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BEING CAYLEY AUTOMATIC IS CLOSED under TAKING WREATH PRODUCT with VIRTUALLY CYCLIC GROUPS

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BEING CAYLEY AUTOMATIC IS CLOSED UNDER TAKING WREATH PRODUCT WITH VIRTUALLY CYCLIC GROUPS

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Abstract

We extend work of Berdinsky and Khoussainov [‘Cayley automatic representations of wreath products’, *International Journal of Foundations of Computer Science* **27**(2) (2016), 147–159] to show that being Cayley automatic is closed under taking the restricted wreath product with a virtually infinite cyclic group. This adds to the list of known examples of Cayley automatic groups.

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1. Introduction

Cayley automatic groups, introduced by Kharlampovich *et al.* in [7], generalise the class of automatic groups while retaining some key algorithmic properties. Namely, the word problem in a Cayley automatic group is decidable in quadratic time, and the first-order theory for a (directed, labelled) Cayley graph of a Cayley automatic group is decidable. The family of Cayley automatic groups is larger than that of automatic groups. For example, it includes all finitely generated nilpotent groups of nilpotency class two [7], the Baumslag–Solitar groups [1, 7], the higher-rank lamplighter groups [3], and restricted wreath products of the form $G \wr \mathbb{Z}$ where G is Cayley automatic [2].

Here we add to this list by extending [2] to restricted wreath products of the form $G \wr H$ where G is Cayley automatic and H is virtually infinite cyclic. While this result is not surprising, the proof contains some subtleties which require care, and we believe is worth recording.

2. Automatic and Cayley automatic groups

We assume that the reader is familiar with the notions of regular languages, finite automata and multi-tape synchronous automata. For more details, we refer the reader

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to [6]. We say that a language $L \subseteq (X^*)^n$ is *regular* if it is accepted by a synchronous n-tape automaton where $n \in$ and X is a finite set, or *alphabet*.

For any group G with finite symmetric generating set $S = S^{-1}$, let $\pi : S^* \to G$ denote the canonical projection map. For $w \in S^*$, let $|w|_S$ denote the length of w as a word in the free monoid S^*, that is, $|w|_S$ denotes the number of letters in the word w.

DEFINITION 2.1. An *automatic structure* for a group G is a pair (S, L) where

(1) S is a finite symmetric generating set for G;
(2) $L \subseteq S^*$ is a regular language;
(3) $\pi|_L : L \to G$ is a bijection;
(4) for each $a \in S$ the binary relation $R_a = \{(u, v) \in L \times L \mid \pi(u)a =_G \pi(v)\} \subseteq S^* \times S^*$ is regular, that is, recognised by a two-tape synchronous automaton.

A group is called *automatic* if it has an automatic structure with respect to some finite generating set.

It is a standard result (see, for example, [6, Theorem 2.4.1]) that if G is automatic then G has an automatic structure with respect to any finite generating set.

Cayley automatic groups were introduced in [7] with the motivation of allowing the language L of normal forms representing group elements to be defined over an arbitrary alphabet Λ rather than a generating set S for G.

DEFINITION 2.2. A *Cayley automatic structure* for a group G is a 4-tuple (S, Λ, L, ψ) where

(1) S is a finite symmetric generating set for G;
(2) Λ is an alphabet and $L \subseteq \Lambda^*$ is a regular language;
(3) $\psi : L \to G$ is a bijection;
(4) for each $a \in S$ the binary relation $R_a = \{(u, v) \in L \times L \,|\, \psi(u)a =_G \psi(v)\} \subseteq \Lambda^* \times \Lambda^*$ is regular, that is, recognised by a two-tape synchronous automaton.

A group is called *Cayley automatic* if it has a Cayley automatic structure (S, Λ, L, ψ) with respect to some finite generating set S.

As for automatic groups, if G has a Cayley automatic structure (S, Λ, L, ψ) and Y is another finite generating set for G, then there exists a Cayley automatic structure $(Y, \Lambda_Y, L_Y, \psi_Y)$ for G. See [7, Theorem 6.9] for a proof of this fact.

