1-1-2010

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Ann. Comb. 13 (2010) 519–532
DOI 10.1007/s00026-009-0034-9
Published online January 12, 2010

Annals of Combinatorics

Relation for Domino Robinson-Schensted lgorithms

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Received October 11, 2006

AMS Subject Classi cation: 05E10

Abstract We describe a map relating hyperoctahedral Robinson-Schensted algorithms on standard domino tableaux of unequal rank. Iteration of this map relates the algorithms de ned by Gar nkle and Stanton-White and when restricted to involutions, this construction answers a question posed by van Leeuwen. The principal technique is derived from operations de ned on standard domino tableaux by Gar nkle which must be extended to this more general setting.

Keywords: domino tableaux, Robinson-Schensted algorithm

1 Introduction

The lassi al Robinson-S hensted algorithm defines a bije tion between the elements of the symmetri group S and the same-shape pairs of standard Young tableaux of size n. The work of Garfinkle [3] defines similar bije tions for H , the hypero tahedral group on n letters, using pairs of ertain same-shape standard *domino* tableaux as parameter sets.

Viewing H as the Weyl group of a simple Lie group of type C, Garfinkle's generalization is a map G whose image is pre isely the set of same-shape pairs of standard domino tableaux of size n and rank 0. When viewing H as the Weyl group of a simple Lie group of type , she defines a more natural map G_1 whose image is the set of same-shape pairs of standard domino tableaux of size n and rank 1. van Leeuwen has observed that Garfinkle's definition an be extended to define bije tive maps G_r from H to same-shape pairs of standard domino tableaux of arbitrary rank r [7]. For r suffi iently large, G_r re overs the bije tion of Stanton and White defined between H and pairs of same-shape standard bitableaux (f. [13] and also [11]).

Consider an element $\in H$ and let $T\ S) = G_r$) and $T\ S\ ) = G_{r\ 1}$). The main result of this paper des ribes a map between the pairs $T\ S)$ and $T\ S\ )$ using te hniques from [3]. In this way, we obtain maps that relate the different members of this family of generalized Robinson-S hensted algorithms as well as the algorithm and Stanton and White. When is an involution, the map sending $T\ S)$ to $T\ S\ )$ has a parti ularly simple des ription and answers a question posed by van Leeuwen in [7, p. 26].

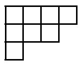
The ombinatorial results of this paper are parti ularly relevant to re ent results in the study of Kazhdan-Lusztig ell stru ture of an unequal parameter He ke algebra . Garfinkle's original work on primitive spe trum of a universal enveloping algebra of a omplex semisimple Lie algebra lassified the Kazhdan-Lusztig ell stru ture of equal parameter He ke algebras of type . In the more general setting of unequal parameter , [1] onje tures a parametrization of ells via domino tableaux of rank r, where the spe ifi hoi e of r depends on the underlying parameters of . In [12], the results of the present paper are used to re on ile the above onje ture and Garfinkle's original work on primitive ideals. In related work, [6] and [5] provide a geometri interpretation of these ombinatorial results in the setting of rational Cherednik algebras.

2 Definitions and Preliminaries

2.1. Generalized Robinson-S hensted Algorithms

Following Garfinkle [3], we view the elements of the hypero tahedral group H as subsets of $\times \quad \times \{\pm 1\}$, with $= \{1 \ 2 \ \ldots \ n\}$ su h that the proje tions onto the first and se ond omponents of are always bije tions onto ([3, Definition 1.1.2]). We will write the element as $\{ \quad _1 \ 1 \ \varepsilon_1) \ \ldots \quad n \ \varepsilon \ )\}$. In this form, orresponds to the signed permutation $\varepsilon_1 \ _1 \ \varepsilon_2 \ _2 \ \ldots \ \varepsilon \quad)$.

For us, Young diagrams will be finite left-justified arrays of squares arranged with non-in reasing row lengths. A square in row i and olumn j of the diagram will be denoted $S_{i \ j}$ so that $S_{1 \ 1}$ is the uppermost left square in the Young diagram below:



Definition 2 1 *Let $r \in$ and λ be a partition of a positive integer m. A* domino tableau of rank r and shape λ *is a Young diagram of shape λ whose squares are labeled by integers from some set M such that* 0 *labels the square $i \ j)$ if and only if $i + j < r + 2$, each element of M labels exactly two adjacent squares, and all labels increase weakly along both rows and columns. A domino tableau is* standard *iff $M =$ for some n.*

We will write $DT_r \ \lambda)$ for the family of all domino tableaux of rank r and shape λ and $DT_r \ n)$ for the family of all domino tableaux of rank r whi h ontain exa tly n dominos. The orresponding families of standard tableaux will be denoted $SDT_r \ \lambda)$ and $SDT_r \ n)$. The set of squares in a tableau T labeled by the integer l will be denoted by $supp \ l \ T)$ and $supp \ 0 \ T)$ will be alled the *core* of T.

