6-1-2013

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Melanie Stein
Trinity College Hartford

Jennifer Taback
Bowdoin College

Michigan Math. J. 62 2013), 365–386

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1. Introduction

We investigate the metric properties of a family of groups whose Cayley graphs with respect to a carefully chosen generating set are the Diestel–Leader graphs $\mathrm{DL}_d(q)$, which are subsets of a product of d infinite trees of valence $q+1$. We call these groups *Diestel–Leader groups* and denote them $\Gamma_d(q)$. More general Diestel–Leader graphs were introduced in [7] as a possible answer to the question: "Is any connected, locally finite, vertex transitive graph quasi-isometric to the Cayley graph of a finitely generated group?" It was first shown in [8] that $\mathrm{DL}_2(m,n)$, the Diestel–Leader graph that is a subset of a product of two trees of respective valences $m+1$ and $n+1$, is not quasi-isometric to the Cayley graph of any such finitely generated group. It is proved in [1] that Diestel–Leader graphs that are subsets of the product of any number of trees of differing valence are not Cayley graphs of finitely generated groups.

It is well known that the Cayley graph of the wreath product $L_n = \mathbb{Z}_n \wr \mathbb{Z}$, often called the *lamplighter group,* with respect to the generating set $\{t, ta, ta^2, \dots, ta^{n-1}\}$ (where a is the generator of $\mathbb{Z}_n$ and t generates $\mathbb{Z}$) is the Diestel–Leader graph $\mathrm{DL}_2(n)$ (see e.g. [2; 14; 15]). This graph is a subset of the product of two trees of constant valence $n+1$. The groups studied in this paper provide a geometric generalization of the family of lamplighter groups because their Cayley graphs generalize the geometry of the lamplighter groups; that is, their Cayley graphs with respect to a natural generating set $S_{d,q}$ are the "larger" Diestel–Leader graphs $\mathrm{DL}_d(q)$, which are subsets of the product of d trees of constant valence $q+1$ (and are defined explicitly in Section 2).

Bartholdi, Neuhauser, and Woess [1] present a construction of a group that we denote $\Gamma_d(q)$, a generating set $S_{d,q}$, and an identification of the graph $\mathrm{DL}_d(q)$ with the Cayley graph $\Gamma(\Gamma_d(q), S_{d,q})$. They also provide a simple criterion for when their construction holds: $d=2$; $d=3$; or, if $d \geq 4$ and $q = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is the prime power decomposition of q, then $p_i > d-1$ for all i. They show that the

Received February 26, 2012. Revision received April 19, 2012.

The second author acknowledges support from National Science Foundation Grant nos. DMS-0604645 and DMS-1105407. Both authors acknowledge support from a Bowdoin College Faculty Research Award. Many thanks to Sean Cleary, David Fisher, Martin Kassabov, Tim Riley, Peter Wong, and Kevin Wortman for helpful conversations during the writing of this paper. The authors wish to thank the anonymous referee for a careful reading of the paper.

groups $\Gamma_d(q)$ are type $F_{d\ \ 1}$ but not F_d when $d \geq 3$ and hence are not automatic. We note that there are still open cases where it is not known whether $\mathrm{DL}_d(q)$ is the Cayley graph of a finitely generated group; the smallest open case is $\mathrm{DL}_4(2)$.

Random walks in the Cayley graph $\Gamma(\Gamma_d(q), S_{d\ q})$ are studied in [1], where a presentation for the group is given explicitly. For example, when $d = 3$ the authors obtain the presentation

$$\Gamma_3(m) \cong \langle a, s, t \mid a^m = 1,\ [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \rangle.$$

When $m = p$ is prime, it is shown in [6] that $\Gamma_3(p)$ is a cocompact lattice in $\mathrm{Sol}_5(\ \ _p((t)))$ and that its Dehn function is quadratic. The Dehn function of $\Gamma_3(m)$ is studied for any m in [10], where it is shown to be at most quartic. It was mentioned to the authors by K. Wortman that arguments analogous to those of Gromov in [9] imply that the Dehn function of $\Gamma_d(q)$ is quadratic regardless of the values of $d \geq 3$ and q. When the relation $a^m = 1$ is removed from the above presentation, one obtains Baumslag s metabelian group Γ—which, in contrast to $\Gamma_3(m)$, has exponential Dehn function [10]. Baumslag defined this group to provide the first example of a finitely presented group with an abelian normal subgroup of infinite rank.

It is noted in [1] that $\Gamma_d(q)$ is in most cases an automata group and hence a self-similar group. Metric properties of self-similar groups are in general not well understood. In this paper we seek to answer some of the standard geometric group-theoretic questions related to metric properties of groups and their Cayley graphs for these Diestel–Leader groups $\Gamma_d(q)$. Such properties often rely on the ability to compute word length of elements within the group; we begin by proving that a particular combinatorial formula yields the word length of elements of $\Gamma_d(q)$ with respect to the generating set $S_{d\ q}$. This formula relies on the symmetry present in the Diestel–Leader graph, and we subsequently use it to prove that $\Gamma_d(q)$ has dead-end elements of arbitrary depth with respect to $S_{d\ q}$. This generalizes a result of Cleary and Riley [4; 5] which proves that $\Gamma_3(2)$ with respect to a generating set similar to $S_{3\ 2}$ has dead-end elements of arbitrary depth, the first example of a finitely presented group with this property. The word-length formula is used in later sections to show that $\Gamma_d(q)$ has infinitely many cone types and thus no regular language of geodesics with respect to $S_{d\ q}$.

2. Definitions and Background

To define $\mathrm{DL}_d(q)$, let T be a homogeneous, locally finite, connected tree in which the degree of each vertex is $q + 1$. This tree has an orientation such that each vertex v has a unique predecessor $v\ \ $ and q successors $w_1, w_2, \ldots, w_q$ where $w_i\ \ = v$ for $1 \leq i \leq q$. The transitive closure of the set of relationships of the form $v\ \ < v$ induces the partial order $\ \ $. In this partial order, any two vertices $v, w \in T$ have a greatest common ancestor $v \curlywedge w$. Choose a base point $o \in T$ and define a height function $h(v) = d(v, o \curlywedge v)\ \ \ d(o, o \curlywedge v)$, where $d(x, y)$ denotes the number of edges on the unique path in T from x to y. With this definition, note that $h(v\ \ ) = h(v)\ \ \ 1$.

Let $T_1, T_2, \dots, T_d$ denote d copies of the tree T with base points o_i and height functions h_i for $1 \le i \le d$. The Diestel–Leader graph $\mathrm{DL}_d(q)$ is the graph whose vertex set $V_d(q)$ is the set of d-tuples $(x_1, x_2, \dots, x_d)$, where x_i is a vertex of T_i for each i and also $h_1(x_1) + \cdots + h_d(x_d) = 0$. Two vertices $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ are connected by an edge if and only if there are two indices i and j, with $i \ne j$, such that x_i and y_i are connected by an edge in T_i; x_j and y_j are connected by an edge in T_j; and $x_k = y_k$ for $k \ne i, j$.

There is a projection $\Pi \colon V_d(q) \to (\mathbb{Z}^2)^d$ given by

$$\Pi(x) = \Pi(x_1, x_2, \dots, x_d) = ((m_1, l_1), (m_2, l_2), \dots, (m_d, l_d)),$$

where $m_i = d(o_i, o_i \curlywedge x_i)$ and $l_i = d(x_i, o_i \curlywedge x_i)$. In particular, $0 \le m_i$ and $0 \le l_i$ for all i. Note that, in T_i, the shortest path from o_i to x_i has length $m_i + l_i$, and recall that $h_i(x_i) = l_i \quad m_i$. The defining conditions of the Diestel–Leader graph ensure that $\sum_{i=1}^{d}(l_i \quad m_i) = 0$.

In [1] it is shown that these graphs are Cayley graphs of certain matrix groups when a simple condition is satisfied. Specifically, let $\ _q$ be a commutative ring of order q with multiplicative unit 1, and suppose $\ _q$ contains distinct elements $l_1, \dots, l_{d\ \ 1}$ such that if $d \ge 3$ then their pairwise differences are invertible. Define a ring of polynomials in the formal variables t and $(t + l_i)^{\ \ 1}$ for $1 \le i \le d \quad 1$ with finitely many nonzero coefficients lying in $\ _q$:

$$\mathcal{R}_d(\ _q) = \ _q[t, (t + l_1)^{\ \ 1}, (t + l_2)^{\ \ 1}, \dots, (t + l_{d\ \ 1})^{\ \ 1}].$$

It is proved in [1] that the group $\Gamma_d(\ _q)$ (which we denote by $\Gamma_d(q)$) of affine matrices of the form

$$\begin{pmatrix} (t + l_1)^{k_1} \cdots (t + l_{d\ \ 1})^{k_{d\ \ 1}} & P \\ 0 & 1 \end{pmatrix} \quad \text{with } k_1, k_2, \dots, k_{d\ \ 1} \in \mathbb{Z} \text{ and } P \in \mathcal{R}_d(\ _q)$$

has Cayley graph $\mathrm{DL}_d(q)$ with respect to the generating set $S_{d\ q}$ consisting of the matrices

$$\begin{pmatrix} t + l_i & b \\ 0 & 1 \end{pmatrix}^{\pm 1} \quad \text{with } b \in \ _q,\ i \in \{1, 2, \dots, d \quad 1\}$$

and

$$\begin{pmatrix} (t + l_i)(t + l_j)^{\ \ 1} & b(t + l_j)^{\ \ 1} \\ 0 & 1 \end{pmatrix} \quad \text{with } b \in \ _q,\ i, j \in \{1, 2, \dots, d \quad 1\},\ i \ne j.$$

It turns out that $\ _q$ always contains distinct elements $l_1, \dots, l_{d\ \ 1}$ satisfying the invertibility conditions for pairwise differences when $d = 2$ or $d = 3$. When $d \ge 4$, however, $\ _q$ contains the desired elements only if all primes in the prime power decomposition of $q = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ satisfy $p_i > d \quad 1$ for all i. We refer the reader to [1] for more details on this construction and on the identification between the group and the Cayley graph $\mathrm{DL}_d(q)$.

In exploring the metric properties of the groups $\Gamma_d(q)$, and hence of the Cayley graphs $\mathrm{DL}_d(q)$, one often needs to keep track of *edge types* along a path in $\mathrm{DL}_d(q)$ rather than the specific generators that label the edges along the path. Given any vertex $x = (x_1, x_2, \dots, x_d)$, by an edge of type $\mathbf{e}_i \quad \mathbf{e}_j$ emanating from vertex x

we mean an edge with one endpoint at x and the other at $y = (y_1, y_2, \ldots, y_d)$, where $y_k = x_k$ for $k \notin \{i, j\}$ and where $y_i \; = x_i$ and $y_j = x_j$. Note that $h_i(y_i) = h_i(x_i) + 1$ and $h_j(y_j) = h_j(x_j) \quad 1$. There are exactly q possible choices for y_i, so there are q distinct edges of type $\mathbf{e}_i \quad \mathbf{e}_j$ emanating from x.