3. Wreath products with virtually infinite cyclic groups

For two groups G and H, let $G^{(H)}$ be the set of all functions $\gamma : H \to G$ with finite support, that is, such that $\gamma(h) \quad 1_G$ for at most finitely many $h \in H$. For a given $\gamma \in G^{(H)}$ and $h \in H$, we denote by γ^h the element of $G^{(H)}$ for which $\gamma^h(x) = \gamma(hx)$ for all $x \in H$. The restricted wreath product $G \wr H$ can be defined as the Cartesian product $G^{(H)} \times H$ with the group multiplication given by the formula

$$(\gamma, h) \quad (\gamma', h') = (\gamma(\gamma')^{h^{-1}}, hh').$$

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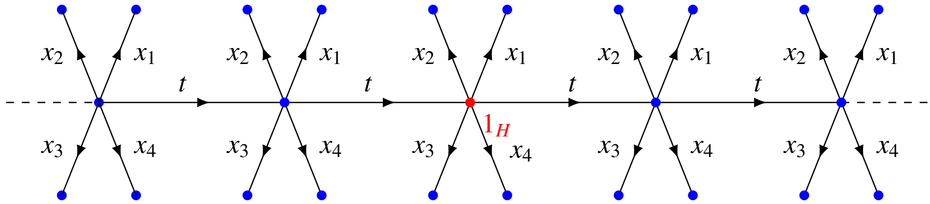

FIGURE 1. Part of a spanning tree $\mathcal{S}$ for $\Gamma(H, T)$, where the index of $\mathbb{Z} = \langle t \rangle$ in H is 5 (colour available online).

Equivalently, we can define $G \wr H$ as

$$\Big\{(\gamma, h) \mid h \in H, \gamma \in \bigoplus_{k \in H} (G)_k \text{ where } \gamma \text{ has finitely many nontrivial entries}\Big\}$$

with multiplication defined as above, where

$$(\gamma')^{h^{-1}} = \Big(\bigoplus_{k \in H} (g)_k\Big)^{h^{-1}} = \bigoplus_{k \in H} (g)_{h^{-1}k}.$$

Note that if G is generated by $S_0 \subseteq G$ and H is generated by $T \subseteq H$ then $G \wr H$ is generated by $S_0 \cup T$.

We prove the following theorem.

THEOREM 3.1 (Wreath products with virtually infinite cyclic groups). *Let G be a Cayley automatic group and H any virtually infinite cyclic group. Then $G \wr H$ is Cayley automatic.*

PROOF. Since G is Cayley automatic, there exist a finite symmetric generating set S_0 for G, an alphabet Λ_0, a regular language $L_0 \subseteq \Lambda_0^*$, a bijection $\psi_0 : L_0 \to G$ and a two-tape automaton ${}_s$ for each $s \in S_0$ with accepted language

$$L({}_s) = \{(u, v) \in L_0 \times L_0 \mid \psi_0(v) =_G \psi_0(u)s\}.$$

Without loss of generality assume $\psi_0(\varepsilon) = 1_G$.

Let H be a finite extension of its cyclic subgroup $\mathbb{Z} = \langle t \rangle$ of index $m + 1$, and denote by $\langle t \rangle x_0, \langle t \rangle x_1, \ldots, \langle t \rangle x_m$ the distinct right cosets of $\mathbb{Z}$, where $x_0 = 1_H$. Let

$$T = \{t, x_1, \ldots, x_m, t^{-1}, x_1^{-1}, \ldots, x_m^{-1}\};$$

then $S = S_0 \cup T$ is a symmetric generating set for $G \wr H$. We identify a particular spanning tree $\mathcal{S}$ of the Cayley graph $\Gamma(H, T)$ which consists of a 'spine' corresponding to $\langle t \rangle$, and at each vertex t^k there are m 'spokes' terminating at the m vertices $t^k x_j$ of H, for $k \in \mathbb{Z}$ and $1 \le j \le m$, as in Figure 1.

As a concrete example, consider the infinite dihedral group

$$H = D_\infty = \langle a, b \mid a^2, b^2 \rangle.$$

In this case we can take $t = ab, x_1 = a$ and $\mathcal{S} = \Gamma(D_\infty, T)$, as shown in Figure 2.