Following [3] and [7], we des ribe the Robinson-S hensted bije tions

$$G_r \colon H \quad \rightarrow SDT_r \ n) \times SDT_r \ n).$$

The algorithm is based on an insertion map α whi h, given an element $i \ j \ \varepsilon)$ of $\in H$, inserts a domino with label i into a domino tableau.

Definition 2 2 *Consider* $\in H$, i j $\varepsilon) \in$, *and a domino tableau* $T \in DT_r$ $k)$. *Write* $\ell = \{l_1 \; l_2 \; \dots \; l_k\}$ *for the set of labels of the dominos of* T *listed in increasing order. When* $i \notin \ell$, *we can define a tableau* $T = \alpha$ i j $\varepsilon)$ T $) \in DT_r$ $k+1)$ *by the following procedure*:

(1) *If* $i > l_k$, T *is formed by*

(a) *adding a new horizontal domino with label i to the end of the first row of* T *if* $\varepsilon = 1$, *or by*
(b) *adding a new vertical domino with label i at the end of the first column of* T *if* $\varepsilon = -1$.

(2) *Otherwise, let* l_m *be the least label in* ℓ *greater than* i. *We inductively define a sequence* $\{T_{m-1} \; T_m \; \dots \; T_k \;\; _1\}$ *of domino tableaux and let* $T = T_k \;\; _1$. *To this effect, construct* T_{m-1} *by removing all dominos with labels greater than or equal to* l_m *from* T . *Let* $T_m = \alpha$ i j $\varepsilon)$ $T_{m-1})$. *For* $p \geq m$,

(a) *if supp* l_p T $) \cap T_p = \emptyset$, *then* $T_p \;\; _1$ *is the tableau obtained from* T_p *by labeling supp* l_p T $)$ *with integer* l_p;
(b) *if supp* l_p T $) \cap T_p =$ $S_{i\,j}\}$, *then* $T_p \;\; _1$ *is the tableau obtained from* T_p *by labeling* $S_{i\,j\;\;1}$ $S_{i\;\;1\,j\;\;1}\}$ *with integer* l_p *if supp* l_p T $)$ *is horizontal, or by labeling* $S_{i\;\;1\,j}$ $S_{i\;\;1\,j\;\;1}\}$ *with integer* l_p *if supp* l_p T $)$ *is vertical;*
( ) *if supp* l_p T $) \cap T_p =$ *supp* l_p T $)$, *then* $T_p \;\; _1$ *is the tableau obtained by adding a horizontal domino with label* l_p *at the end of row* $\iota + 1$ *of* T_p *if supp* l_p T $)$ *is horizontal and lies in row* ι *of* T , *or by adding a vertical domino with label* l_p *at the end of column* $\iota + 1$ *of* T_p *if supp* l_p T $)$ *is vertical and lies in column* ι *of* T .

That this pro edure is well-defined and indeed produ es a domino tableau is verified in [3, Se tion 2]. To des ribe the generalized Robinson-S hensted algorithm itself, we start by onstru ting the left tableau. Let T $0)$ be the only tableau in SDT_r $0)$. Define T $1) = \alpha$ $_1$ 1 $\varepsilon_1)$ T $0))$ and ontinue indu tively by letting

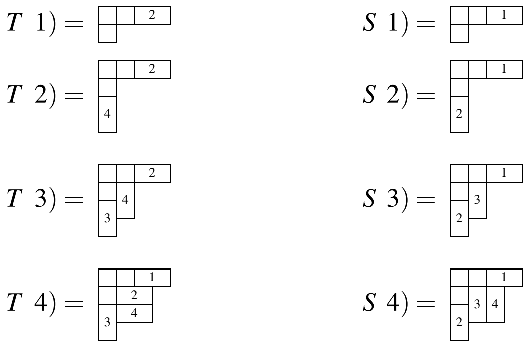
$$T \;\; k+1) = \alpha \quad _{k \;\; 1} \;\; k+1 \;\; \varepsilon_{k \;\; 1}) \;\; T \;\; k)) \, .$$

