, suppose that amongst the words of this form which are the identity in G_n for infinitely many n and are not the identity in G, the word w' has k minimal. Next, let $t = max(n_1, n_1 + n_2, \ldots, n_1 + n_2 + \quad + n_{k-1})$, and since w' is the identity (in G_n or G) if and only if $c^{-t} w' c^t$ is the identity, we may assume w' is of the form:

$$(c^{-m_1} w_1 c^{m_1})(c^{-m_2} w_2 c^{m_2}) \qquad (c^{-m_k} w_k c^{m_k}),$$

where $m_i \geq 0$ for all i, and at least one $m_i = 0$. For each value of i where $m_i = 0$, the subword $c^{-m_i} w_i c^{m_i} = w_i = p_i q_i$, where p_i (resp. q_i) is a positive word (resp. a negative word) in normal form involving no $x_0^{\pm 1}$. Thus, in G, using finitely many relators of the form

$$x_j (c^{-i} x_{s+1} c^i) = (c^{-i} x_s c^i) x_j$$

for $j \geq 1$, $i \geq 1$, we can move p_i to the left and q_i to the right of the expression, until eventually w can be written (reindexing) as $p(\Pi_{i=1}^{l} c^{-m_i} w_i c^{m_i})q$, where $l < k$ and p (resp. q) is a positive (resp. negative) word in infinite normal form. Since this can be done in G using only finitely many relations, for sufficiently large n it can be done in G_n as well, so we may assume it can be done in G_n for infinitely many n. Now notice that by choosing the minimal value of the index n to be perhaps even larger, we can ensure that once we replace c by x_0^n in G_n, the product $\Pi_{i=1}^{l} c^{-m_i} w_i c^{m_i}$, when written in the standard infinite normal form, involves only the generators x_j with j much larger that the subscripts of the generators in p or in q. But since w is the identity for infinitely many n, it follows that $p = q^{-1}$. Hence, $\Pi_{i=1}^{l} c^{-m_i} w_i c^{m_i}$, with $l < k$, is the identity in G_n for infinitely many n, but is not the identity in G, which is a contradiction, as we assumed that k was minimal. □

Using the notation in the definition of the generalized double over a limit group, we remark that when $A = F$, and $C = \mathbb{Z}$, where both inclusions of C in A map C to the subgroup generated by x_0, then $A *_C$ is a generalized double over the limit group G obtained in Theorem 4.5.

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