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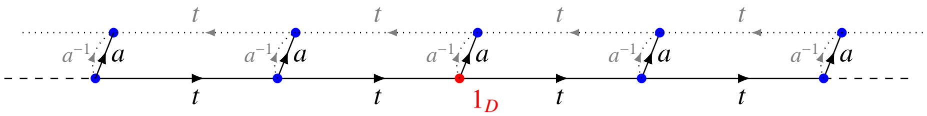

FIGURE 2. Part of the spanning tree $\mathcal{S}$ (shown in black) drawn inside $\Gamma(D_\infty, T)$, where $T = \{t, a, t^{-1}, a^{-1}\}$. Note that $a = a^{-1}$ and $ata = aaba = ba = t^{-1}$ (colour available online).

We borrow some terminology from the lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ to describe elements of $G \wr H$. An element $v \in G \wr H$ can be thought of in two equivalent ways:

(1) algebraically, as an element (γ, h) where $\gamma \in \bigoplus_{k \in H}(G)_k$ has finitely many nontrivial entries and $h \in H$;
(2) geometrically, as a copy of $\mathcal{S}$ (or $\Gamma(H, T)$) where each vertex is marked by some element of G, with all but finitely many vertices marked by 1_G, and the vertex h of $\mathcal{S}$ is also marked with a *pointer* indicating the final position of the 'lamplighter'. We refer to this marking as a *configuration* of $\mathcal{S}$.

Write $v = (\gamma, h)$ where $\gamma \in \bigoplus_{h_i \in H}(G)_{h_i}$ and $h \in H$. Since every element h of H can be written uniquely as $h = t^k x_q$ for some $k \in \mathbb{Z}$ and $0 \leq q \leq m$, the map $\xi : H \to \mathbb{Z}$ defined by $\xi(t^k x_q) = k$ is well defined. Then the vertex corresponding to $h \in H$ is an endpoint of a spoke attached to the vertex $t^{\xi(h)}$. For $v = (\gamma, h)$ with γ as above, let

(1) $k_* = \xi(h)$,
(2) $k_1 = \min\{0, \xi(h_i) \mid (g)_{h_i} \in \gamma, (g)_{h_i} \quad 1_G\}$, and
(3) $k_2 = \max\{0, \xi(h_i) \mid (g)_{h_i} \in \gamma, (g)_{h_i} \quad 1_G\}$.

Additionally, let $m_1 = \min(k_*, k_1)$ and $m_2 = \max(k_*, k_2)$. Define the *integer support* of v, denoted isupp(v), to be the interval $[m_1, m_2]$. The left endpoint of the integer support is the smallest k so that either:

(1) v has a nontrivial entry among the copies of $\Gamma(G, S_0)$ attached to the spine at the vertex $t^k x_i$ for some i with $0 \leq i \leq m$, or
(2) the final position of the lamplighter is $t^k x_i$ for some i with $0 \leq i \leq m$, or
(3) all of k_* and $\xi(h_i)$ are positive, that is, the lamplighter is never in a position along the spine with negative index, so that $m_1 = 0$ denotes the starting position of the lamplighter.

The right endpoint of the integer support is defined analogously, where the 0 is included in the definition of k_2 to account for the possibility that k_* and all the $\xi(h_i)$ are negative.

To define our normal form, we mimic the standard 'left-first' representation of elements of the lamplighter group $\mathbb{Z}_n \wr \mathbb{Z}$ (see [4]). Given $v = (\gamma, h) \in G \wr H$, we describe a path traversed by the lamplighter from the vertex 1_H in $\mathcal{S}$ to its final vertex $h \in \mathcal{S}$. If $m_1 < 0$, the lamplighter first moves left along the spine of $\mathcal{S}$ to the vertex labelled t^{m_1} and marks it with a possibly trivial element of G.

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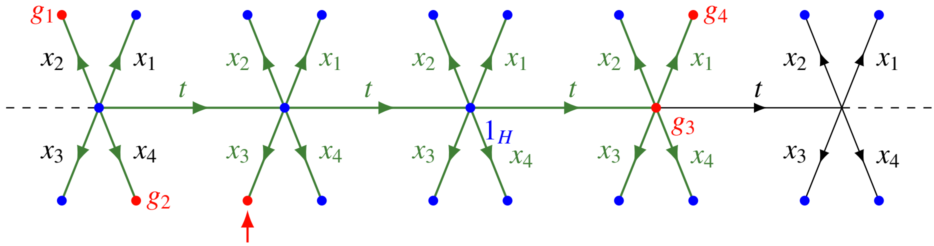