The left domino tableau T $n)$ will be standard and of rank r. The right tableaux tra k the shapes of the left tableaux. Begin by forming a domino tableau S $1)$ by adding a domino with label 1 to T $0)$ in su h a way that S $1)$ and T $1)$ have the same shape. Continue adding dominos by requiring that at ea h step S $k)$ lie in SDT_r $k)$ and have the same shape as T $k)$. Again, the domino tableau S $n)$ will be standard and of rank r. Finally, the image of under G_r is defined as the tableau pair T $n)$ S $n))$. To simplify notation, we will write G_r^k $)$ for the pair T $k)$ S $k))$. We will also sometimes simplify notation slightly and write α_m $T)$ instead of α $_m$ m $\varepsilon_m)$ $T)$ and α_m T $S))$ for the domino tableau pair obtained by following the above shape-tra king pro edure for α_m $T)$.

When $r = 0$ or 1, G_r are pre isely Garfinkle's algorithms; for $r > 1$ they are natural extensions to larger-rank tableaux. In all ases, G_r define a bije tion from H to pairs of same-shape tableaux in SDT_r $n)$ [7]. These generalizations of the Robinson-S hensted algorithm share a number of properties with the original algorithm. We state the following:

Proposition 2.3 ([7, (4.2)]) *$G_r(\sigma^{-1}) = (S, T)$ whenever $G_r(\sigma) = (T, S)$. In particular, if σ is an involution, $G_r(\sigma) = (T, T)$ for some standard domino tableau T.*

Example 2.4. Consider the signed permutation $(2\ -4\ -3\ 1)$. It corresponds to the set $\sigma = \{(2, 1, 1), (4, 2, -1), (3, 3, -1), (1, 4, 1)\} \in H_4$. If $r = 2$, then successive insertion of elements of σ into the empty tableau of rank zero yields the following sequence of tableau pairs



Consequently, $G_2(\sigma) = (T(4), S(4))$.

2.2. Cycles

The notion of a cycle in a domino tableau appears in a number of references. See, for instance, [2], [8], or [9]. We now review its definition.

Definition 2.5 *For a standard domino tableau T of arbitrary rank r, we call a square in position (i, j)* fixed *when $i + j$ has the opposite parity as r; otherwise, we will call it* variable.

It is possible to choose the sets of fixed and variable squares differently, as in [3, Definition 1.5.4]; however, we refrain from defining the more general possibilities as only this choice will be necessary for our results.

If $T \in SDT_r(n)$, we will write $D(k, T)$ for the domino labeled by the positive integer k in T viewed as a set of labeled squares, and $supp\,D(k, T)$ will denote its underlying squares. Write $label\,S_{i,j}$ for the label of square $S_{i,j}$ in T. We extend this notion slightly by letting $label\,S_{i,j} = 0$ if either i or j is less than or equal to zero, and $label\,S_{i,j} = \infty$ if i and j are positive but $S_{i,j}$ is not a square in T.

Definition 2.6 *Suppose that $supp\,D(k, T) = \{S_{i,j}, S_{i+1,j}\}$ or $\{S_{i,j-1}, S_{i,j}\}$ and the square $S_{i,j}$ is fixed. Define $D'(k)$ to be a domino labeled by integer k with $supp\,D'(k, T)$ equal to*

(1) $\{S_{i,j}, S_{i-1,j}\}$ *if $k < label\,S_{i-1,j+1}$;*
(2) $\{S_{i,j}, S_{i,j+1}\}$ *if $k > label\,S_{i-1,j+1}$.*

Alternately, suppose that $supp\,D(k, T) = \{S_{i,j}, S_{i-1,j}\}$ or $\{S_{i,j+1}, S_{i,j}\}$ and the square $S_{i,j}$ is fixed. Define $supp\,D'(k, T)$ to be

(1) $S_{i\ j}\ S_{i\ j-1}\}$ *if* $k < label\, S_{i\ \ 1\ j-1}$*;*
(2) $S_{i\ j}\ S_{i\ \ 1\ j}\}$ *if* $k > label\, S_{i\ \ 1\ j-1}$.

