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A note on convexity properties of Thompson's group F

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A note on convexity properties of Thompson s group

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Abstract. We prove that Thompson's group F is not minimally almost convex with respect to any generating set which is a subset of the standard infinite generating set for F and which contains x_1. We use this to show that F is not almost convex with respect to any generating set which is a subset of the standard infinite generating set, generalizing results in [4].

1 Introduction

Convexity properties of a group G with respect to a finite generating set S yield information about the configuration of spheres within the Cayley graph (G, S) of G with respect to S. A finitely generated group G is *almost convex*(k), or $AC(k)$ with respect to a finite generating set X if there is a constant $L(k)$ satisfying the following property. For every positive integer n, any two elements x and y in the ball $B(n)$ of radius n with $d_X(x, y)$ k can be connected by a path of length $L(k)$ which lies completely within this ball. J. Cannon, who introduced this property in [2], proved that if a group G is $AC(2)$ with respect to a generating set X then it is also $AC(k)$ for all $k \geqslant 2$ with respect to that generating set. Thus if (G, X) is $AC(2)$, it is called *almost convex* with respect to that generating set. C. Thiel showed that almost convexity is generating set dependent [6].

Clearly, any two points in $B(n)$ can always be connected by a path of length $2n$. A weaker convexity condition is *minimal almost convexity*, which asks whether any two points in $B(n)$ at distance two can be connected by a path of length at most $2n - 1$ lying within this ball. A group G is said to be minimally almost convex with respect to a finite generating set X if the Cayley graph (G, X) has this property. If G is not minimally almost convex with respect to a finite generating set X, then (G, X) contains isometrically embedded loops of arbitrarily large circumference. I. Kapovich

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proved in [5] that any group which is minimally almost convex is also finitely presented.

J. Meier posed a conjecture relating these two notions of convexity. Namely, he conjectured that if a finitely generated group G is not minimally almost convex with respect to one finite generating set, then it cannot be almost convex with respect to any finite generating set. We prove the following special case of this conjecture. Suppose X and Y are two finite generating sets for a group G. Then G can be viewed as a metric space using the word length metric with respect to either generating set; we write (G, X) for G viewed as a metric space using length with respect to X. The identity map on G is a quasi-isometry between (G, X) and (G, Y). We prove this conjecture in the case that this quasi-isometry is a coarse isometry, that is, has multiplicative constant equal to one, in Theorem 3.1:

Theorem 3.1. *Let $f : (G, X_G) \to (H, X_H)$ be a C-coarse-isometry. If (G, X_G) is not minimally almost convex, then (H, X_H) is not almost convex.*

Convexity properties have been studied for Thompson's group F with respect to various generating sets. This group can be viewed either as a finitely or infinitely presented group, using the two standard presentations:

$$\langle x_k, k \geqslant 0 \mid x_i^{-1} x_j x_i = x_{j+1} \text{ if } i < j \rangle$$

or, as it is clear that x_0 and x_1 are sufficient to generate the entire group, since powers of x_0 conjugate x_1 to x_i for $i \geqslant 2$,

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$$

Thompson's group F is shown to be not almost convex with respect to $X_1 = \{x_0, x_1\}$ in [3] and not minimally almost convex with respect to X_1 in [1], and not almost convex with respect to the generating sets $X_n = \{x_0, x_1, \quad, x_n\}$ in [4]. In this note we extend these results in the following:

Theorem 4.2. *Let $X = \{x_0, x_1, x_{i_1}, x_{i_2}, \quad, x_{i_j}\}$, where $1 < i_1 < \quad < i_j$, be a generating set for F. Then F is not minimally almost convex with respect to X.*

We then apply Theorem 3.1 to prove:

Theorem 4.4. *Let X be any subset of the standard infinite generating set for F which includes x_0. Then F is not almost convex with respect to X.*

2 Computing word length in Thompson s group

In this section we summarize the method for computing word length of elements of F with respect to the generating sets $X_n = \{x_0, x_1, \quad, x_n\}$ which was introduced in [4], and refer the reader to that paper for complete details.

We follow standard terminology when describing elements of F as tree pair diagrams, that is, pairs of finite binary rooted trees comprised of *carets*. If a caret c has no children, we call c an *exposed* caret. If a caret c has a right (respectively left) child, we call the subtree rooted at that child the right (respectively left) subtree of caret c. Define the *level* of a caret inductively as follows. The root caret is defined to be at level 1, and the child of a level k caret has level $k+1$, for $k \geqslant 1$. We number the carets of each tree in infix order from 1 through n. Each element $g \quad F$ can be represented by an equivalence class of tree pair diagrams, among which there is a unique reduced tree pair diagram. We say that a pair of trees is *unreduced* if there are exposed carets in both trees with the same infix numbers. Such pairs of carets are removed until the tree pair diagram is *reduced*. This procedure produces the unique reduced tree pair diagram representing g. When we write $g = (T, S)$, we are assuming that this is the unique reduced tree pair diagram representing $g \quad F$.