FIGURE 3. The element $t^{-2}x_2g_1x_2^{-1}x_4g_2x_4^{-1}t^3g_3x_1g_4x_1^{-1}t^{-2}x_3$ as a configuration of $\mathcal{S}$. The integer support of this element is $[-2, 1]$ and the red arrow denotes the final position of the lamplighter. The positive path for this element, $x_1x_1^{-1}x_2g_1x_2^{-1}x_3x_3^{-1}x_4g_2x_4^{-1}(tx_1x_1^{-1}x_2x_2^{-1}x_3x_3^{-1}x_4x_4^{-1})^2tg_3x_1g_4x_1^{-1}x_2x_2^{-1}x_3x_3^{-1}x_4x_4^{-1}$, is shown in green (colour available online).

The lamplighter then visits $t^{m_1}x_1$ and marks it with a possibly trivial element of G and returns to t^{m_1}. This procedure is repeated for the vertices $t^{m_1}x_2, \dots, t^{m_1}x_m$. The lamplighter then proceeds to the vertex corresponding to t^{m_1+1} and repeats the process of visiting the vertex at the end of each spoke in order and marking it with a possibly trivial element of G. This continues until the lamplighter reaches the vertex corresponding to t^{m_2}, where the process is repeated one last time. If $m_1 = 0$, the lamplighter begins the process of marking the vertices with possibly trivial elements of G at $1_H \in \mathcal{S}$, and then visits the spokes as described above, until it reaches the vertex labelled t^{m_2} and marks the vertices $t^{m_2}x_j$ for $0 \le j \le m$ with possibly trivial elements of G.

We refer to the subpath which starts at the vertex t^{m_1} and ends at the vertex t^{m_2} after having marked the vertices t^ix_j for $m_1 \le i \le m_2, 0 \le j \le m$ with possibly trivial elements of G as the *positive path*, because when written as a word in the group generators, the exponents of t are all positive. See Figure 3 for an example of a configuration with integer support $[-2, 1]$ where the positive path is marked.

Upon completing the positive path, one of two things will occur. It may be that the lamplighter is in its final position and the path simply ends. If not, the lamplighter moves to its final position via a subpath of the form t^k or t^kx_q where $k \in \mathbb{Z}, k \le 0$. Note that since m_2, the right endpoint of isupp(v), is the maximum of k_2 and k_*, the lamplighter will never be in a position along $\mathbb{Z} = \langle t \rangle$ to the right of m_2, so the exponent k is nonpositive.

As the lamplighter travels along its positive path, we will wish to indicate two special positions: the first time the lamplighter is at the vertex corresponding to $1_H = t^0$, and the first time the lamplighter is at the vertex which will be its final position. The integer support and the positive path are defined so that these are unique positions along the positive path.

The normal form for the Cayley automatic structure on $G \wr H$ will be constructed in stages. We first define a normal form $\mathcal{N}_0 \subseteq (\Lambda_0 \cup T)^*$ for elements of $G \wr H$ as follows. Given $v \in G \wr H$ with isupp$(v) = [m_1, m_2]$, the above description allows us to represent

v uniquely as a word either of the form

$$v = t^n x_q \quad \text{or} \quad v = t^{m_1} v_1 t v_2 \qquad t v_s t^j x_q, \tag{3.1}$$

where $n, j, q, s \in \mathbb{Z}$, $j \leq 0$, $s \geq 1$, $0 \leq q \leq m$, $m_1 + (s-1) = m_2$ and

$$v_\kappa = v_{\kappa,0} x_1 v_{\kappa,1} x_1^{-1} x_2 v_{\kappa,2} \qquad x_{m-1}^{-1} x_m v_{\kappa,m} x_m^{-1} \tag{3.2}$$

with $v_{\kappa,t} \in L_0$. If $m_1 = k_*$ then we allow v_1 to be trivial, otherwise v_1 must be nontrivial. If $m_2 = k_*$ we allow v_κ to be trivial, otherwise v_κ must be nontrivial. Each word v_κ encodes a sequence of words $(v_{\kappa,0}, \ldots, v_{\kappa,m}) \in L_0^{m+1}$ with $\psi_0(v_{\kappa,0})$ labelling the vertex at position $t^{m_1+\kappa-1}$ in $\mathcal{S}$ and $\psi_0(v_{\kappa,i})$ labelling the end of the spoke at position $t^{m_1+\kappa-1} x_i$ for $1 \leq i \leq m$. Note that in Equation (3.1), the v_κ are separated by instances of t as the lamplighter moves along the positive path. Let $\mathcal{N}_0 \subseteq (\Lambda_0 \cup T)^*$ denote the set of words of this form.