Since the vertices of $\mathrm{DL}_d(q)$ are identified with the elements of $\Gamma_d(q)$, we abuse notation and consider the projection map Π to be a map from the group $\Gamma_d(q)$ to $(\mathbb{Z}_2)^d$; thus we write

$$\begin{aligned}\Pi(g) &= \Pi(x = (x_1, x_2, \ldots, x_d)) \\ &= \; (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g)))\end{aligned}$$

when $g \in \Gamma_d(q)$ is identified with the vertex x in $\mathrm{DL}_d(q)$. We remark that the base point vertex $o = (o_1, \ldots, o_d)$ in $\mathrm{DL}_d(q)$ is identified with the identity element in $\Gamma_d(q)$.

3. Computing Word Length in $\Gamma_d \; q)$ with Respect to $S_{d \; q}$

Let $S_{d \; q}$ be the generating set for $\Gamma_d(q)$, so that the Cayley graph $\Gamma(\Gamma_d(q), S_{d \; q})$ is $\mathrm{DL}_d(q)$. We will show that the word length of an element with respect to $S_{d \; q}$ depends only on $\Pi(g)$ and not on g itself. In the course of establishing the formula for word length, it is often sufficient to keep track of a path s edge types rather than its edge labels. Given a vertex $v \in \mathrm{DL}_d(q)$, we have defined edges of type $\mathbf{e}_i \quad \mathbf{e}_j$ emanating from v. By a path of type $\;_1 \;_2 \cdots \;_r$ starting at v, where $\;_k = (\mathbf{e}_{i_k} \quad \mathbf{e}_{j_k})^{p_k}$ with $p_k \geq 0$ for each k, we mean a path beginning at v that follows p_1 edges of type $\mathbf{e}_{i_1} \quad \mathbf{e}_{j_1}$, then p_2 edges of type $\mathbf{e}_{i_2} \quad \mathbf{e}_{j_2}$, and so on.

We begin by defining a function f from $\Gamma_d(q)$ to the natural numbers that is a candidate for the word length function $l\colon \Gamma_d(q) \to \mathbb{N}$ for elements of $\Gamma_d(q)$ with respect to the generating set $S_{d \; q}$. For an element $g \in \Gamma_d(q)$, the value of $f(g)$ actually depends only on the d-tuple of ordered pairs

$$\Pi(g) = \; (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g))).$$

In order to define this function, one first considers all permutations of these ordered pairs. For each permutation the goal is to construct a path in $\mathrm{DL}_d(q)$, from the identity vertex to g, in an order specified by the permutation. For a given permutation $\sigma \in \Sigma_d$, one component of the length $f_\sigma(g)$ of this path is found by maximizing several quantities related to σ. To obtain the word length of g, we minimize over the lengths of these paths.

For a given permutation $\sigma \in \Sigma_d$, define the following quantities related to the length of the path determined by σ from the identity to g.

DEFINITION 1. Let $g \in \Gamma_d(q)$, with

$$\Pi(g) = \; (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g)))$$

and with σ in Σ_d, the symmetric group on d letters. Define

- $A_{\sigma(d)}(g) = \sum_{j=1}^{d} m_{\sigma(j)}(g)$ and $A_{\sigma(i)}(g) = \sum_{j=2}^{i} m_{\sigma(j)}(g) + \sum_{k=i}^{d \; 1} l_{\sigma(k)}(g)$ for $2 \leq i \leq d \quad 1$,
- $f_\sigma(g, i) = m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(i)}(g)$ for $2 \leq i \leq d$, and
- $f_\sigma(g) = \max_{2 \leq i \leq d} f_\sigma(g, i)$.

We use these quantities to define the function f, which we shall prove yields the word length of $g \in \Gamma_d(q)$ with respect to the generating set $S_{d\ q}$.

DEFINITION 2. For $g \in \Gamma_d(q)$, let $f(g) = \min_{\sigma \in \Sigma_d} f_\sigma(g)$.

EXAMPLE. Let $d = 3$, choose any g with $\Pi(g) = ((5,5),(3,10),(7,0))$, and consider $\sigma \in \Sigma_3$.

- When $\sigma = \text{id}$ is the identity permutation, the preceding definitions yield $f_{\text{id}}(g,2) = 5 + 0 + (3 + 10) = 18$ and $f_{\text{id}}(g,3) = 5 + 0 + (5 + 3 + 7) = 20$. Thus, $f_{\text{id}}(g) = 20$.
- Choosing $\sigma = (1\ 2)$, we have $f_{(1\ 2)}(g,2) = 3 + 0 + (5 + 5) = 13$ and $f_{(1\ 2)}(g,3) = 3 + 0 + (5 + 3 + 7) = 18$; hence $f_{(1\ 2)}(g) = 18$.
- For any σ we have $f_\sigma(g,3) = m_{\sigma(i)}(g) + l_{\sigma(j)}(g) + 15 \geq 18$, so $f_\sigma(g) \geq 18$ for all $\sigma \in \Sigma_3$.

Minimizing over all $\sigma \in \Sigma_3$, we conclude that $f(g) = 18$.

In order to establish that the function f just defined is the word-length function, we use the following general lemma.

LEMMA 1. *Given a group G with generating set S, let $l\colon G \to \mathbb{N} \cup \{0\}$ be the word length with respect to S. Let $f\colon G \to \mathbb{N} \cup \{0\}$ be another function that satisfies the following statements*:

(1) $f(g) = 0$ *if and only if g is the identity element*;
(2) *for every $g \in G$, $l(g) \geq f(g)$*;
(3) *for every $g \in G$, there exists some $s \in S$ with $f(gs) = f(g) \quad 1$.*

Then $l(g) = f(g)$ for every $g \in G$.

Proof. Let $g \in G$, and suppose $f(g) = n$. Then by property (3) there exist $s_1, s_2, \ldots, s_n \in S$ satisfying $f(gs_1s_2 \cdots s_n) = 0$. By property (1), $g = s_n^{\ 1} \cdots s_2^{\ 1} s_1^{\ 1}$ and so $l(g) \leq f(g)$. Hence by property (2) we have $l(g) = f(g)$. □

Clearly, for the function f defined in Definition 1 we have $f(g) = 0$ if and only if g is the identity element. The other two properties of the function f will be verified in Propositions 2 and 8. It then will follow from Lemma 1 that the function f defined in Definition 1 is the word-length function for $\Gamma_d(q)$ with respect to the generating set $S_{d\ q}$.

PROPOSITION 2. *Let $g \in \Gamma_d(q)$ with*

$$\Pi(g) = \ (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g))),$$

let $f(g)$ be as in Definition 1, and let $l(g)$ be the word length of g with respect to the generating set $S_{d\ q}$. Then $l(g) \geq f(g)$.

Proof. Let γ be a path of length n in $\mathrm{DL}_d(q)$ from o to the vertex x identified with g; thus γ corresponds naturally to a word $a_1a_2a_3 \ldots a_n$ with $a_i \in S_{d\ q}$ for $1 \leq i \leq n$. We will show that for some choice of $\sigma \in \Sigma_d$, we have $n \geq f_\sigma(g,i)$ for every $2 \leq i \leq d$. It follows that $n \geq f_\sigma(g) \geq f(g)$ and thus $l(g) \geq f(g)$.

We begin by choosing the permutation $\sigma \in \Sigma_d$. Along the path γ from o to x, there must be points where the kth coordinate is $y_k = o_k \curlywedge x_k$ for $1 \leq k \leq d$. Let v^1 be the first such point, so that $v^1_{i_1} = y_{i_1}$ for some i_1 with $1 \leq i_1 \leq d$. By the definition of v^1, we know that $v^1_k \curlywedge x_k = y_k$ for $k \neq i_1$. Thus, on the portion of the path from v^1 to x, there must be points where the kth coordinate is $y_k = o_k \curlywedge x_k$ for each $1 \leq k \leq d$, $k \neq i_1$. Let v^2 be the first such point; then $v^2_{i_2} = y_{i_2}$ for some i_2 with $1 \leq i_2 \leq d$, $i_2 \neq i_1$. Continuing in this manner, we define points $v^1, v^2, \dots, v^d$, each with a distinct associated coordinate $i_1, i_2, \dots, i_d$ and such that the (i_k)th coordinate of v^k is y_{i_k}. Let $\sigma \in \Sigma_d$ be the unique permutation defined by $\sigma(k) = i_k$ for $1 \leq k \leq d$.

First we consider the point v^j, $2 \leq j \leq d-1$, and suppose that the prefix $a_1 \dots a_r$ corresponds to the subpath of γ starting at o and ending at v^j. Then, for every p with $1 \leq p \leq j$, the path $a_1 \dots a_r$ must contain at least $m_{\sigma(p)}(g)$ edges of type $\mathbf{e}_t - \mathbf{e}_{\sigma(p)}$; here $t \neq \sigma(p)$ may vary by edge, so $r \geq \sum_{p=1}^{j} m_{\sigma(p)}(g)$. However, for every p with $j \leq p \leq d$, the path $a_{r+1} \dots a_n$ must contain at least $l_{\sigma(p)}(g)$ edges of type $\mathbf{e}_{\sigma(p)} - \mathbf{e}_t$ (where again $t \neq \sigma(p)$ may vary by edge), so $n - r \geq \sum_{p=j}^{d} l_{\sigma(p)}(g)$. Thus

$$\begin{aligned} n = r + (n - r) &\geq \sum_{p=1}^{j} m_{\sigma(p)}(g) + \sum_{p=j}^{d} l_{\sigma(p)}(g) \\ &= m_{\sigma(1)}(g) + A_{\sigma(j)}(g) + l_{\sigma(d)}(g) \\ &= f_\sigma(g, j) \end{aligned}$$

for every $2 \leq j \leq d-1$.

For the case $j = d$, we use a slightly different argument. In this case, let $a_1 \dots a_r$ be the path from o to v^1 and let $a_{r+1} \dots a_s$ be the path from v^1 to v^d. Then the path $a_1 \dots a_r$ must contain at least $m_{\sigma(1)}(g)$ edges of type $\mathbf{e}_t - \mathbf{e}_{\sigma(1)}$, so $r \geq m_{\sigma(1)}(g)$. Similarly, the path $a_{s+1} \dots a_n$ must contain at least $l_{\sigma(d)}(g)$ edges of type $\mathbf{e}_{\sigma(d)} - \mathbf{e}_t$, so $n - s \geq l_{\sigma(d)}(g)$.