Definition 2 7 *The cycle $c = c\ \ k\ \ T)$ through k in a standard domino tableau T is a union of labels of T defined by the condition that $l \in c$ if either*

(1) *$l = k$,*
(2) *$supp\, D\ \ l\ \ T) \cap supp\, D\ \ \ m\ \ T) \neq \emptyset$ for some $m \in c$, or*
(3) *$supp\, D\ \ \ l\ \ T) \cap supp\, D\ \ m\ \ T) \neq \emptyset$ for some $m \in c$.*

We will often identify the labels ontained in the y le with their underlying dominos. For a standard domino tableau T of rank r and a y le c in T, we an define a domino tableau $MT\ \ T\ \ c)$ by repla ing every domino $D\ \ l\ \ T) \in c$ by the orresponding domino $D\ \ \ l\ \ T)$. That the resulting tableau $MT\ \ T\ \ c)$ is standard follows from [3, Proposition 1.5.27]. In general, the shape of $MT\ \ T\ \ c)$ will either equal the shape of T, or one square will be removed (or added to the ore) and one will be added. The y le c is alled *closed* in the former ase and *open* in the latter. For an open y le c of a tableau T, we will write $S_b\ \ c\ \ T)$ and $S_f\ \ c\ \ T)$ for the squares that have been removed (or added to the ore) and added by moving through c; we will often abbreviate this notation to $S_b\ \ c)$ and $S_f\ \ c)$ when no onfusion an result. Let U be a set of y les in T. A ording to [3, Corollary 1.5.29], the order in whi h one moves through a set of y les does not matter, allowing us to unambiguously write $MT\ \ T\ \ U)$ for the tableau obtained by moving through all of the y les in U.

We next define the set of y les that it will be ne essary to move through to des ribe the relationship between G_r and $G_{r\ \ 1}$.

For $T \in SDT_r\ \ n)$, we will write $\delta = \delta\ \ T)$ for the set of squares $S_{i\ j}$ that satisfy $i + j = r + 2$. These are the squares with positive labels adja ent to the ore of T. All are variable in our hoi e of fixed and variable squares. In order to obtain a domino tableau of rank $r + 1$, it will be ne essary to lear all of the squares in δ. Simply moving through $\Delta\ \ T)$, the y les in T that pass through δ, will a hieve this effe t. However, when applied to a pair of tableaux of the same shape, the resulting pair of tableaux may not be of the same shape. To orre t this, we would like to define a minimal set of y les in a pair of domino tableaux that will ensure the resulting pair is of same shape. More pre isely, for a pair $T\ \ S)$, we would like to find sets of y les $\gamma = \ \ \gamma\ \ T)\ \ \gamma\ \ S))$ in both T and S with $\Delta\ \ T) \subset \gamma\ \ T)$ and $\Delta\ \ S) \subset \gamma\ \ S)$ su h that $MT\ \ T\ \ \gamma\ \ T))$ and $MT\ \ S\ \ \gamma\ \ S))$ have the same shape.

The natural notion to onsider is an extended y le ([4, Definition 2.3.1]), whi h we now re onstru t.

Definition 2 8 *Consider $T\ \ S)$ a pair of same-shape domino tableaux, k a label of a domino in T, and c the cycle in T through k. The extended cycle $\tilde{c}$ of k in T relative to S is a union of cycles in T which contains c. Further, the union of two cycles $c_1 \cup c_2$ lies in $\tilde{c}$ if either is contained in $\tilde{c}$ and, for some cycle d in S, $S_b\ \ d)$ coincides with a square of c_1 and $S_f\ \ d)$ coincides with a square of $MT\ \ T\ \ c_2)$. The symmetric notion of an extended cycle in S relative to T is defined in the natural way.*

Let $\tilde{c}$ be an extended y le in T relative to S. A ording to the definition, it is possible to write $\tilde{c} = c_1 \cup \cdots \cup c_m$ and find y les $d_1\ \ldots\ d_m$ in S su h that $S_b\ \ c_i) =$

$S_b\ d_i)$ for all i, $S_f\ d_m) = S_f\ c_1)$, and $S_f\ d_i) = S_f\ c_{i\ \ 1})$ for $1 \leq i < m$. The union $\tilde{d} = d_1 \cup \cdots \cup d_m$ is an extended y le in S relative to T alled the *extended cycle corresponding to* $\tilde{c}$. Symmetri ally, $\tilde{c}$ is the extended y le orresponding to $\tilde{d}$.