For example, the element in Figure 3 has $\mathcal{N}_0$ normal form

$$t^{-2} x_1 x_1^{-1} x_2 v_{1,2} x_2^{-1} x_3 x_3^{-1} x_4 v_{1,4} x_4^{-1} t x_1 x_1^{-1} x_2 x_2^{-1} x_3 x_3^{-1} x_4 x_4^{-1} t x_1 x_1^{-1} x_2 x_2^{-1} \\ x_3 x_3^{-1} x_4 x_4^{-1} t v_{4,0} x_1 v_{4,1} x_1^{-1} x_2 x_2^{-1} x_3 x_3^{-1} x_4 x_4^{-1} t^{-2} x_3$$

where $\psi_0(v_{1,2}) = g_1, \psi_0(v_{1,4}) = g_2$, $\psi_0(v_{4,0}) = g_3$, $\psi_0(v_{4,1}) = g_4$ and, in all other cases, $v_{i,j} = \varepsilon$ where $\psi_0(\varepsilon) = 1_G$.

Next, we insert special symbols into the words in $\mathcal{N}_0$ to obtain the intermediate language $\mathcal{N}_1$.

Let $\Lambda_1 = \Lambda_0 \cup T \cup \{B, C, B_0, C_*\}$ and $\Lambda = \Lambda_0 \cup \{B, C, B_0, C_*\}$. Let $v = (\gamma, h) \in G \wr H$ be written in the form of Equation (3.1). Notice that all terms of the form v_κ are part of the positive path. With v_κ as in Equation (3.2), before each $v_{\kappa,j}$ we place the symbol C, with one exception. If $v_{\kappa,j}$ is the label of the vertex h of $\mathcal{S}$ which is the final position of the lamplighter, then precede $v_{\kappa,j}$ by the symbol C_*. Before each term v_κ we place the symbol B, with one exception. If $m_1 + \kappa - 1 = 0$ we place the symbol B_0 in front of v_κ, indicating the unique position along the positive path where the lamplighter is at the vertex $1_H \in \mathcal{S}$.

Let $\mathcal{N}_1 \subseteq \Lambda_1^*$ denote the set of all words in $\mathcal{N}_0$ where the symbols $\{B, C, B_0, C_*\}$ have been inserted as described. The word in $\mathcal{N}_1$ for the element in Figure 3 is then

$$t^{-2} B C x_1 C x_1^{-1} x_2 C v_{1,2} x_2^{-1} x_3 C x_3^{-1} x_4 C v_{1,4} x_4^{-1} t B C x_1 C x_1^{-1} x_2 C x_2^{-1} x_3 C_* x_3^{-1} \\ x_4 C x_4^{-1} t B_0 C x_1 C x_1^{-1} x_2 C x_2^{-1} x_3 C x_3^{-1} x_4 C x_4^{-1} t B C v_{4,0} x_1 C v_{4,1} x_1^{-1} x_2 C x_2^{-1} \\ x_3 C x_3^{-1} x_4 C x_4^{-1} t^{-2} x_3.$$

To obtain the final normal form which will be the basis of the Cayley automatic structure for $G \wr H$, let $\mathcal{N} \subseteq \Lambda^*$ denote the set of words in $\mathcal{N}_1$ where all instances of the letters $t, x_1, \ldots, x_m, t^{-1}, x_1^{-1}, \ldots, x_m^{-1}$ from the set T are removed. The word in $\mathcal{N}$ for the element in Figure 3 is then

$$BCCCv_{1,2}CCv_{1,4}BCCCC_*CB_0CCCCCBCv_{4,0}Cv_{4,1}CCC.$$

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Define the language

$$L_1 = \left\{ \prod_{i=1}^{p} (\beta_i \Gamma_{i,0} v_{i,0} \Gamma_{i,1} v_{i,1} \Gamma_{i,2} v_{i,2} \quad \Gamma_{i,m} v_{i,m}) \;\middle|\; \begin{array}{l} v_{i,j} \in L_0, \\ \beta_i \in \{B, B_0\}, \\ \Gamma_{i,j} \in \{C, C_*\}, \\ p \geq 1 \end{array} \right\}. \tag{3.3}$$

As L_0 is a regular language, it follows that L_1 is a regular language.