For each $p \neq 1$ we have $y_{\sigma(p)} \curlywedge v^1_{\sigma(p)} = y_{\sigma(p)}$ and so, for each such p, there must be at least $h_{\sigma(p)}(v^1_{\sigma(p)}) - h_{\sigma(p)}(y_{\sigma(p)})$ letters corresponding to generators of type $\mathbf{e}_t - \mathbf{e}_{\sigma(p)}$ for various choices of t in the word $a_{r+1} \dots a_s$. Therefore, $s - r \geq \sum_{p=2}^{d} h_{\sigma(p)}(v^1_{\sigma(p)}) - h_{\sigma(p)}(y_{\sigma(p)})$. Now, since $\sum_{p=1}^{d} h_{\sigma(p)}(v^1_{\sigma(p)}) = 0$ and $h_{\sigma(1)}(v^1_{\sigma(1)}) = -m_{\sigma(1)}(g)$, it follows that $\sum_{p=2}^{d} h_{\sigma(p)}(v^1_{\sigma(p)}) = -h_{\sigma(1)}(v^1_{\sigma(1)}) = m_{\sigma(1)}(g)$. Furthermore, $h_{\sigma(p)}(y_{\sigma(p)}) = -m_{\sigma(p)}(g)$ for every $2 \leq p \leq d$. Hence

$$\begin{aligned} s - r &\geq \sum_{p=2}^{d} (h_{\sigma(p)}(v^1_{\sigma(p)}) - h_{\sigma(p)}(y_{\sigma(p)})) \\ &= m_{\sigma(1)}(g) - \sum_{p=2}^{d} h_{\sigma(p)}(y_{\sigma(p)}) \\ &= \sum_{p=1}^{d} m_{\sigma(p)}(g) = A_{\sigma(d)}(g). \end{aligned}$$

Thus we have

$$\begin{aligned} n &= r + (s - r) + (n - s) \\ &\geq m_{\sigma(1)}(g) + A_{\sigma(d)}(g) + l_{\sigma(d)}(g) \\ &= f_\sigma(g, d). \end{aligned}$$

Hence we have shown that $n \geq f_\sigma(g, j)$ for every $2 \leq j \leq d$, as desired. $\square$

To complete the argument, we must prove that f satisfies the third and final property of Lemma 1. In doing so, it is often necessary to keep track of which values of $l_{\chi(i)}(g)$ in $\Pi(g)$ are zero for a given permutation χ, so we first prove several preliminary lemmas.

LEMMA 3. *Let $\chi \in \Sigma_d$ be any permutation and $g \in \Gamma_d(q)$ any nontrivial element.*

(1) *If $l_{\chi(d)}(g) = 0$, let n be the maximal value of j with $1 \leq j \leq d - 1$ such that $l_{\chi(j)}(g) \neq 0$. Then*

$$\max_{2 \leq i \leq d} A_{\chi(i)}(g) = \max_{2 \leq i \leq n \ i=d} A_{\chi(i)}(g).$$

(2) *If $l_{\chi(1)}(g) = 0$, let k be the minimum value of j with $2 \leq j \leq d$ such that $l_{\chi(j)}(g) \neq 0$. Then*

$$\max_{2 \leq i \leq d} A_{\chi(i)}(g) = \max_{k \leq i \leq d} A_{\chi(i)}(g).$$

Proof. Since g is nontrivial, the values of n and k as defined in (1) and (2) both exist. The proof of (1) follows because, if $l_{\chi(n+1)}(g) = l_{\chi(n+2)}(g) = \cdots = l_{\chi(d-1)}(g) = 0$, then for $n+1 \leq i \leq d - 1$ we have $A_{\chi(i)}(g) \leq A_{\chi(d)}(g)$. Similarly, to prove (2), if $l_{\chi(2)}(g) = \cdots = l_{\chi(k-1)}(g) = 0$ for $2 \leq i \leq k - 1$ then $A_{\chi(i)}(g) \leq A_{\chi(k)}(g)$. $\square$

LEMMA 4. *Fix $g \in \Gamma_d(q)$. Let $\sigma \in \Sigma_d$ with $l_{\sigma(1)}(g) = 0$. Let $\tau \in \Sigma_d$ be defined by $\tau(i) = \sigma(i+1)$ for $1 \leq i < d$ and $\tau(d) = \sigma(1)$. Then $f_\sigma(g) \geq f_\tau(g)$.*

Proof. First note that, for $2 \leq i \leq d - 2$,

$$A_{\sigma(i+1)}(g) = m_{\sigma(2)}(g) + \cdots + m_{\sigma(i+1)}(g) + l_{\sigma(i+1)}(g) + \cdots + l_{\sigma(d-1)}(g)$$

and

$$\begin{aligned} A_{\tau(i)}(g) &= m_{\tau(2)}(g) + \cdots + m_{\tau(i)}(g) + l_{\tau(i)}(g) + \cdots + l_{\tau(d-1)}(g) \\ &= m_{\sigma(3)}(g) + \cdots + m_{\sigma(i+1)}(g) + l_{\sigma(i+1)}(g) + \cdots + l_{\sigma(d)}(g). \end{aligned}$$

Hence for $2 \leq i \leq d - 2$ we have

$$A_{\sigma(i+1)}(g) = A_{\tau(i)}(g) + m_{\sigma(2)}(g) - l_{\sigma(d)}(g). \tag{$*$}$$

The lemma is clearly true if g is the identity element, so we may assume for the rest of the proof that g is nontrivial. Using the definition of k given in Lemma 3, we have $\max_{2 \leq i \leq d} A_{\sigma(i)}(g) = \max_{k \leq i \leq d} A_{\sigma(i)}(g)$. We may therefore assume that $f_\sigma(g) = f_\sigma(g, i)$ for $k \leq i \leq d$; that is, $f_\sigma(g) = m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(i)}(g)$ for some i with $k \leq i \leq d$. We must show for each j, $2 \leq j \leq d$, that $f_\sigma(g, i) \geq f_\tau(g, j)$. From this it will follow that $f_\sigma(g) \geq f_\tau(g)$, as desired. We consider three subcases.

Case 1: $2 \leq j \leq d - 2$. In this case we see that

$$\begin{aligned} f_\sigma(g,i) &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(i)}(g) \\ &\geq m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(j+1)}(g) \\ &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\tau(j)}(g) + m_{\sigma(2)}(g) - l_{\sigma(d)}(g) \\ &\geq m_{\sigma(2)}(g) + A_{\tau(j)}(g) \\ &= m_{\tau(1)}(g) + l_{\tau(d)}(g) + A_{\tau(j)}(g) \\ &= f_\tau(g,j). \end{aligned}$$

Here the first inequality holds because $f_\sigma(g) = f_\sigma(g,i)$ and so $A_{\sigma(i)}(g) \geq A_{\sigma(j)}(g)$ for $i \neq j$, the next line follows from $(*)$, and the penultimate equality holds because

$$l_{\tau(d)}(g) = l_{\sigma(1)}(g) = 0.$$

Case 2: $j = d$. Then

$$\begin{aligned} f_\sigma(g,i) &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(i)}(g) \\ &\geq m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(k)}(g) \\ &= m_{\sigma(1)}(g) + m_{\sigma(2)}(g) + \cdots + m_{\sigma(k)}(g) + l_{\sigma(k)}(g) \\ &\quad + l_{\sigma(k+1)}(g) + \cdots + l_{\sigma(d)}(g) \\ &\geq m_{\sigma(2)}(g) + 0 + l_{\sigma(k)}(g) + l_{\sigma(k+1)}(g) + \cdots + l_{\sigma(d)}(g) \\ &= m_{\tau(1)}(g) + l_{\tau(d)}(g) + \sum_{r=1}^{d} m_{\tau(r)(g)} = f_\tau(g,d). \end{aligned}$$

The last line relies on the facts that $l_{\tau(d)}(g) = l_{\sigma(1)}(g) = 0$ and that, by our choice of k,

$$\sum_{r=1}^{d} m_{\tau(r)}(g) = \sum_{r=1}^{d} m_{\sigma(r)}(g) = \sum_{r=1}^{d} l_{\sigma(r)}(g) = \sum_{r=k}^{d} l_{\sigma(r)}(g).$$

Case 3: $j = d - 1$. In this case we differentiate between $2 \leq i \leq d - 1$ and $i = d$. Recall that $f_\sigma(g) = f_\sigma(g,i)$.

First let $2 \leq i \leq d - 1$ and recall that $l_{\tau(d)}(g) = l_{\sigma(1)}(g) = 0$ by assumption. In this case,

$$\begin{aligned} f_\tau(g,d-1) &= m_{\tau(1)}(g) + m_{\tau(2)}(g) + \cdots + m_{\tau(d-1)}(g) + l_{\tau(d-1)}(g) + l_{\tau(d)}(g) \\ &= m_{\sigma(2)}(g) + \cdots + m_{\sigma(d)}(g) + l_{\sigma(d)}(g). \end{aligned}$$

We also assume that $A_{\sigma(i)} \geq A_{\sigma(d)}$. Writing out this inequality and canceling identical terms from both sides yields

$$\begin{aligned} &l_{\sigma(i)}(g) + l_{\sigma(i+1)}(g) + \cdots + l_{\sigma(d-1)}(g) \\ &\qquad \geq m_{\sigma(1)}(g) + m_{\sigma(i+1)}(g) + m_{\sigma(i+2)}(g) + \cdots + m_{\sigma(d)}(g). \end{aligned}$$

Therefore,

$$\begin{aligned}
f_\sigma(g,i) &= m_{\sigma(1)}(g) + m_{\sigma(2)}(g) + \cdots + m_{\sigma(i)}(g) + l_{\sigma(i)}(g) + \cdots + l_{\sigma(d)}(g) \\
&\geq (m_{\sigma(1)}(g) + \cdots + m_{\sigma(i)}(g)) \\
&\quad + (m_{\sigma(1)}(g) + m_{\sigma(i+1)}(g) + m_{\sigma(i+2)}(g) + \cdots + m_{\sigma(d)}(g)) + l_{\sigma(d)}(g) \\
&\geq m_{\sigma(2)}(g) + \cdots + m_{\sigma(d)}(g) + l_{\sigma(d)}(g) \\
&= m_{\tau(1)}(g) + m_{\tau(2)}(g) + \cdots + m_{\tau(d \quad 1)}(g) + l_{\tau(d \quad 1)}(g) + l_{\tau(d)}(g) \\
&= f_\tau(g, d \quad 1).
\end{aligned}$$

Now let $i = d$. In this case,

$$\begin{aligned}
f_\sigma(g,d) &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + \sum_{r=1}^{d} m_{\sigma(i)}(g) \\
&\geq m_{\sigma(2)}(g) + \cdots + m_{\sigma(d)}(g) + l_{\sigma(d)}(g) \\
&= m_{\tau(1)}(g) + m_{\tau(2)}(g) + \cdots + m_{\tau(d \quad 1)}(g) + l_{\tau(d \quad 1)}(g) \\
&= m_{\tau(1)}(g) + m_{\tau(2)}(g) + \cdots + m_{\tau(d \quad 1)}(g) + l_{\tau(d \quad 1)}(g) + l_{\tau(d)}(g) \\
&= f_\tau(g, d \quad 1). \qquad \square
\end{aligned}$$

If $g \in \Gamma_d(q)$ is nontrivial, let $\Theta_g = \{\sigma \in \Sigma_d \mid f(g) = f_\sigma(g)\}$ and let $\Theta'_g = \{\sigma \in \Theta_g \mid l_{\sigma(1)}(g) \neq 0\}$. Then Lemma 4 has the following corollary.