It is now possible to define a moving through operation for a pair of same-shape domino tableaux. If we let b be the ordered pair $(\tilde{c}\ \tilde{d})$ of extended y les in $T\ S)$ that orrespond to ea h other, then we define

$$MT\quad T\ S)\ b) = \left(MT\ \ T\ \tilde{c})\ \ MT\left(S\ \tilde{d}\right)\right).$$

As desired, this operation produ es another pair of same-shape domino tableaux ([4, Definition 2.3.1]). If is a family of ordered pairs of extended y les that orrespond to ea h other, then we an unambiguously define $MT\quad T\ S)\quad)$, the operation of moving through all of the pairs simultaneously.

3 A Domino Tableau Correspondence

From the definitions of the previous se tion, it is apparent that moving through all of the extended y les that pass through $\delta\ T)$ and $\delta\ S)$ of a same-shape domino tableau pair $T\ S)$ will not only in rease the rank of the resulting tableau pair by one, but the two tableaux will also be of the same shape. What is perhaps surprising is that this map, whi h merely eva uates δ in the simplest manner that will keep the domino tableau pair of the same shape, des ribes the relationship between the Robinson-S hensted maps G_r and $G_{r\ \ 1}$.

3.1. Main Theorem

We first simplify our notation slightly. Consider a pair of domino tableaux $T\ S)$ of rank r and define $\gamma\ T)$ to be the set of extended y les in T through $\delta\ T)$ relative to S. Similarly, let $\gamma\ S)$ be the set of extended y les in S through $\delta\ S)$ relative to T. If we write γ for the ordered pair of sets of extended y les $\gamma\ T)\ \ \gamma\ S))$, then let

$$MMT\quad T\ S)) = MT\quad T\ S)\ \ \gamma)$$

be the minimal moving through map that lears all of the squares in $\delta\ T)$ and $\delta\ S)$.

Theorem 3 1 *Consider an element* $\in H$. *The Robinson-Schensted maps* G_r *and* $G_{r\ \ 1}$ *for rank r and* $r\ 1$ *domino tableaux are related by*

$$G_{r\ \ 1}\quad) = MMT\ \ G_r\quad)).$$

The proof is a dire t onsequen e of the following lemma; we show that domino insertion ommutes with moving through the set of extended y les whi h pass through the squares adja ent to the ores of a domino tableau pair. We note that the lemma is not true when more general sets of y les are onsidered.

Lemma 3 2 *Consider* $\in H$. *Then*

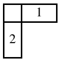
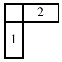
$$MMT\left(\alpha_{k\ \ 1}\left(G_r^k\quad\right)\right) = \alpha_{k\ \ 1}\left(MMT\left(G_r^k\quad\right)\right).$$

When $r = 0$, the result is reminis ent of [4, Proposition 2.3.2]. We follow a similar approa h and redefine the s ope of a number of te hni al statements to over the situations possible in the set of rank r standard domino tableaux when $r \geq 0$.

Example 3.3. Consider $= \ 2 \ 1 \ -1) \ \ 1 \ 2 \ 1))$ in H_2. If $T \ S) = G \quad)$, then

$T =$  $S =$ 

The y les in T are $c_1 = \{1\}$ and $c_2 = \{2\}$ and the y les in S are $d_1 = \{1\}$ and $d_2 = \{2\}$. Note that $\Delta \ T) = c_1$ and $\Delta \ S) = d_1$. However, $\gamma \ T) = c_1 \cup c_2$ and $\gamma \ S) = d_1 \cup d_2$, so that $MMT \ G \quad))$ is the pair of tableaux

$T =$  $S =$ 

As stated in the theorem, $MMT \ G \quad)) \equiv \ T \quad S \ )$ equals $G_1 \quad)$.

3.2. Te hni al Lemmas

It is possible to des ribe the open y les in $T \ k+1)$ in terms of the open y les in $T \ k)$. Garfinkle's [4, Theorem 2.2.3] des ribes this relationship when $r = 0$. With only minor hanges, this result an be stated for arbitrary rank tableaux. We will write $OC \ T)$ for the set of open y les in T. To be pre ise, let's re all a definition:

Definition 3 4 *If $T_1 \ T_2 \in SDT_r \ n)$, and U_1 and U_2 are sets of open cycles in T_1 and T_2, then a map μ: $U_1 \to U_2$ is a* y le stru ture preserving bije tion *if for every $c \in U_1$, $S_b \ \mu \ c)) = S_b \ c)$ and $S_f \ \mu \ c)) = S_f \ c)$.*

In general, there is no y le stru ture preserving bije tion between the open y les in $T \ k+1)$ and those in $T \ k)$. However, their relationship is only slightly more subtle.