Recall that when $v \quad t^k x_q$, if $m_1 = k_*$ we allow v_1 to be trivial, otherwise v_1 must be nontrivial, and if $m_2 = k_*$ we allow v_s to be trivial, otherwise v_s must be nontrivial. These conditions are easily verified by a finite-state automaton inspecting, respectively, the first and last expressions in the product representing an element of L_1 according to the following rules.

(1) If C_* occurs in the first factor in the product as expressed in Equation (3.3), then all $v_{i,j}$ may be ε for $0 \leq j \leq m$; if not, at least one $v_{i,j}$ must be nontrivial.
(2) If the C_* occurs in the last factor in the product as expressed in Equation (3.3), then all $v_{i,j}$ may be ε for $0 \leq j \leq m$; if not, at least one $v_{i,j}$ must be nontrivial.

Note that a finite-state automaton can also easily verify that when all v_κ are trivial, we have a normal form corresponding to $t^k x_q$. Let $L_2 \subseteq L_1$ be the set of all strings in L_1 for which all three of these conditions are satisfied. Since these conditions are easily checked by a finite-state automaton and L_1 is a regular language, it follows that L_2 is a regular language.

Finally, we verify that the string has only one occurrence each of B_* and C_*. Let

$$L = L_2 \cap \{pB_0qC_*r, pC_*qB_0r \mid p, q, r \in (\Lambda \setminus \{B_0, C_*\})^*\}.$$

It follows that L is regular and that $L = \mathcal{N}$.

As a further example, note that if $v = t^n x_q$, the corresponding word in L is as follows:

(1) when $n > 0$, we have $\operatorname{isupp}(t^n x_q) = [0, n]$ and the corresponding word is

$$B_0 C^{m+1} (BC^{m+1})^{n-1} BC^q C_* C^{m-q};$$

(2) when $n = 0$, we have $\operatorname{isupp}(x_q) = [0, 0]$ and the corresponding word is

$$B_0 C^q C_* C^{m-q};$$

(3) when $n < 0$, we have $\operatorname{isupp}(t^n x_q) = [n, 0]$ and the corresponding word is

$$BC^q C_* C^{m-q} (BC^{m+1})^{n-1} B_0 C^{m+1}.$$

Given a word $\sigma \in L$, the symbols B_0 and C_* allow us to reconstruct the integer support of the corresponding element, as well as the final position of the lamplighter, that is, the coordinate h. The words $v_{i,j}$ correspond (via ψ_0) to elements of G listed in a specified order. That is, we can deterministically reconstruct $\gamma \in \bigoplus_{h \in H} (G)_h$ and $h \in H$

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from σ. Formally, let $\psi : L \to G \wr H$ be the bijective map defined by

$$f\psi(w) = \psi\Big(\prod_{k=1}^{s}(\beta_k\Gamma_{k,0}u_{k,0}\Gamma_{k,1}u_{k,1}\Gamma_{k,2}u_{k,2} \quad \Gamma_{k,m}u_{k,m})\Big) = t^{m_1}p_1tp_2 \quad tp_st^j,$$

where

$$p_i = \psi_0(v_{i,0})x_1\psi_0(v_{i,1})x_1^{-1}x_2\psi_0(v_{i,2}) \quad x_{m-1}^{-1}x_m\psi_0(v_{i,m})x_m^{-1}$$

with $u_{i,j} = v_{i,j}$, $\beta_k \in \{B, B_0\}$, $\Gamma_{k,j} \in \{C, C_*\}$ and m_1 calculated from the positions of B_0 and C_* as described above.

We claim that (S, Λ, L, ψ) is a Cayley automatic structure for $G \wr H$. To prove this, we must show that for every generator $s \in S = S_0 \cup T$ the set

$$R_s = \{(u, v) \in L \times L \mid \psi(u)s =_{G\wr H} \psi(v)\}$$

is a regular language, that is, recognised by a two-tape synchronous automaton. It suffices to do this for $s \in S_0 \cup \{x_1, \dots, x_m, t\}$ (see, for example, [5, Lemma 9]).