COROLLARY 5. *If $g \in \Gamma_d(q)$ is not the identity element, then Θ'_g is not empty. In other words, there exists a $\sigma \in \Sigma_d$ such that $f(g) = f_\sigma(g)$ and $l_{\sigma(1)}(g) \neq 0$.*

Proof. Suppose that $\chi \in \Theta_g$ and $l_{\chi(1)}(g) = 0$. Let k be defined as in Lemma 3. Then $k \quad 1$ applications of Lemma 4 yields the corollary. $\square$

LEMMA 6. *Let $g \in \Gamma_d(q)$ and $\sigma \in \Sigma_d$ with $m_{\sigma(d)}(g) = l_{\sigma(d)}(g) = 0$. Define $\tau \in \Sigma_d$ such that $\tau(1) = \sigma(d)$ and $\tau(i) = \sigma(i \quad 1)$ for $i \geq 2$. Then $f_\sigma(g) = f_\tau(g)$.*

Proof. One can argue as in the previous lemma to verify directly that $f_\sigma(g,d) = f_\tau(g,2)$ and that $f_\sigma(g,i) = f_\tau(g,i+1)$ for $2 \leq i \leq d \quad 1$. $\square$

We immediately obtain the following corollary.

COROLLARY 7. *If $g \in \Gamma_d(q)$ is not the identity element, then there exists $\sigma \in \Theta_g$ such that either $l_{\sigma(d)}(g) \neq 0$ or $m_{\sigma(d)}(g) \neq 0$.*

The next proposition uses the preceding lemmas and corollaries to prove that the function f satisfies condition (3) of Lemma 1.

PROPOSITION 8. *Let $g \in \Gamma_d(q)$ be a nontrivial group element, and let $f(g)$ be as in Definition 1. Then there exists $s \in S_{d\ q}$ with $f(gs) = f(g) \quad 1$.*

Proof. If $g \in S_{d\ q}$, that is, if g is a generator of $\Gamma_d(q)$, then it is easy to see that $f(g) = 1$ and (choosing $s = g^{\ 1}$) that $f(gs) = 0$; hence the condition of the proposition is satisfied. From now on we assume that $g \notin S_{d\ q}$, which means that for any $s \in S_{d\ q}$ we know that gs is nontrivial.

Let x be the vertex in $\mathrm{DL}_d(q)$ identified with g, and recall that we write $\Pi(g)$ for $\Pi(x)$.

Case 1. There exists a $\sigma \in \Theta_g$ with $l_{\sigma(d)}(g) \neq 0$. If, in addition, $l_{\sigma(1)}(g) > 0$ or $l_{\sigma(1)}(g) = m_{\sigma(1)}(g) = 0$, then choose s to be any generator corresponding to an edge of type $\mathbf{e}_{\sigma(1)} \quad \mathbf{e}_{\sigma(d)}$. If $l_{\sigma(1)}(g) = 0$ and $m_{\sigma(1)}(g) > 0$, let w be the vertex in $T_{\sigma(1)}$ adjacent to $x_{\sigma(1)}$ on the unique shortest path from $o_{\sigma(1)}$ to $x_{\sigma(1)}$. Choose s to be any generator of type $\mathbf{e}_{\sigma(1)} \quad \mathbf{e}_{\sigma(d)}$, so that if z is the vertex in $\mathrm{DL}_d(q)$ identified with gs then $z_{\sigma(1)} \neq w$. Consequently,

(1) $(m_{\sigma(d)}(gs), l_{\sigma(d)}(gs)) = (m_{\sigma(d)}(g), l_{\sigma(d)}(g) \quad 1)$,
(2) $(m_{\sigma(1)}(gs), l_{\sigma(1)}(gs)) = (m_{\sigma(1)}(g), l_{\sigma(1)}(g) + 1)$, and
(3) $(m_{\sigma(i)}(gs), l_{\sigma(i)}(gs)) = (m_{\sigma(i)}(g), l_{\sigma(i)}(g))$ for $i \neq 1, d$.

However, this implies that $A_{\sigma(i)}(gs) = A_{\sigma(i)}(g)$ for every $2 \leq i \leq d$, and hence

$$\begin{aligned} f_\sigma(gs) &= m_{\sigma(1)}(gs) + l_{\sigma(d)}(gs) + \max_{2 \leq i \leq d} A_{\sigma(i)}(gs) \\ &= m_{\sigma(1)}(g) + (l_{\sigma(d)}(g) \quad 1) + \max_{2 \leq i \leq d} A_{\sigma(i)}(g) \\ &= f_\sigma(g) \quad 1. \end{aligned}$$

First we note that the inequality $f(g) \quad 1 \geq f(gs)$ is not hard to verify, given that $f(g) \quad 1 = f_\sigma(g) \quad 1 = f_\sigma(gs) \geq f_\tau(gs)$ for any $\tau \in \Theta_{gs}$. Hence $f(g) \quad 1 \geq f(gs)$.

Now, for any $\tau \in \Sigma_d$, it follows that: $m_{\tau(i)}(gs) = m_{\tau(i)}(g)$ for every $1 \leq i \leq d$; $l_{\tau(i)}(gs) \neq l_{\tau(i)}(g)$ for only two choices of i; and $l_{\tau(i)}(gs) = l_{\tau(i)}(g) \quad 1$ for one of those choices and $l_{\tau(i)}(gs) = l_{\tau(i)}(g) + 1$ for the other. For any value of i, $l_{\tau(i)}(g)$ (resp. $l_{\tau(i)}(gs)$) appears at most in the formula for $f_\tau(g, i)$ (resp. $f_\tau(gs, i)$); therefore, $f_\tau(g) \quad 1 \leq f_\tau(gs)$. Thus for $\tau \in \Theta_{gs}$ we have $f(gs) = f_\tau(gs) \geq f_\tau(g) \quad 1 \geq f(g) \quad 1$ and so $f(gs) \geq f(g) \quad 1$ as well. Hence $f(gs) = f(g) \quad 1$, as desired.

Case 2. For every $\chi \in \Theta_g$, assume that $l_{\chi(d)}(g) = 0$. We can now apply Corollary 7 to choose $\sigma \in \Theta_g$ such that $m_{\sigma(d)}(g) \neq 0$.

Let w be the vertex in $T_{\sigma(d)}$ adjacent to $x_{\sigma(d)}$ on the unique shortest path from $o_{\sigma(d)}$ to $x_{\sigma(d)}$, and let n be as defined in Lemma 3. Choose the generator $s \in S_{d\ q}$ of type $\mathbf{e}_{\sigma(d)} \quad \mathbf{e}_{\sigma(n)}$ such that, if z is the vertex in $\mathrm{DL}_d(q)$ identified with gs, then $z_{\sigma(d)} = w$. Then we have, for the pair σ and s:

(1) $(m_{\sigma(d)}(gs), l_{\sigma(d)}(gs)) = (m_{\sigma(d)}(g) \quad 1, l_{\sigma(d)}(g))$, where $l_{\sigma(d)}(gs) = l_{\sigma(d)}(g) = 0$;
(2) $(m_{\sigma(n)}(gs), l_{\sigma(n)}(gs)) = (m_{\sigma(n)}(g), l_{\sigma(n)}(g) \quad 1)$;
(3) $(m_{\sigma(i)}(gs), l_{\sigma(i)}(gs)) = (m_{\sigma(i)}(g), l_{\sigma(i)}(g))$ for $i \neq n, d$.

For the preceding choice of σ and s, we claim that $f_\sigma(gs) = f_\sigma(g) \quad 1$. Applying Lemma 3 to g reveals that

$$f_\sigma(g) = m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + \max_{2 \leq i \leq n\ i = d} A_{\sigma(i)}(g).$$

Because the only index for which $l_{\sigma(i)}(gs) \neq l_{\sigma(i)}(g)$ is $i = n$, we again see that $l_{\sigma(j)}(gs) = 0$ for $j > n$. Applying Lemma 3 to gs now yields

$$f_\sigma(gs) = m_{\sigma(1)}(gs) + l_{\sigma(d)}(gs) + \max_{2 \leq i \leq n \ i=d} A_{\sigma(i)}(gs).$$

It follows from the definition of s that $A_{\sigma(d)}(gs) = A_{\sigma(d)}(g) \quad 1$. Similarly, for $2 \leq i \leq n$ we have $A_{\sigma(i)}(gs) = A_{\sigma(i)}(g) \quad 1$, since neither expression contains $m_{\sigma(d)}$ and both contain $l_{\sigma(n)}$. Therefore,

$$\max_{2 \leq i \leq n \ i=d} A_{\sigma(i)}(gs) = \max_{2 \leq i \leq n \ i=d} (A_{\sigma(i)}(g) \quad 1) = \Big(\max_{2 \leq i \leq n \ i=d} A_{\sigma(i)}(g) \Big) \quad 1.$$

Combining the arguments so far, we obtain

$$\begin{aligned} f_\sigma(gs) &= m_{\sigma(1)}(gs) + l_{\sigma(d)}(gs) + \max_{2 \leq i \leq n \ i=d} A_{\sigma(i)}(gs) \\ &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + \max_{2 \leq i \leq n \ i=d} A_{\sigma(i)}(g) \quad 1 = f_\sigma(g) \quad 1. \end{aligned}$$

Since $f_\sigma(g) = f(g)$ and $f(gs) \leq f_\sigma(gs)$, it follows immediately from the equality $f_\sigma(gs) = f_\sigma(g) \quad 1$ that $f(gs) \leq f(g) \quad 1$.

To complete the proof of Proposition 8, we must show that $f(gs) \geq f(g) \quad 1$. First note that, for any $\chi \in \Sigma_d$, it follows from the definition of f that $f_\chi(gs) \geq f_\chi(g) \quad 3$. We now show that this inequality can be improved slightly for $\tau \in \Theta'_{gs}$; for such τ we will show that $f_\tau(gs) \geq f_\tau(g) \quad 2$. Suppose to the contrary that $\tau \in \Theta'_{gs}$ and $f_\tau(gs) = f_\tau(g) \quad 3$. This can occur in only one way; namely, all three of the following conditions must be met:

(1) $\tau(1) = \sigma(d)$,
(2) $\tau(d) = \sigma(n)$, and
(3) $\max_{2 \leq i \leq n \ i=d} A_{\tau(i)}(gs) = A_{\tau(d)}(gs)$.