Definition 3 5 *A cycle $c \in OC \ T \ k+1))$ corresponds to a cycle $c \quad \in OC \ T \ k))$ if either $S_b \ c \ ) = S_b \ c)$ or $S_f \ c) = S_f \ c \ )$.*

We will des ribe the open y le orresponden es and y le stru ture preserving bije tions between $T \ k+1)$ and $T \ k)$. The first lemma is a generalized version of [4, Theorem 2.2.3], extended by the ase here labeled as Case 2 $a) \ ii)$. Before stating it, let's introdu e some notation that will be used throughout this se tion. We will write T for $T \ k+1)$, T for $T \ k)$, and $\overline{U}$ for the tableau U with its highest-labeled domino removed. Let P be the squares in T that are not in T and $\overline{P}$ be the squares in $\overline{T}$ that are not in $\overline{T}$. If e is the highest label in T, let P_e be the squares of $D \ e \ T \ )$, and P_e be the squares of $D \ e \ T)$.

Lemma 3 6 *Consider $T \ k)$ and $T \ k+1) \in SDT_r \ n)$. Suppose that P is horizontal and consists of the squares $\{S_{i \ j} \ S_{i \ j \ 1}\}$. When P is vertical instead, the obvious transpositions of the statements below are true. The relationship of the open cycle structure of $T \ k)$ to the open cycle structure of $T \ k+1)$ is described by the following cases*:

(1) *Suppose that $S_{i \ j \ 1}$ is variable.*

(a) *First assume that* $j > 1$ *and* $S_{i\ \ 1\ j-1}$ *is not contained in the diagram underlying* $T\ \ k)$. *Let* $c\ $ *be the open cycle in* $T\ \ k)$ *with* $S_b\ \ c\ \ ) = S_{i\ j-1}$. *Then there is an open cycle* c *in* $T\ \ k+1)$ *with* $S_f\ \ c) = S_f\ \ c\ \ )$ *and* $S_b\ \ c) = S_{i\ j\ \ 1}$. *Furthermore, there is a cycle structure preserving bijection between the remaining open cycles of* $T\ \ k)$ *and* $T\ \ k+1)$.

(b) *Otherwise, either* $j = 1$ *or* $S_{i\ \ 1\ j-1}$ *is contained in the diagram underlying* $T\ \ k)$. *Then there are two possibilities. Either*

(i) *there is an open cycle* c *in* $T\ \ k+1)$ *with* $S_b\ \ c) = S_{i\ j\ \ 1}$ *and* $S_f\ \ c) = S_{i\ \ 1\ j}$ *and a cycle structure preserving bijection between* $OC\ \ T\ \ k))$ *and* $OC\ \ T\ \ k+1)) \setminus \{c\}$, *or*

(ii) *there is an open cycle* $c\ $ *in* $T\ \ k)$ *and cycles* $c_1\ \ c_2$ *in* $T\ \ k+1)$ *such that* $S_f\ \ c_1) = S_f\ \ c\ \ )$, $S_b\ \ c_1) = S_{i\ j\ \ 1}$, $S_f\ \ c_2) = S_{i\ \ 1\ j}$, *and* $S_b\ \ c_2) = S_b\ \ c\ \ )$. *In this case, there is a cycle structure preserving bijection between* $OC\ \ T\ \ k)) \setminus \{c\ \ \}$ *and* $OC\ \ T\ \ k+1)) \setminus \{c_1\ \ c_2\}$.