First let $s \in S_0$ and suppose $(u, v) \in R_s$. Viewing $\psi(u)$ as a configuration of $\mathcal{S}$ with finitely many vertices marked with elements of G and a distinguished position for the lamplighter, we can easily see the effect of multiplication by s on the normal form. Let $t^k x_q$ denote the vertex of $\mathcal{S}$ which is the final position of the lamplighter in u, marked by the element $g_u \in G$. Let $\rho_u \in L_0$ be such that $\psi_0(\rho_u) = g_u$. To obtain the normal-form word for $\psi(u)s$ we simply multiply ρ_u by s and verify that the multiplication is correct using the multiplier automaton $\ _s$ given as part of the given Cayley automatic structure on G. Therefore we need to accept pairs of strings $(u, v) \in L \times L$ of the form

$$u = (\Pi_{i=1}^{p}\beta_i\Pi_{j=0}^{m}C \quad _{i,j}) \quad _u \ (\Pi_{i=p+2}^{\kappa}\beta_i\Pi_{j=0}^{m}C \quad _{i,j})$$

and

$$v = (\Pi_{i=1}^{p}\beta_i\Pi_{j=0}^{m}C \quad _{i,j}) \quad _v \ (\Pi_{i=p+2}^{\kappa}\beta_i\Pi_{j=0}^{m}C \quad _{i,j}),$$

where $\beta_i \in \{B, B_0\}$, $\ _{i,j} \in L_0$,

$$_u = \beta_{p+1}C \quad _{p+1,0} \quad C_* \quad _{\mathbf{p+1,r}} \quad C \quad _{p+1,m}$$

and

$$_v = \beta_{p+1}C \quad _{p+1,0} \quad C_* \quad '_{\mathbf{p+1,r}} \quad C \quad _{p+1,m}$$

and $(\ _{p+1,r}, \ '_{p+1,r})$ is accepted by the multiplier automaton $\ _s$ which forms part of the given Cayley automatic structure on G. The bold highlighted symbols represent the only difference between the two words.

By [7] (see also [5, Lemma 8]) the language L_0 is necessarily quasigeodesic. It follows that the difference between the lengths of $\ _{p+1,r}$ and $\ '_{p+1,r}$ is uniformly bounded. As it is regular to check that two words are identical with a bounded shift, it follows that we can construct a two-tape automaton which checks that the prefixes of u and v are identical, then calls $\ _s$ to read $(\ _{p+1,r}, \ '_{p+1,r})$, and finally checks that the suffixes of u and v are identical (with a bounded shift). Thus R_s is a regular language.

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Next let $x_i \in \{x_1, \ldots, x_m\}$, and suppose $(u, v) \in R_{x_i}$. Writing $\psi(u)$ as in Equation (3.1), we see that $\psi(u)x_i$ ends in the letters $x_q x_i$. The product $x_q x_i \in H$ is an element of some right coset $\langle t \rangle x_r$. That is, $x_q x_i = t^k x_r$ for some k and r. Viewing $\psi(u)$ and $\psi(v)$ as configurations in $\mathcal{S}$, this means that the configurations are identical except for the final position of the lamplighter which is indicated by C_* in the normal form. Note that as x_q and x_i vary among the finite set of coset representatives, there are only a finite number of possible values of (k, r) which arise.

The elements $\psi(u)$ and $\psi(v)$ may or may not have identical integer support. For example, if $\psi(u) = t^{-10} g_1 t^{20} g_2 t^{-5} x_q$ and $x_q x_i = t^{-7} x_r$ then $\mathrm{isupp}(\psi(u)) = \mathrm{isupp}(\psi(v))$, whereas if $\psi(u) = t^{-10} g_1 t^{20} g_2 t^{-5} x_q$ and $x_q x_i = t^{-17} x_r$ then $\mathrm{isupp}(\psi(u)) \quad \mathrm{isupp}(\psi(v))$.