Now $\tau \in \Theta'_{gs}$ implies that $l_{\tau(1)}(gs) \neq 0$, but condition (1) requires that $l_{\tau(1)}(gs) = l_{\sigma(d)}(gs) = l_{\sigma(d)}(g) = 0$, a contradiction. Thus, for all $\tau \in \Theta'_{gs}$ we must have $f_\tau(gs) \geq f_\tau(g) \quad 2$.

It follows from Corollary 5 that Θ'_{gs} is not empty, so we may choose $\chi \in \Theta'_{gs}$. If $\chi \notin \Theta_g$ then $f_\chi(g) > f_\sigma(g)$. Thus we have $f(gs) = f_\chi(gs) \geq f_\chi(g) \quad 2 > f_\sigma(g) \quad 2$ and hence $f(gs) \geq f_\sigma(g) \quad 1 = f(g) \quad 1$, as desired.

On the other hand, if $\chi \in \Theta_g$ then we make the following claim.

CLAIM. *If $\chi \in \Theta_g$, then there exists a $\tau \in \Theta'_{gs}$ with $f_\tau(gs) \geq f_\tau(g) \quad 1$.*

Proposition 8 follows immediately from the claim, as follows. Let τ be as in the claim, so that $f_\tau(gs) \geq f_\tau(g) \quad 1$. Then $f(gs) = f_\tau(gs) \geq f_\tau(g) \quad 1 \geq f(g) \quad 1$ and hence $f(gs) \geq f(g) \quad 1$, as desired.

To prove the claim, if $f_\chi(gs) \geq f_\chi(g) \quad 1$ then simply let $\tau = \chi$. If $f_\chi(gs) = f_\chi(g) \quad 2$ then we use χ to construct $\tau \in \Theta'_{gs}$ such that $f_\tau(gs) \geq f_\tau(g) \quad 1$, as follows.

There exist distinct $u, v \in \{1, 2, \ldots, d\}$ such that $\chi(u) = \sigma(d)$ and $\chi(v) = \sigma(n)$. We now show that $1 < u < v < d$. To see that $1 < u$, observe that $l_{\sigma(d)}(gs) = 0$ but $l_{\chi(1)}(gs) \neq 0$ since $\chi \in \Theta'_{gs}$; hence $\sigma(d) \neq \chi(1)$ (i.e., $u \neq 1$). To see that

$v < d$, observe that $l_{\sigma(n)}(g) = l_{\chi(v)}(g) \neq 0$. Recall that, since $\chi \in \Theta_g$, we must have $l_{\chi(d)}(g) = 0$ and hence $v \neq d$.

Finally, we need to show that $u < v$. Since $\chi(1) \neq \sigma(d)$, it follows that $m_{\chi(1)}(gs) = m_{\chi(1)}(g)$. Also, since $\chi(v) \neq \chi(d)$ we have $l_{\chi(d)}(gs) = l_{\chi(d)}(g)$. Thus, for $f_\chi(gs) = f_\chi(g) - 2$ it must be that

$$\max_{2 \le i \le d} A_{\chi(i)}(g) - \max_{2 \le i \le d} A_{\chi(i)}(gs) = 2.$$

The only way this can happen is if $\max_{2 \le i \le d} A_{\chi(i)}(g)$ is realized by an expression that contains both $m_{\sigma(d)}(g)$ and $l_{\sigma(n)}(g)$, that is, both $m_{\chi(u)}(g)$ and $l_{\chi(v)}(g)$. By construction of the terms $A_{\chi(i)}(g)$, we must have $u < v$ for this to occur; for if $u > v$ and if $l_{\chi(v)}(g)$ were a term in the expression that realized $\max_{2 \le i \le d} A_{\chi(i)}(g)$, then this expression would also contain $l_{\chi(u)}(g)$ and not $m_{\chi(u)}(g)$ as required. Therefore, $u < v$.

We now construct $\tau \in \Theta'_{gs}$ satisfying $f_\tau(gs) \geq f_\tau(g) - 1$. We let u and v be as before, and we set the following definitions:

- for $i < u$, let $\tau(i) = \chi(i)$;
- for $u \le i < v$, let $\tau(i) = \chi(i+1)$;
- for $i = v$, let $\tau(v) = \chi(u)$;
- for $i > v$, let $\tau(i) = \chi(i)$.

We first show that $\tau \in \Theta'_{gs}$. Since $\chi(1) = \tau(1)$ and $\chi(d) = \tau(d)$, we claim that for any i with $2 \le i \le d$ we have $A_{\tau(i)}(gs) \le \max_{2 \le j \le d} A_{\chi(j)}(gs)$. This is clearly true when $i = d$, since $A_{\tau(d)}(gs) = A_{\chi(d)}(gs)$. We next consider the four remaining cases, and we abuse our notation by writing $m_{\chi(i)}$, $l_{\chi(i)}$, and $A_{\chi(i)}$ instead of $m_{\chi(i)}(gs)$, $l_{\chi(i)}(gs)$, and $A_{\chi(i)}(gs)$, respectively.

Case I: $i < u$. Then

$$\begin{aligned} A_{\chi(i)} &= m_{\chi(2)} + \cdots + m_{\chi(i)} + l_{\chi(i)} + \cdots + l_{\chi(u)} + \cdots + l_{\chi(v)} + \cdots + l_{\chi(d-1)} \\ &= m_{\chi(2)} + \cdots + m_{\chi(i)} + l_{\chi(i)} + \cdots + l_{\chi(u-1)} + l_{\chi(u+1)} + \cdots + l_{\chi(v)} \\ &\quad + l_{\chi(u)} + l_{\chi(v+1)} + \cdots + l_{\chi(d-1)} \\ &= m_{\tau(2)} + \cdots + m_{\tau(i)} + l_{\tau(i)} + \cdots + l_{\tau(d-1)} \\ &= A_{\tau(i)}. \end{aligned}$$

Here the second equality follows from the first via rearranging terms and is then rewritten with equivalent indices for τ in the third equality.

Case II: $u \le i < v$. Then

$$\begin{aligned} A_{\tau(i)} &= m_{\tau(2)} + \cdots + m_{\tau(u)} + \cdots + m_{\tau(i)} + l_{\tau(i)} + \cdots + l_{\tau(d-1)} \\ &= m_{\chi(2)} + \cdots + m_{\chi(u-1)} + m_{\chi(u+1)} + \cdots + m_{\chi(i+1)} + l_{\chi(i+1)} + \cdots + l_{\chi(v)} \\ &\quad + l_{\chi(u)} + l_{\chi(v+1)} + \cdots + l_{\chi(d-1)} \\ &< m_{\chi(2)} + \cdots + m_{\chi(u-1)} + m_{\chi(u)} + m_{\chi(u+1)} + \cdots + m_{\chi(i+1)} \\ &\quad + l_{\chi(i+1)} + \cdots + l_{\chi(v)} + l_{\chi(v+1)} + \cdots + l_{\chi(d-1)} \\ &= A_{\chi(i+1)}; \end{aligned}$$

here the inequality is obtained by including the "missing" term $m_{\chi(u)} = m_{\sigma(d)} > 0$ and omitting $l_{\chi(u)} = 0$ from the expression.

Case III: $i = v$. Recall that $\tau(v) = \chi(u) = \sigma(d)$, and note that $\tau(v \quad 1) = \chi(v)$ by the definition of τ. Then

$$\begin{aligned}
A_{\tau(v)} &= m_{\tau(2)} + \cdots + m_{\tau(u \quad 1)} + m_{\tau(u)} + \cdots + m_{\tau(v \quad 1)} + m_{\tau(v)} \\
&\quad + l_{\tau(v)} + \cdots + l_{\tau(d \quad 1)} \\
&= m_{\chi(2)} + \cdots + m_{\chi(u \quad 1)} + m_{\chi(u+1)} + \cdots + m_{\chi(v)} + m_{\chi(u)} \\
&\quad + l_{\chi(u)} + l_{\chi(v+1)} + \cdots + l_{\chi(d \quad 1)} \\
&\leq m_{\chi(2)} + \cdots + m_{\chi(u \quad 1)} + m_{\chi(u)} + m_{\chi(v)} + l_{\chi(v)} + \cdots + l_{\chi(d \quad 1)} = A_{\chi(v)},
\end{aligned}$$

where the final line is obtained from the preceding equality by rearranging the existing terms, omitting $l_{\chi(u)} = 0$, and adding in the term $l_{\chi(v)}$.

Case IV: $i > v$. Then

$$\begin{aligned}
A_{\chi(i)} &= m_{\chi(2)} + \cdots + m_{\chi(u)} + \cdots + m_{\chi(v)} + \cdots + m_{\chi(i)} + l_{\chi(i)} + \cdots + l_{\chi(d \quad 1)} \\
&= m_{\chi(2)} + \cdots + m_{\chi(u \quad 1)} + m_{\chi(u+1)} + \cdots + m_{\chi(v)} + m_{\chi(u)} \\
&\quad + m_{\chi(v+1)} + \cdots + m_{\chi(i)} + l_{\chi(i)} + \cdots + l_{\chi(d \quad 1)} \\
&= A_{\tau(i)}.
\end{aligned}$$

Combining these cases shows that, for all $2 \leq i \leq d$, we have $A_{\tau(i)}(gs) \leq \max_{2\leq j\leq d} A_{\chi(j)}(gs)$. Thus $f_\tau(gs) \leq f_\chi(gs) = f(gs)$ and hence $f(gs) = f_\tau(gs)$; that is, $\tau \in \Theta_{gs}$. Now, since $\chi \in \Theta'_{gs}$, this implies that $l_{\chi(1)}(gs) \neq 0$. But $\tau(1) = \chi(1)$ and so $l_{\tau(1)}(gs) \neq 0$ as well; therefore, $\tau \in \Theta'_{gs}$.

Finally, it remains to show that $f_\tau(gs) = f_\tau(g) \quad 1$. Recall from the definition of τ that $\tau(v) = \chi(u) = \sigma(d)$ and $\tau(v \quad 1) = \chi(v) = \sigma(n)$. From the choice of s, recall that

(1) $(m_{\sigma(d)}(gs), l_{\sigma(d)}(gs)) = (m_{\sigma(d)}(g) \quad 1, l_{\sigma(d)}(g))$,
(2) $(m_{\sigma(n)}(gs), l_{\sigma(n)}(gs)) = (m_{\sigma(n)}(g), l_{\sigma(n)}(g) \quad 1)$, and
(3) $(m_{\sigma(i)}(gs), l_{\sigma(i)}(gs)) = (m_{\sigma(i)}(g), l_{\sigma(i)}(g))$ for $i \neq n, d$.