Let T be a finite rooted binary tree. We often use the infix numbers of the carets as labels for the carets. A caret is said to be a right (resp. left) caret if one of its sides lies on the right (resp. left) side of T. The root caret can be considered either left or right. All other carets are called interior carets. The generator x_n in the presentations above is represented by a tree pair diagram (T_n, S_n), where S_n consists of a string of $n+1$ right carets, in which the final right caret also has a single exposed left child, and the tree T_n is simply a string of $n+2$ right carets.

To multiply two elements $g = (T_1, T_2)$ and $h = (S_1, S_2)$ of F we add carets to create unreduced representatives for the two elements, $g = (T_1', T_2')$ and $h = (S_1', S_2')$ in which $S_2' = T_1'$. The product gh is then given by the (possibly unreduced) tree pair diagram (S_1', T_2').

Our formula for the word length of an element $g \quad F$ with respect to the generating set $X_n = \{x_0, x_1, \quad , x_n\}$ has two components. The first we call $l_\infty(g)$, as it is the word length of g with respect to the standard infinite generating set $\{x_i \mid i \geqslant 0\}$ for F. This quantity is simply the number of carets in the unique reduced tree pair diagram representing g which are not right carets. The second component in the word length formula is twice what we term the *penalty weight* $p_n(g)$ of the element. This nonnegative quantity is determined by the combinatorial relationships between a set of distinguished caret pairs in the tree pair diagram for g. We will need a consequence of the following theorem, which computes $l_n(g)$:

Theorem 2.1 ([4, Theorem 3.3]). *For every $g \quad F$, the word length of g with respect to the generating set $X_n = \{x_0, x_1, \quad , x_n\}$ is given by the formula*

$$l_{X_n}(g) = l_n(g) = l_\infty(g) + 2p_n(g)$$

where $l_\infty(g)$ is the number of carets in the reduced tree pair diagram for g which are not right carets and $p_n(g)$ is the penalty weight of g.

In particular, since $p_n(g) \geqslant 0$ it is always the case that $l_n(g) \geqslant l_\infty(g)$.

3 Coarse isometries and convexity

Recall that a map f between two metric spaces G and H is a *quasi-isometry* if there are positive constants K and C so that for every pair of points $g_1, g_2 \in G$,

$$\frac{1}{K} d_G(g_1, g_2) - C \leq d_H(f(g_1), f(g_2)) \leq K d_G(g_1, g_2) + C$$

If the constant K can be chosen to be 1, we call f a *C-coarse isometry*. Given a group G and a finite generating set X, G can be regarded as a metric space using the word length metric, namely, $d_X(g,h) = \min\{n \mid gh^{-1} = a_1 a_2 \cdots a_n, a_i^{\pm 1} \in X\}$. We denote G, viewed as a metric space in this way, by (G, X). Equivalently, one can view the Cayley graph $\Gamma(G, X)$ as a metric space by declaring each edge to have length 1. Recall that for any finitely generated group G with finite generating sets X and Y, the identity map between (G, X) as (G, Y) is a quasi-isometry. In general, it is unknown to what extent quasi-isometries preserve convexity properties, but in the special case of a coarse-isometry, we obtain the following:

Theorem 3.1. *Let $f : (G, X_G) \to (H, X_H)$ be a C-coarse-isometry. If (G, X_G) is not minimally almost convex, then (H, X_H) is not almost convex.*

Proof. Let g be any coarse inverse for f, which is easily seen to be a coarse isometry as well. Without loss of generality, we may assume that g is also a C-coarse isometry.

Suppose that (H, X_H) is almost convex. Then for each $n \geqslant 2$, there is an almost convexity constant $K(n)$. Fix $M > 3C + 1$, and let $K = K(2M + C)$. Let $n > K + M + KC$.

Since (G, X_G) is not minimally almost convex, we can find $x, y \in B(n) \subset \Gamma(G, X_G)$ with $d_G(x, y) = 2$ so that the shortest path from x to y which remains in $B(n)$ has length $2n$. Since we can always construct a path of this length passing through the identity, let γ be such a path containing the identity.