If $\mathrm{isupp}(\psi(u)) = \mathrm{isupp}(\psi(v))$, then we simply need to check that the two strings are identical except for the location of C_*. If $\psi(u)$ ends in x_q when written as in Equation (3.1), we have $x_q x_i = t^k x_r$. Let $\pi : \Lambda^* \to \{C, C_*, B_0\}^*$ be a homomorphism which is the identity on C, C_*, B_0 and sends all other letters to ε. Then $\pi(u)$ and $\pi(v)$ are identical strings except for the location of C_* in each string. Observe that the letter B_0 is in the same position in each string since the integer supports of $\psi(u)$ and $\psi(v)$ are the same. Further observe that there exists an integer $s_{q,i}$ such that for every pair $(u, v) \in R_{x_i}$ with the same integer support, if C_* is the xth letter of $\pi(u)$ and the yth letter of $\pi(v)$, then $x - y = s_{q,i}$.

Consider the language $X_{q,i} \subseteq \{C, C_*, B_0\}^* \times \{C, C_*, B_0\}^*$ consisting of all pairs of strings, each of which contains exactly one C_* letter and one B_0 letter, where C_* is the xth letter of the first string and the yth letter of the second string with $x - y = s_{q,i}$, and B_0 is in the same position in both strings. Since these conditions are regular to check, $X_{q,i}$ is a regular language.

Let

$$\kappa \colon \Lambda^* \times \Lambda^* \to \{C, C_*, B_0\}^* \times \{C, C_*, B_0\}^*$$

be the map which in each coordinate is the identity on C, C_* and B_0 and sends all other letters to ε. Let $Y \subseteq \Lambda^* \times \Lambda^*$ be the language consisting of all pairs of strings such that for every positive integer z the zth letter of the first string and the second string is the same unless one of these letters is C_* and the other is C. The language Y is regular. Thus the language

$$\kappa^{-1}(X_{q,i}) \cap (L \times L) \cap Y$$

is regular, and the union of these languages for $0 \leq q \leq m + 1$ is exactly the subset of R_{x_i} for which multiplication by x_i does not change the integer support for the first entry.

Now consider all the possible ways that the integer support of $\psi(u)$ can change upon multiplication by x_i. Again assume $\psi(u)$ ends in x_q when written as in Equation (3.1), and $x_q x_i = t^k x_r$. We must consider the following cases:

(1) $\psi(u) = t^n x_q$ and $k \quad 0$;

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(2) $\psi(u) = t^{m_1} v_1 t v_2 \quad t v_s t^j x_q$ with $\mathrm{isupp}(\psi(u)) = [m_1, m_2]$, $j \leq 0$ and either

(a) $k > -j$, in which case the integer support of v extends further to the right of m_2, or

(b) $k < m_1 - m_2 - j$, in which case the integer support of v extends further to the left of m_1.

Each of these cases can be handled in a manner similar to the above case, by considering the relative positions of C_* and B_0 in $\pi(u), \pi(v)$. For the first case, if $n > 0, k > -n$ then

$$u = B_0 C^{m+1} (BC^{m+1})^{n-1} BC^q C_* C^{m-q} \quad \text{and} \quad v = B_0 C^{m+1} (BC^{m+1})^{n+k-1} BC^r C_* C^{m-r};$$

if $n > 0, k < -n$ then

$$u = B_0 C^{m+1} (BC^{m+1})^{n-1} BC^q C_* C^{m-q} \quad \text{and} \quad v = BC^r C_* C^{m-r} (BC^{m+1})^{-k-n-1} B_0 C^{m+1}.$$

Analogous pairs of expressions can be worked out for $n \leq 0$; clearly all such pairs can be recognised by two-tape automata since q, i, k, r are fixed. We leave details of the remaining cases to the reader.

Finally, suppose $(u, v) \in R_t$. Writing $\psi(u)$ as in Equation (3.1), we see that $\psi(u)t$ ends in the letters $x_q t$. Once again, we can consider the case where the integer support of $\psi(u)$ does not change, in which case we merely need to check the location of the C_* letters in each word, and separately the case where the integer support of $\psi(u)$ differs at one endpoint from the integer support of $\psi(v)$. We follow the same reasoning as in the previous case of multiplication by x_i; note that $x_q t$ is in some right coset of $\mathbb{Z}$ in H, so we can write $x_q t = t^k x_r$, for a possibly different coset representative x_r. We can therefore show that R_t is a regular language as well. The regular languages R_s, R_{x_i} and R_t complete the construction of the Cayley automatic structure (S, Λ, L, ψ) for $G \wr H$.

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