(2) *Suppose that* $S_{i\ j\ \ 1}$ *is fixed.*

(a) *First assume that either* $i = 1$ *or* $S_{i-1\ j\ \ 2}$ *is contained in the diagram underlying* $T\ \ k+1)$. *There are two possibilities. Either*

(i) *there is an open cycle* $c\ $ *in* $T\ \ k)$ *with* $S_f\ \ c\ \ ) = S_{i\ j}$ *and an open cycle* c *in* $T\ \ k+1)$ *with* $S_f\ \ c) = S_{i\ j\ \ 2}$ *and* $S_b\ \ c) = S_b\ \ c\ \ )$; *in this case there is a cycle structure preserving bijection between the remaining open cycles of* $T\ \ k)$ *and* $T\ \ k+1)$, *or*

(ii) $S_{i\ j} \in \delta\ \ T\ \ k))$, *there is a cycle* c *in* $T\ \ k+1)$ *with* $S_b\ \ c) = S_{i\ j}$ *and* $S_f\ \ c) = S_{i\ j\ \ 2}$, *and a cycle structure preserving bijection between* $OC\ \ T\ \ k))$ *and* $OC\ \ T\ \ k+1)) \setminus \{c\}$.

(b) *Otherwise, both* $i > 1$ *and* $S_{i-1\ j\ \ 2}$ *is not contained in the diagram underlying* $T\ \ k+1)$. *Then there is an integer* $u >\ \ _{k\ \ 1}$ *such that the domino with label* u *forms a cycle* $c\ $ *in* $T\ \ k)$ *with* $S_f\ \ c\ \ ) = S_{i\ j}$ *and* $S_b\ \ c\ \ ) = S_{i-1\ j\ \ 1}$. *In this case, there is a cycle structure preserving bijection between* $OC\ \ T\ \ k)) \setminus \{c\ \ \}$ *and* $OC\ \ T\ \ k+1))$.

To verify the above, it is ne essary to understand how the y le stru ture of a domino tableau U is related to the y le stru ture of $\overline{U}$. When $r = 0$, this is des ribed in [4, Proposition 2.2.4]. Again for ompleteness, we state our version for arbitrary rank tableaux in full, whi h differs in the additional ase 2(a)(ii). The proof of this lemma follows from an easy, but tedious, inspe tion.

Lemma 3 7 *Suppose that* $T \in SDT_r\ \ n)$, *e is the label of its highest domino D, and* $\overline{T}$ *is the domino tableau with D removed. Suppose D occupies the squares* $\{S_{i\ j}\ \ S_{i\ j\ \ 1}\}$ *in* T. *Again, the obvious transpositions of the statements below are true for vertical D.*

(1) *Suppose that* $S_{i\ j\ \ 1}$ *is variable.*

(a) *First assume that* $j > 1$ *and* $S_{i\ \ 1\ j-1}$ *is not contained in the diagram underlying* $\overline{T}$. *Let* $\overline{c}$ *be the open cycle in* $\overline{T}$ *with* $S_b\ \ \overline{c}) = S_{i\ j-1}$. *Then there is an open cycle* c *in* T *with* $S_f\ \ c) = S_f\ \ \overline{c})$ *and* $S_b\ \ c) = S_{i\ j\ \ 1}$. *Furthermore,* *there is a cycle structure preserving bijection between the remaining open cycles of $\overline{T}$ and T.*

(b) *Otherwise, either $j = 1$ or $S_{i\ \ 1\ j-1}$ is contained in the diagram underlying $\overline{T}$. Then $c = \{e\}$ is an open cycle in T and there is a cycle structure preserving bijection between $OC\ \overline{T})$ and $OC\ \ T) \setminus \{c\}$.*

(2) *Suppose that $S_{i\ j\ \ 1}$ is fixed.*

(a) *First assume that either $i = 1$ or $S_{i-1\ j\ \ 2}$ is contained in the diagram underlying T.Then there are two possibilities. Either*

(i) *there exists an open cycle $\overline{c}$ in $\overline{T}$ with $S_f\ \ \overline{c}) = S_{i\ j}$, and $c = \overline{c} \cup \{e\}$ is an open cycle in T; in this case there is a cycle structure preserving bijection between $OC\ \overline{T}) \setminus \{\overline{c}\}$ and $OC\ \ T) \setminus \{c\}$, or*

(ii) *$S_{i\ j} \in \delta\ \ T)$, there is a cycle c in T with $S_b\ \ c) = S_{i\ j}$ and $S_f\ \ c) = S_{i\ j\ \ 2}$, and a cycle structure preserving bijection between $OC\ \overline{T})$ and $OC\ \ T) \setminus \{c\}$.*

(b) *Otherwise, both $i > 1$ and $S_{i-1\ j\ \ 2}$ is not contained in the diagram underlying T. Then either*