Consider the closed loop η obtained by concatenating γ with the path of length two between x and y. Let z denote the point in $B(n+1)$ at distance one from x and y. Choose a and b on γ, with a on the subpath of γ from x to the identity, and b between y and the identity, so that $d_G(a, Id) = d_G(b, Id)$ and $d_G(a, z) = d_G(b, z) = M$. Let η_1 be the subpath of γ containing a, b and the identity, and η_2 is the remaining subpath of η.

Consider $f(a)$ and $f(b)$, elements of the Cayley graph $\Gamma(H, X_H)$. We know that $d_H(f(a), f(b)) \leq 2M + C$. Since we are assuming that (H, X_H) is almost convex, there must be a path ξ from $f(a)$ to $f(b)$ whose length is at most K, and which remains in the ball $B(D)$, where $D = \max\{d_H(f(a), id), d_H(f(b), id)\} \leq d_G(a, id) + C$.

Consider the image of ξ under g, the coarse inverse to f. Since

$$\text{length}(\eta_1) = 2n - 2M + 2 > 2(K + KC + M) - 2M + 2 > 2K + 2KC$$

and

$$\text{length}(g(\xi)) \leq K + KC,$$

we see that length$(g(\xi)) <$ length(η_1). We now show that this path stays in $B(n)$, contradicting the fact that any path from x to y in $B(n)$ has length $2n$.

The maximum distance of any point on ξ from the identity in H is D. Thus the maximum distance of any point on $g(\xi)$ from the identity of G is

$$D + 2C \quad d_G(a, id) + 3C = n - M + 1 + 3C.$$

Since $M > 3C + 1$, it follows that $g(\xi) \quad B_G(n)$.

By concatenating the portion of η_2 from x to a, $g(\xi)$, and the portion of η_2 from b to y, we obtain a path from x to y which remains inside of $B(n)$ and has length less than $2n$, a contradiction since (G, X_G) is not minimally almost convex. □

4 Convexity results

The main goal of this section is to show that F is not almost convex with respect to any generating set which is a subset of the standard infinite generating set; we note that in order for a subset of the standard infinite generating set to generate F, it must contain x_0.

We begin with the following:

Theorem 4.1. *Let $X_n = \{x_0, x_1, \quad , x_n\}$ be a generating set for F with $n \geqslant 2$. Then F is not minimally almost convex with respect to X_n.*

Proof. We prove this by providing, for any $k > 0$, a pair of group elements $g = g_k$ and $h = h_k$ satisfying $l_n(g) = l_n(h) = 2k + 2$ and $l_n(h^{-1}g) = 2$, so that any path γ from g to h that lies entirely within the ball of radius $2k + 2$ must have length at least $4k + 4$.

Let $g = g_n = x_1^{k+1} x_{k+n+1} x_0^{-k} = x_n x_1^{k+1} x_0^{-k}$ and $h = h_n = g x_0^{-1} x_n^{-1} = x_1^{k+1} x_0^{-(k+1)}$. The tree pair diagrams for these elements are given in Figure 1. In the tree pair diagrams (T_-, T_+) for g and (H_-, H_+) for h, we observe that $l_\infty(g) = l_\infty(h) = 2k + 2$. Furthermore, $l_\infty(g x_0^{-1}) = 2K + 3$. Recall that $l_n(a) \geqslant l_\infty(a)$ for all $a \quad F$ is a consequence of Theorem 2.1, and since we have provided words above of length $2k + 2$ for both g and h, it follows that $l_n(g) = l_n(h) = 2k + 2$, and $l_n(g x_0^{-1}) = 2K + 3$.

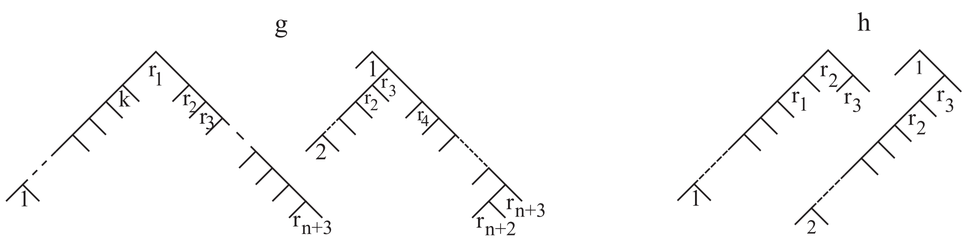

Figure 1. The tree pair diagrams representing the elements g and h used in the proof of Theorem 4.1.

Suppose there is a path γ from g to h which lies within the ball of radius $2k+2$. In T_-, the caret r_{n+2} it is a right caret at level $n+2$. Our argument relies on noting the level of this caret at successive vertices along the path γ.