Comparing $A_{\tau(i)}(gs)$ and $A_{\tau(i)}(g)$ shows that $A_{\tau(i)}(gs) = A_{\tau(i)}(g) \quad 1$ for all possible values of i.

From the definition of τ we see that

$$m_{\tau(1)}(g) = m_{\chi(1)}(g) = m_{\chi(1)}(gs) = m_{\tau(1)}(gs)$$

and

$$l_{\tau(d)}(g) = l_{\chi(d)}(g) = l_{\chi(d)}(gs) = l_{\tau(d)}(gs).$$

Thus $f_\tau(gs) = f_\tau(g) \quad 1$, which concludes the proof of the claim and hence of Proposition 8. □

4. Comparing Word Length in Γ_d q) and Distance in the Product of Trees

Since the Diestel–Leader graph $\mathrm{DL}_d(q)$ is a subset of the product of d trees of valence $q+1$, it is natural to compare the word metric on the Cayley graph $\mathrm{DL}_d(q)$ to the product metric on the product of trees. This product metric assigns every edge length 1 and simply counts edges in each tree between the coordinates corresponding to two different group elements. It is a straightforward consequence of the word-length formula that these two metrics are quasi-isometric. In Corollary 10 we extend the word-length function f to compute the distance in the word metric (with respect to the generating set $S_d(q)$) between arbitrary group elements. We conclude with a corollary that constructs a family of quasi-geodesic paths from the vertex corresponding to the identity to a vertex corresponding to any group element.

THEOREM 9. *Let $l(g)$ denote the word length of $g \in \Gamma_d(q)$ with respect to the generating set $S_{d\ q}$. Let $d_T(g)$ be the distance in the product metric on the product of trees between g in $\mathrm{DL}_d(q)$ and ε, the fixed base point corresponding to the identity in $\Gamma_d(q)$. Then*

$$\frac{1}{2}d_T(g) \leq l(g) \leq 2d_T(g);$$

that is, the word length is quasi-isometric to the distance from the identity in the product metric on the product of trees.

Proof. Let $\Pi(g) = ((m_1,l_1),(m_2,l_2),\ldots,(m_d,l_d))$. It follows that $d_T(g) = \sum_{i=1}^d m_i + l_i = 2\sum_{i=1}^d m_i$. Using the word-length formula from Section 3, we see that, for some $\sigma \in \Sigma(d)$,

$$\begin{aligned} l(g) = f_\sigma(g) &= (m_{\sigma(1)} + l_{\sigma(d)}) + \max_{2\leq i\leq d} A_{\sigma(i)} \\ &\leq \Big(\sum_{i=1}^d m_i + \sum_{i=1}^d l_i\Big) + \sum_{i=1}^d m_i + \sum_{i=1}^d l_i = 2d_T(g). \end{aligned}$$

To obtain a lower bound, note that

$$\begin{aligned} l(g) = \min_{\sigma\in\Sigma_d} f_\sigma(g) &= \min_{\sigma\in\Sigma_d}\Big(m_{\sigma(1)} + l_{\sigma(d)} + \max_{2\leq i\leq d} A_{\sigma(i)}(g)\Big) \\ &\geq \min_{\sigma\in\Sigma_d}\Big(\max_{2\leq i\leq d} A_{\sigma(i)}(g)\Big). \end{aligned}$$

Yet for every $\sigma \in \Sigma_d$ we have $\max_{2\leq i\leq d} A_{\sigma(i)}(g) \geq A_{\sigma(d)}(g) = \sum_{i=1}^d m_i$, so

$$l(g) \geq \sum_{i=1}^d m_i = \frac{1}{2}d_T(g).$$

Combining these inequalities proves the theorem. □

The first corollary to Theorem 9 requires that we extend the techniques of Section 3 in order to compute the distance in the word metric between arbitrary group elements.

COROLLARY 10. *Let $g,h \in \Gamma_d(q)$, and let $d_T(g,h)$ denote the distance between the two vertices in $\mathrm{DL}_d(q)$ corresponding to g and h with respect to the product metric on the product of trees. Then*

$$\frac{1}{2}d_T(g,h) \leq l(g^{\ \ 1}h) \leq 2d_T(g,h).$$

Proof. In Section 3 we show that $l(g) = f(g)$ for the function f defined there. The calculation of the value of $f(g)$ depends only on the coordinates of $\Pi(g) = (m_1(g), l_1(g)), \dots, (m_d(g), l_d(g)))$. Recall that if g corresponds to the vertex $(g_1, \dots, g_d)$ in $\mathrm{DL}_d(q)$ then, for $1 \leq i \leq d$,

$$(m_i(g), l_i(g)) = (d_{T_i}(o_i, o_i \curlywedge g_i), d_{T_i}(g_i, o_i \curlywedge g_i)),$$

where $(o_1, \dots, o_d)$ is the vertex in $\mathrm{DL}_d(q)$ corresponding to the identity element of $\Gamma_d(q)$. Define an analogous relative projection function $\Pi_h(g) = \ (m_{h\ 1}(g), l_{h\ 1}(g)), \dots, (m_{h\ d}(g), l_{h\ d}(g)))$, where for $1 \leq i \leq d$ we have

$$(m_{h\ i}(g), l_{h\ i}(g)) = (d_{T_i}(h_i, h_i \curlywedge g_i), d_{T_i}(g_i, h_i \curlywedge g_i)).$$

Now define $f_h(g)$ as in Section 3, replacing $\Pi(g)$ with $\Pi_h(g)$. Because the proof that $l(g) = f(g)$ is strictly combinatorial, the arguments in Section 3 imply that $f_h(g)$ computes the word length of $g^{\ \ 1}h$ with respect to the generating set $S_{d\ q}$; the corollary then follows directly from Theorem 9. $\square$

The component of the word-length function that computes the maximum of the quantities $A_{\sigma(i)}$ over $\sigma \in \Sigma_d$ presents a combinatorial obstruction to writing down a family of geodesic paths representing elements of $\Gamma_d(q)$. The symmetry present in the Diestel–Leader graphs gives rise to a natural family of paths, described by edge labels, with the property that any path with these edge labels is a quasi-geodesic path in the Cayley graph $\mathrm{DL}_d(q)$. Although it is often not difficult to write down a family of quasi-geodesic paths in a Cayley graph, the paths we describe are especially natural to traverse and the construction is valid when the trees are permuted, which captures the symmetry of the Diestel–Leader graphs. Hence we note in what follows that they are quasi-geodesics.

Let $g \in \Gamma_d(q)$ have projection $\Pi(g) = ((m_1, l_1), (m_2, l_2), \dots, (m_d, l_d))$. Consider the sequence of edge labels

$$(\mathbf{e}_d \quad \mathbf{e}_1)^{m_1}(\mathbf{e}_d \quad \mathbf{e}_2)^{m_2} \dots (\mathbf{e}_d \quad \mathbf{e}_{d\ 1})^{m_{d\ 1}}(\mathbf{e}_1 \quad \mathbf{e}_d)^{l_1}(\mathbf{e}_2 \quad \mathbf{e}_d)^{l_2} \\ \dots (\mathbf{e}_{d\ 1} \quad \mathbf{e}_d)^{l_{d\ 1}}(\mathbf{e}_1 \quad \mathbf{e}_d)\ (\mathbf{e}_d \quad \mathbf{e}_1)^{l_d},$$

where $\ = m_d + (m_1 + \cdots + m_{d\ 1}) \quad (l_1 + \cdots + l_{d\ 1}) = m_d + \ \sum_{i=1}^d m_i \quad m_d) \ \sum_{i=1}^d m_i \quad l_d) = l_d$. We claim there is such a path ζ_g from the base point ε to the point $\gamma \in \mathrm{DL}_d(q)$ identified with g; in general, there are many possible choices of a path with the preceding edge labels. Moreover, this construction holds under permutation of the trees $T_1, T_2, \dots, T_d$.

Corollary 11. *Let $g \in \Gamma_d(q)$, and let ζ_g be any path from ε to γ with the edge labels listed above. Then ζ_g is a quasi-geodesic path.*

Proof. The corollary follows from combining Theorem 9 and Corollary 10 and checking that, for any two points h_1 and h_2 along ζ_g, the distance between them along the path ζ_g is coarsely equivalent to the distance between them in the product metric on the product of trees. $\square$

5. Dead-end Elements

In a group G with finite generating set S, an element that corresponds to a vertex $x \in \Gamma(G, S)$ is a *dead-end element* if no geodesic ray in $\Gamma(G, S)$ from can be extended past x and remain geodesic. Intuitively, the *depth* of the dead-end element g is the length of the shortest path in $\Gamma(G, S)$ from g to any point in the complement of the ball of radius $l(g)$. Both the existence and depth of dead-end elements depend on the generating set; in [13] an example is given of a finitely generated group that has dead-end elements of finite depth with respect to one generating set yet of unbounded depth with respect to another. Theorem 12 generalizes the main result of [4; 5]—namely, that $\Gamma_3(2)$ has dead-end elements of arbitrary depth with respect to a generating set similar to $S_{3\ 2}$.

DEFINITION 3. An element g in a finitely generated group G is a *dead-end element with respect to a finite generating set S for G* if $l(g) = n$ and $l(gs) \leq n$ for all generators s in $S \cup S^{\ 1}$, where $l(g)$ denotes the word length of $g \in G$ with respect to S.

DEFINITION 4. A dead-end element g in a finitely generated group G with respect to a finite generating set S has *depth k* if k is the largest integer with the following property. If the word length of g is n, then $l(gs_1s_2\cdots s_r) \leq n$ for $1 \leq r < k$ and all choices of generators $s_i \in S \cup S^{\ 1}$.

The goal of this section is to prove the following theorem.

THEOREM 12. *The group $\Gamma_d(q)$ has dead-end elements of arbitrary depth with respect to the generating set $S_{d\ q}$.*

The outline of the proof of Theorem 12 mimics the outline of the proof in [4; 5]. However, the details of the proofs are quite different. In [4; 5], the lamplighter model of an element of $\Gamma_3(2)$ is used to compute word length as well as lemmas analogous to those that follow here. This model extends the well-known lamplighter model of an element in $L_n = \mathbb{Z}_n \wr \mathbb{Z}$ (due to J. Cannon) in which a group element of L_n is visualized using a bi-infinite string of multi-state light bulbs placed at integer points on a number line along with a "lamplighter". Then $g \in L_n$ corresponds to a finite collection of illuminated bulbs and an integral position of the lamplighter. However, in $\Gamma_3(2)$ the "lampstand" (analogous to $\mathbb{Z}$ for L_n) consists of three bi-infinite rays, the illuminated bulbs are obtained using a series of relations derived from Pascal s triangle modulo 2, and the "lamplighter" moves over a $\mathbb{Z} \times \mathbb{Z}$ grid. A precise extension of this model to describe elements of $\Gamma_d(q)$ for $d > 3$ seems ambiguous. The proofs given here rely instead on the geometry of the Diestel–Leader graphs and their inherent symmetry.