In order for the path γ to terminate at h, there is a point at which the pair of carets numbered r_{n+2} in each tree must be removed as part of a reduction along γ. This requires caret r_{n+2} from T_- to be an interior caret at the point of reduction. Given the effect of multiplication by each generator on the tree pair diagram, we observe that the generators in X_n cannot move any right caret off the right side of the left tree in a tree pair diagram unless its level is between 1 and n. Hence, there is a smallest nontrivial prefix γ_0 of γ so that in $g\gamma_0 = f$ the caret r_{n+2} in the left tree for f is a right caret at level $n+1$.

Let (S_-, S_+) be the tree pair diagram for $f = g\gamma_0$ which is constructed from the tree pair diagram (T_-, T_+) for g by altering these trees according to multiplication by each generator of γ_0, but without performing any possible reductions. During this process, the carets in T_+ remain unchanged, though additional carets may be added to T_+ to form S_+. Hence, S_+ contains T_+ as a subtree, and the tree pair diagram (S_-, S_+) may be unreduced.

We first show that the tree pair diagram (S_-, S_+) constructed in this way must be unreduced, and that when the reduction is accomplished, some of the original carets from T_+ will be removed from S_+. If this was not the case, then in S_- there would be at least $k+1$ carets with smaller infix numbers than r_{n+1} which were not right carets, and thus counted towards $l_\infty(f)$. Additionally, in S_+ there would also be $k+1$ interior carets with infix numbers less than r_{n+2}, and caret r_{n+2} itself is also an interior caret. This implies that $l_\infty(f) \geqslant 2k+3$, contradicting the fact that $f \in B(2k+2)$. Thus there must be some reduction of the carets of T_+, viewed as a subtree of S_+, in order to obtain the reduced tree pair diagram for $g\gamma_0 = f$.

We now consider which carets of T_+, viewed as a subtree of S_+ might be removed to reduce the tree pair diagram; in order for a caret to be reduced after multiplication by a particular generator, it must be exposed. The only exposed carets of T_+ itself are carets 2 and r_{n+2}. Since caret r_{n+2} is a right caret in S_-, and not the final right caret, it is not exposed in S_-. Therefore, it must be that in reducing (S_-, S_+), the original caret 2 from the infix ordering on T_+ must be removed. We claim that in S_-, caret 2 must be in the right subtree of caret 1. If, in forming S_+, no carets were added to either leaf of caret 2, then caret 2 is exposed in S_+, and hence it is exposed in S_-, which implies that caret 2 is in the right subtree of caret 1 in S_-. If, on the other hand, carets were added to the leaves of caret 2 in forming S_+, then they must all be removed in (S_-, S_+) before caret 2 is removed. But this means that in S_-, once these carets are removed, caret 2 is again exposed, and once again, caret 2 must be in the right subtree of caret 1 in S_-.

The fact that caret 2 is in the right subtree of caret 1 in S_- provides a lower bound on $l_n(h^{-1}f)$ as follows. To form the tree pair diagram for $h^{-1}f$, consider the unreduced tree pair diagram (S_-, S_+). If $h = (H_-, H_+)$, to form this product we consider these trees in the order $S_-\ S_+\ H_+\ H_-$, and add carets to each pair to ensure that the middle trees are identical. Thus we must at least add the string of right carets $r_4, \ldots, r_{n+1}, r_{n+3}$, with caret r_{n+2} the left child of r_{n+3}, from S_+ to both trees in the diagram (H_+, H_-) in order to perform this multiplication. Since in S_-, caret 2 must be removed before caret 1, but in H_- the reverse is true, caret 1 cannot be removed to reduce the product $h^{-1}f$. Hence, because of their configuration in H_-, the entire string of carets $1, 2, \quad, k, r_1$ are not removed to reduce the product $h^{-1}f$. Also, as we remarked above, caret r_{n+2} is not removed through reduction in this product. Hence we obtain the following lower bound on the word length of $h^{-1}f : l_n(h^{-1}f) \geqslant l_\infty(h^{-1}f) \geqslant 2(k+1) + 1 = 2k+3$

Let γ_1 be the subpath of γ from $f = g\gamma_0$ to h. Since $l_n(h^{-1}f) \geqslant 2k+3$, it follows that $|\gamma_1| \geqslant 2k+3$. But traversing γ_0 in reverse, followed by x_0^{-1} and then x_n^{-1} yields another path from f to h, so similarly $|\gamma_0| + 2 \geqslant 2k+3$, and hence $|\gamma_0| \geqslant 2k+1$. This implies that $|\gamma| = |\gamma_0| + |\gamma_1| \geqslant 4k+4$. □

In the proof above, both g and h are be represented by words of length $2k+2$ involving only the generators $x_0^{\pm 1}, x_1^{\pm 1}$, and $x_n^{\pm 1}$, namely, $g = x_n x_1^{k+1} x_0^{-k}$ and $h = x_1^{k+1} x_0^{-(k+1)}$. Hence, the above result can be extended to any generating set for F which is a finite subset of the standard infinite generating set containing x_0 and x_1.