Begin by defining, for any $n \in \mathbb{Z}^+$, the set

$$H_n = \big\{g \in \Gamma_d(q) \mid \Pi(g) = \ \ (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g))\big)$$
$$\text{with } 0 \leq m_i(g) \leq n \text{ and } 0 \leq l_i(g) \leq m_i(g) + n \text{ for all } 1 \leq i \leq d\big\}.$$

In the next two lemmas we show that the word length of any point in H_n with respect to $S_{d\ q}$ is bounded and describe a set of vertices in H_n at maximal distance from the identity. Proofs of both lemmas follow easily from the word-length formula proved in Section 3.

LEMMA 13. *If $g \in H_n$, then $l(g) \leq (d+2)n$.*

Proof. Let $g \in H_n$ with $\Pi(g) = ((m_1,l_1),(m_2,l_2),\ldots,(m_d,l_d))$. Choose $\sigma \in \Sigma_d$ so that $l_{\sigma(1)} \geq l_{\sigma(2)} \geq \cdots \geq l_{\sigma(d)}$. We claim that $m_{\sigma(1)} + A_{\sigma(i)}(g) + l_{\sigma(d)} \leq (d+2)n$ for every $2 \leq i \leq d$ and hence $f_\sigma(g) \leq (d+2)n$. It then follows from the word-length formula that $l(g) \leq f_\sigma(g) \leq (d+2)n$.

Choose k such that $l_{\sigma(k)} > n$ but $l_{\sigma(k+1)} \leq n$ and such that $k = 0$ if $l_{\sigma(i)} \leq n$ for every i. Since

$$\sum_{i=1}^{d} l_{\sigma(i)}(g) = \sum_{i=1}^{d} m_{\sigma(i)}(g) \leq dn,$$

it follows that $k < d$. Furthermore, we claim that $l_{\sigma(i)} + \cdots + l_{\sigma(d)} \leq (d \quad i+1)n$ for $1 \leq i \leq d$. This is clear if $i \geq k+1$, since then each term in the sum is less than n. But if $1 \leq i \leq k+1$ then $l_{\sigma(1)} + \cdots + l_{\sigma(i\ \ 1)} \geq (i \quad 1)n$, so $l_{\sigma(i)} + \cdots + l_{\sigma(d)} \leq dn \quad (\ \ i+1)n = (d \quad i+1)n$.

For $2 \leq j \leq d \quad 1$, we see that

$$\begin{aligned} m_{\sigma(1)} + A_{\sigma(j)}(g) + l_{\sigma(d)} &= \sum_{i=1}^{j} m_{\sigma(i)} + \sum_{i=j}^{d} l_{\sigma(i)} \\ &\leq jn + (d \quad j+1)n = (d+1)n. \end{aligned}$$

But $A_\sigma(d)(g) = \sum_{i=1}^{d} m_{\sigma(d)} \leq dn$ and so $m_{\sigma(1)} + A_{\sigma(d)}(g) + l_{\sigma(d)} \leq (d+2)n$. Thus $m_{\sigma(1)} + A_{\sigma(i)}(g) + l_{\sigma(d)} \leq (d+2)n$ for every $2 \leq i \leq d$, as claimed, and the lemma follows. □

The next lemma is an immediate consequence of the word-length formula from Section 3.

LEMMA 14. *If $g_n \in H_n$ and $\Pi(g_n) = ((n,n),(n,n),\ldots,(n,n))$, then $l(g) = (d+2)n$.*

The proof of Theorem 12 follows easily from Lemmas 13 and 14.

Proof of Theorem 12. Let $g_n \in H_n$ be any element with $\Pi(g_n) = ((n,n),(n,n),\ldots,(n,n))$. In Lemma 14 it is shown that $l(g) = (d+2)n$. It follows immediately from Lemma 13 that g_n is a dead-end element because all vertices adjacent to g_n lie in H_n.

To see that the depth of g_n is at least n, note that the length of a path from g_n to a point outside H_n must contain a subpath of at least n edges. Thus the depth of g_n is at least n, and we conclude that $\Gamma_d(q)$ has dead-end elements of arbitrary depth with respect to the generating set $S_{d\ q}$. □

6. Cone Types and Geodesic Languages

We now prove that $\Gamma_d(q)$ has no regular language of geodesics with respect to the generating set $S_{d\ q}$; in other words, there is no collection of geodesic representatives for elements of $\Gamma_d(q)$ that is accepted by a finite-state automaton. The existence of a regular language of geodesics for a finitely generated group G is equivalent to the finiteness of the set of cone types of G (see e.g. [11, Thm. 9.28] for a proof of this equivalence). We prove that $\Gamma_d(q)$ has infinitely many cone types with respect to the generating set $S_{d\ q}$, and it follows that $\Gamma_d(q)$ has no regular language of geodesics with respect to $S_{d\ q}$.

We begin by defining the *cone* and the *cone type* of an element $g \in G$, where G is a group with finite generating set S. Cannon [3] defined the cone type of an element $w \in G$ to be the set of geodesic extensions of w in the Cayley graph $\Gamma(G, S)$.

DEFINITION 5. A path p is *outbound* if $d(1, p(t))$ is a strictly increasing function of t. For a given $g \in G$, the *cone* at g, denoted $C'(g)$, is the set of all outbound paths starting at g. Define the *cone type* of g, denoted $C(g)$, to be $g\ \ {}^1C'(g)$.

This definition applies both in the discrete setting of the group and in the one-dimensional metric space that is the Cayley graph. A subtlety is that if the presentation for G includes odd-length relators, then the cone type of an element in the Cayley graph may include paths that end at the middle of an edge. If the presentation for G consists entirely of even-length relators, then every cone type viewed in the Cayley graph consists entirely of full edge paths. We refer the reader to [11] or [12] for a more detailed discussion of cone types.

THEOREM 15. *The group $\Gamma_d(q)$ has infinitely many cone types with respect to the generating set $S_{d\ q}$.*

The following corollary is an immediate consequence of Theorem 15.

COROLLARY 16. *The group $\Gamma_d(q)$ has no regular language of geodesics with respect to the generating set $S_{d\ q}$.*

We begin with a lemma stating sufficient but not necessary conditions on $\sigma \in \Sigma_d$ to ensure that $f(g) = f_\sigma(g)$; this lemma will be extremely useful in the proof of Theorem 15, given that realizing when $f(g) = f_\sigma(g)$ for a particular $g \in \Gamma_d(q)$ and $\sigma \in \Sigma_d$ can be difficult. Recall that we identify $g \in \Gamma_d(q)$ with the vertex $x \in \mathrm{DL}_d(q)$ corresponding to it and that we abuse notation by writing $\Pi(g)$ for $\Pi(x)$.

LEMMA 17. *Let $g \in \Gamma_d(q)$ have projection*

$$\Pi(g) = \ (m_1(g), l_1(g)), (m_2(g), l_2(g)), \ldots, (m_d(g), l_d(g))).$$

If $\sigma \in \Sigma_d$ satisfies

(1) $\min_{\tau \in \Sigma_d} m_{\tau(1)}(g) + l_{\tau(d)}(g) = m_{\sigma(1)}(g) + l_{\sigma(d)}(g)$ *and*
(2) $\max_{2 \le i \le d} A_{\sigma(i)}(g) = A_{\sigma(d)}(g)$,

then $f(g) = f_\sigma(g)$.

Proof. Let σ be as in the statement of the lemma, and let τ be any element of Σ_d. It is always true that $\max_{2\le i\le d} A_{\tau(i)}(g) \ge A_{\tau(d)}(g)$ and, by the choice of σ, that $m_{\tau(1)}(g) + l_{\tau(d)}(g) \ge m_{\sigma(1)}(g) + l_{\sigma(d)}(g)$. Therefore,

$$\begin{aligned} f_\tau(g) &= m_{\tau(1)}(g) + l_{\tau(d)}(g) + \max_{2\le i\le d} A_{\tau(i)}(g) \\ &\ge m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\tau(d)}(g) \\ &= m_{\sigma(1)}(g) + l_{\sigma(d)}(g) + A_{\sigma(d)}(g) \\ &= f_\sigma(g). \end{aligned}$$

Hence, by the definition of $f(g)$, we must have $f(g) = f_\sigma(g)$. □

To prove Theorem 15 we define a sequence of elements $\{g_n\}$ such that there is a geodesic path of length n from g_n terminating at a dead-end element and such that no shorter geodesic path from g_n reaches any other dead-end element of the group. Thus each g_n lies in a different cone type, and the theorem follows.

Proof of Theorem 15. Let g_n for $n \in \mathbb{Z}^+$ be any element with projection

$$\Pi(g_n) = ((2n,3n),(3n,4n),(4n,5n),(5n,6n), \ldots,((d-1)n,dn),(dn,3n),(2n,n)).$$

We start by showing that $f(g_n) = f_\varepsilon(g_n)$ for ε the identity permutation and specifically that $f(g_n) = 3n + \sum_{i=1}^d m_i(g_n)$.

First note that $\min_{\tau\in\Sigma_d} m_{\tau(1)}(g_n) + l_{\tau(d)}(g_n) = 3n = m_{\varepsilon(1)}(g_n) + l_{\varepsilon(d)}(g_n)$. Second, consider $A_{\varepsilon(d)}(g_n) = 2n + \sum_{j=2}^d jn = 4n + \sum_{j=3}^d jn$ and compare this value to $A_{\varepsilon(i)}(g_n)$ for $i \ne d$. When $2 \le i < d-1$,

$$\begin{aligned} A_{\varepsilon(i)}(g_n) &= [m_2(g_n) + m_3(g_n) + \cdots + m_i(g_n)] + [l_i(g_n) + \cdots + l_{d-1}(g_n)] \\ &= [3n + 4n + \cdots + (i+1)n] + [(i+2)n + \cdots + dn + 3n] \\ &= 3n + \sum_{j=3}^d jn < 4n + \sum_{j=3}^d jn = A_{\varepsilon(d)}(g_n). \end{aligned}$$

When $i = d-1$ we see that

$$A_{d-1}(g_n) = 3n + \sum_{j=3}^d jn < 4n + \sum_{j=3}^d jn = A_{\varepsilon(d)}(g_n).$$

It then follows from Lemma 17 that $f(g_n) = f_\varepsilon(g_n) = 3n + \sum_{i=1}^d m_i(g_n)$. We note for later use that $A_{\varepsilon(d)}(g_n) - A_{\varepsilon(i)}(g_n) = n$ when $i \ne d$.

Let h_n be any point connected to g_n by a path of length at most n in $\mathrm{DL}_d(q)$. Then h_n has projection

$$\Pi(h_n) = ((2n,3n-r_1),(3n,4n-r_2),(4n,5n-r_3),(5n,6n-r_4), \ldots,((d-1)n,dn-r_{d-2}),(dn,3n-r_{d-1}),(2n,n-r_d)),$$

where the r_i satisfy the following statements:

(1) $\sum_{i=1}^{d} r_i = 0$; and
(2) the sum of the positive r_i is at most n and so the sum of the negative r_i is at least $-n$.