Theorem 4.2. *Let $X = \{x_0, x_1, x_{i_1}, x_{i_2}, \quad, x_{i_j}\}$, where $1 < i_1 < \quad < i_j$, be a generating set for F. Then F is not minimally almost convex with respect to X.*

Proof. The identity map on G is a quasi-isometry between the metric spaces (G, X) and (G, X_{i_j}), where $X_{i_j} = \{x_0, x_1, x_2, x_3 \quad, x_{i_j}\}$. Since $X \quad X_{i_j}$, we remark that $d_{X_{i_j}}(a,b) \quad d_X(a,b)$ for any $a, b \quad F$. In particular, $d_{X_{i_j}}(a, Id) \quad d_X(a, Id)$ for any $a \quad F$.

Assume that (F, X) is minimally almost convex. It is proven in Theorem 4.1 that (F, X_{i_j}) is not minimally almost convex. Let $h = h_k = x_1^{k+1} x_0^{-(k+1)}$ and $g = g_k = x_1^{k+1} x_{k+i_j+1} x_0^{-k}$ be the group elements used in the proof of Theorem 4.1. It is clear that $2k+2 = d_{X_{i_j}}(h, id) = d_X(h, id)$ and $2k+2 = d_{X_{i_j}}(g, id) = d_X(g, id)$; if there was a shorter expression for either g or h with respect to X, then there would be one with respect to X_{i_j} as well. In addition, it is clear that since $g^{-1}h = x_{i_j+1}^{-1} x_0^{-1} = x_0^{-1} x_{i_j}^{-1}$, we have $d_{X_{i_j}}(g, h) = d_X(g, h) = 2$.

Since (F, X) is assumed to be minimally almost convex, there is a path γ of length at most $4k+3$ connecting g and h which lies within the ball of radius $2k+2$ relative to X. Since each group element a along this path satisfies

$$d_{X_{i_j}}(a, id) \quad d_X(a, id) \quad 2k+2,$$

this contradicts the assumption that (F, X_{i_j}) is not minimally almost convex. Thus we conclude that (F, X) cannot be minimally almost convex. □

To show that F is not almost convex with respect to arbitrary finite subsets of the infinite generating set containing x_0, we show first that word length with respect to one of these arbitrary generating sets differs from word length with respect to some generating set containing x_1 only by an additive constant.

Lemma 4.3. *Let* $X = \{x_0, x_{i_1}, x_{i_2}, \quad , x_{i_j}\}$ *be a generating set for* F, *and form a new generating set* $Y = \{x_0, x_1, x_{i_2-i_1+1}, x_{i_3-i_1+1}, \quad x_{i_j-i_1+1}\}$. *Then* (F, X) *and* (F, Y) *are coarsely isometric.*

Proof. Let $g \quad F$, and suppose $g = {}_1\,{}_2 \quad {}_m$, where ${}_k^{\pm 1} \quad Y$. Then

$$g = x_0^{i_1-1}(x_0^{1-i_1} g x_0^{i_1-1}) x_0^{1-i+1} = x_0^{i_1-1}\overline{_1}\,\overline{_2} \quad \overline{_m} x_0^{i_1-1},$$

where $\overline{_k} = x_0^{1-i_1} \; {}_k x_0^{i_1-1}$. Now in the cases where ${}_k = x_0^{\pm 1}$, we have $\overline{_k} = {}_k$, and in the cases where ${}_k = x_l^{\pm 1}$ with $l \geqslant 1$, then $\overline{_k} = x_{l+i_1-1}^{\pm 1} \quad X$. Hence

$$l_X(g) \quad l_Y(g) + 2(i_1 - 1).$$

Similarly, one sees that $l_Y(g) \quad l_X(g) + 2(i_1 - 1)$. Hence,

$$l_X(g) - 2(i_1 - 1) \quad l_Y(g) \quad l_X(g) + 2(i_1 - 1). \qquad \square$$

Finally, we apply Theorem 3.1 to F with the two generating sets X and Y of the preceding theorem to obtain:

Theorem 4.4. *Let* X *be any subset of the standard infinite generating set for* F *which includes* x_0. *Then* F *is not almost convex with respect to* X.

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