We use Lemma 17 again to calculate $f(h_n)$. Observe that, for any $\tau \in \Sigma_d$,

$$\min_{\tau \in \Sigma_d} m_{\tau(1)}(h_n) + l_{\tau(d)}(h_n) = 3n - r_d = m_{\varepsilon(1)}(h_n) + l_{\varepsilon(d)}(h_n)$$

and that $2n \leq 3n - r_d \leq 4n$. Moreover, $A_{\varepsilon(d)}(h_n) = A_{\varepsilon(d)}(g_n)$. We now compare $A_{\varepsilon(i)}(g_n)$ and $A_{\varepsilon(i)}(h_n)$ for $i \neq d$ and obtain

$$\begin{aligned} A_{\varepsilon(i)}(h_n) &= 3n + 4n + \cdots + (i+1)n + (i+2)n - r_i \\ &\qquad + (i+3)n - r_{i+1} + \cdots + dn - r_{d-2} + 3n - r_{d-1} \\ &= A_{\varepsilon(i)}(g_n) - (r_i + \cdots + r_{d-1}) \leq A_{\varepsilon(i)}(g_n) + n. \end{aligned}$$

We have already shown that $A_{\varepsilon(d)}(g_n) - A_{\varepsilon(i)}(g_n) = n$ for $2 \leq i < d$. Combining this with the preceding inequality yields

$$A_{\varepsilon(i)}(h_n) \leq A_{\varepsilon(i)}(g_n) + n = A_{\varepsilon(d)}(g_n) - n + n = A_{\varepsilon(d)}(g_n) = A_{\varepsilon(d)}(h_n);$$

hence $\max_{2 \leq i \leq d} A_{\varepsilon(i)}(h_n) = A_{\varepsilon(d)}(h_n)$. Lemma 17 then implies that

$$f(h_n) = f_\varepsilon(h_n) = 3n - r_d + A_{\varepsilon(d)}(h_n).$$

Next, we choose h_n to be a point of the form just described that is connected to g_n by a path of length at most $n-1$. We show that h_n is not a dead-end element by exhibiting a generator s such that $f(h_n s) = f(h_n) + 1$. Let $s \in S_{d,q}$ be a generator corresponding to an edge of type $\mathbf{e}_d - \mathbf{e}_1$ emanating from h_n, so that

$$\begin{aligned} \Pi(h_n s) = ((2n, 3n - r_1 - 1), (3n, 4n - r_2), (4n, 5n - r_3), (5n, 6n - r_4), \\ \ldots, ((d-1)n, dn - r_{d-2}), (dn, 3n - r_{d-1}), (2n, n - r_d + 1)). \end{aligned}$$

As the ordered pairs in the projection are unchanged between $\Pi(h_n)$ and $\Pi(h_n s)$ except in the second coordinate of the first and last ordered pairs, it is still the case that $\max_{2 \leq i \leq d} A_{\varepsilon(i)}(h_n s) = A_{\varepsilon(d)}(h_n s)$. Note in addition that

$$\min_{\tau \in \Sigma_d} m_{\tau(1)}(h_n s) + l_{\tau(d)}(h_n s) = 3n - r_d + 1 = m_{\varepsilon(1)}(h_n s) + l_{\varepsilon(d)}(h_n s).$$

The maximum value of $3n - r_d + 1$ is $4n$; it may be possible to achieve a value of $4n$ using another permutation in Σ_d, but if $3n - r_d + 1 = 4n$ then the value of $m_{\tau(1)}(h_n s) + l_{\tau(d)}(h_n s)$ can never be less than $4n$ with any nonidentity permutation. Thus we can achieve the minimum value of this quantity by using ε. Lemma 17 now implies that $f(h_n s) = f_\varepsilon(h_n s) = 3n - r_d + 1 + A_{\varepsilon(d)}(h_n s) = 3n - r_d + 1 + A_{\varepsilon(d)}(h_n) = f(h_n) + 1$; hence h_n is not a dead-end element in $\Gamma_d(q)$ with respect to the generating set $S_{d,q}$.

We now show that there is a geodesic path of length n from g_n that terminates at a dead-end element, which we denote $g_{n,n}$. Namely, consider any path of length n originating at g_n with the property that the ith point on the path, denoted $g_{n,i}$, has projection

$$\begin{aligned} \Pi(g_{n,i}) = ((2n, 3n - i), (3n, 4n), (4n, 5n), (5n, 6n), \\ \ldots, ((d-1)n, dn), (dn, 3n), (2n, n + i)) \end{aligned}$$

for $1 \le i \le n$. Letting $r_1 = i$, $r_d = \quad i$, and $r_j = 0$ for $1 < j < d$, the preceding argument implies that

$$\begin{aligned} f(g_{n\ i}) = f_\varepsilon(g_{n\ i}) &= 3n \quad r_d + A_{\varepsilon(d)}(g_{n\ i}) = 3n \quad r_d + A_{\varepsilon(d)}(g_n) \\ &= f(g_n) \quad r_d = f(g_n) + i. \end{aligned}$$

Therefore, this path is geodesic.

We now show that the endpoint $g_{n\ n}$ of this path, which has projection

$$\begin{aligned} \Pi(g_{n\ n}) = ((2n, 2n), (3n, 4n), (4n, 5n), (5n, 6n), \\ \ldots, ((d \quad 1)n, dn), (dn, 3n), (2n, 2n)), \end{aligned}$$

is a dead-end element in $\Gamma_d(q)$ with respect to the generating set $S_{d\ q}$.

We know that $f(g_{n\ n}) = 4n + A_{\varepsilon(d)}(g_{n\ n})$. Let $s \in S_{d\ q}$ be any generator such that $g_{n\ n}s \ne g_{n\ n\quad 1}$. We must show that $f(g_{n\ n}s) \le f(g_{n\ n})$. Since $l_i(g_{n\ n}) > 0$ for all i, there must be indices $j \ne k$ with

(1) $(m_j(g_{n\ n}s), l_j(g_{n\ n}s)) = (m_j(g_{n\ n}), l_j(g_{n\ n}) + 1)$,
(2) $(m_k(g_{n\ n}s), l_k(g_{n\ n}s)) = (m_k(g_{n\ n}), l_k(g_{n\ n}) \quad 1)$, and
(3) $(m_r(g_{n\ n}s), l_r(g_{n\ n}s)) = (m_r(g_{n\ n}), l_r(g_{n\ n}))$ for $r \ne j, k$.

Case 1: $k = d$. Using the identity permutation ε, observe that

$$\min_{\tau \in \Sigma_d} m_{\tau(1)}(g_{n\ n}s) + l_{\tau(d)}(g_{n\ n}s) = 4n \quad 1 = m_{\varepsilon(1)}(g_{n\ n}s) + l_{\varepsilon(d)}(g_{n\ n}s).$$

It may now be the case that $A_{\varepsilon(i)}(g_{n\ n}s) = A_{\varepsilon(i)}(g_{n\ n}) + 1$ for some i; however, it is always true that for $2 \le i \le d \quad 1$ we have $A_{\varepsilon(i)}(g_{n\ n}s) \le A_{\varepsilon(i)}(g_{n\ n}) + 1$. Since $A_{\varepsilon(d)}(g_{n\ n}) \quad A_{\varepsilon(i)}(g_{n\ n}) = n$, for $2 \le i \le d \quad 1$ it follows that

$$A_{\varepsilon(i)}(g_{n\ n}s) \le A_{\varepsilon(i)}(g_{n\ n}) + 1 \le A_{\varepsilon(i)}(g_{n\ n}) + n = A_{\varepsilon(d)}(g_{n\ n}) = A_{\varepsilon(d)}(g_{n\ n}s).$$

Then, by Lemma 17, $f(g_{n\ n}s) = f_\varepsilon(g_{n\ n}s) = 4n \quad 1 + A_{\varepsilon(d)}(g_{n\ n}s)$. Since $A_{\varepsilon(d)}(g_{n\ n}s) = A_{\varepsilon(d)}(g_{n\ n})$ we see that $f(g_{n\ n}s) = f(g_{n\ n}) \quad 1$.

Case 2: $k = 1$. Replacing ε with the permutation $\sigma = (1\ d) \in \Sigma_d$, the argument in Case 1 shows that $f(g_{n\ n}s) = f_\sigma(g_{n\ n}s) = f(g_{n\ n}) \quad 1$.

Case 3: $2 \le k \le d \quad 1$ *and* $j \ne d$. First note that

$$\min_{\tau \in \Sigma_d} m_{\tau(1)}(g_{n\ n}s) + l_{\tau(d)}(g_{n\ n}s) = 4n = m_{\varepsilon(1)}(g_{n\ n}s) + l_{\varepsilon(d)}(g_{n\ n}s)$$

and that $A_{\varepsilon(d)}(g_{n\ n}s) = A_{\varepsilon(d)}(g_{n\ n})$. As in the preceding cases, $A_{\sigma(i)}(g_{n\ n}s) \le A_{\varepsilon(i)}(g_{n\ n}) + 1$ for $2 \le i \le d \quad 1$ and the same reasoning as before yields $A_{\varepsilon(i)}(g_{n\ n}s) \le A_{\varepsilon(d)}(g_{n\ n}s)$. Altogether, then, we have $f(g_{n\ n}s) = f_\varepsilon(g_{n\ n}s) = 4n + A_{\varepsilon(d)}(g_{n\ n}s)$. Since $A_{\varepsilon(d)}(g_{n\ n}s) = A_{\varepsilon(d)}(g_{n\ n})$ it follows that $f(g_{n\ n}s) = f(g_{n\ n})$.

Case 4: $2 \le k \le d \quad 1$ *and* $j = d$. Replacing ε with the permutation $\sigma = (1\ d) \in \Sigma_d$, the argument in Case 3 shows that $f(g_{n\ n}s) = f_\sigma(g_{n\ n}s) = f(g_{n\ n})$.

Combining Cases 1–4 shows that $f(g_{n\ n}s) \le f(g_{n\ n})$ for all $s \in S_{d\ q}$. Therefore, $g_{n\ n}$ is a dead-end element in $\Gamma_d(q)$ with respect to this generating set.

Thus, there is a geodesic path of length n from g_n that terminates at a dead-end element of $\Gamma_d(q)$, and no shorter path from g_n reaches a dead-end element. Hence each g_n lies in a distinct cone type, from which the theorem follows. □

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M. Stein
Department of Mathematics
Trinity College
Hartford, CT 06106

melanie.stein@trincoll.edu

J. Taback
Department of Mathematics
Bowdoin College
Brunswick, ME 04011

jtaback@bowdoin.edu