(i) *there is a cycle $\overline{c}$ in $\overline{T}$ with $S_b\ \ \overline{c}) = S_{i-1\ j\ \ 1}$ and $S_f\ \ \overline{c}) = S_{i\ j}$, $c = \overline{c} \cup \{e\}$ is a closed cycle in T, and $OC\ \ T) = OC\ \overline{T}) \setminus \{\overline{c}\}$, or*

(ii) *there are two open cycles $\overline{c}_1\ \ \overline{c}_2$ in $\overline{T}$ such that $S_b\ \ \overline{c}_1) = S_{i-1\ j\ \ 1}$, $S_f\ \ \overline{c}_2) = S_{i\ j}$, the set $c = c_1 \cup c_2 \cup \{e\}$ is an open cycle in T and $OC\ \ T) \setminus \{c\} = OC\ \overline{T}) \setminus \{\overline{c}_1\ \ \overline{c}_2\}$.*

Armed with this observation, we an now prove Lemma 3.6.

Proof of Lemma 3.6. Lemma 3.7 des ribes the relationships between the y le stru tures of $\overline{T}\ \ k)$ and $T\ \ k)$, as well as $\overline{T}\ \ k+1)$ and $T\ \ k+1)$. If we use indu tion on the size of the tableaux, we an relate the y le stru tures of $\overline{T}\ \ k)$ and $\overline{T}\ \ k+1)$. Together, this allows us to des ribe the desired relationship between the y le stru tures of $T\ \ k)$ and $T\ \ k+1)$.

If a pair of squares in a domino tableau satisfies the hypotheses of a ase of Lemma 3.6 or Lemma 3.7, we will say that the pair lies in the situation labeled by that ase. The proof of the lemma divides into different ases des ribed by the situations of $\overline{P}$ and P_e and their relative positions. When $r = 0$, this is exhaustively arried out in the proof of [4, Theorem 2.2.3], whi h in ludes a des ription of the possibilities for $\overline{P}$ and P_e. We will use the same labels for these possibilities. To verify the lemma for arbitrary rank tableaux, we must he k that the on lusions still hold in the ases originally onsidered, as well as examine the new ases that arise for larger rank tableaux. The former follows from a lengthy inspe tion of the proof of [4, Theorem 2.2.3]. We examine the new ases.

We have to onsider situations where either $P\ \overline{P}\ \ P_e$ or P_e is in situation 2(a)(ii). Most of the ases are essentially trivial. We treat two of them in detail; the rest follow along similar lines. The ases are labeled to mimi similar ases onsidered in [4, Theorem 2.2.3].

Case K . Here $\overline{P} = P_e$ is in situation 2(a)(ii). We have a y le stru ture preserving bije tion between $OC\ \overline{T})$ and $OC\ \ T\ \ )$. Note that $P_e = P$, and they both must be in situation 2(a)(ii) or 1(b). In both ases, the desired relationship between $OC\ T)$ and $OC\ T\ )$ exists between $OC\ T)$ and $OC\ \overline{T})$ by Lemma 3.7. Sin e we already have a y le stru ture preserving bije tion between $OC\ \overline{T})$ and $OC\ T\ )$, we are done.

Case L . Here $\overline{P}$ is in situation 2(a)(ii) and $P_{\underline{e}} = \ \ S_{i\ j}\ \ S_{i\ \ 1\ j}\}$, so that P_{e} is in situation 2(a)(ii) as well. If D is the domino in $\overline{T}$ in position $\overline{P}$ with label f, then we have a y le stru ture preserving bije tion between $OC\ \overline{T}) \setminus \{f\}$ and $OC\left(\overline{T}\ \right) \setminus \{e\}$. Note that $P = \ \ S_{i\ j\ \ 1}\ \ S_{i\ \ 1\ j\ \ 1}\}$ is in situation 1(b) of Lemma 3.6 and $P_e = \ \ S_{i\ \ 1\ j}\ \ S_{i\ \ 1\ j\ \ 1}\}$ is in situation 1(b) of Lemma 3.7. Be ause of the latter, we know that there is a y le stru ture preserving bije tion between $OC\ \overline{T})$ and $OC\ T) \setminus \{e\}$. From this, we an onstru t a y le stru ture preserving bije tion between $OC\ T\ ) \setminus \{c\ \}$ and $OC\ T) \setminus \{c_1\ \ c_2\}$ where $c\ \ = \{e\}$ is in T , $c_1 = \{e\}$ is in T, and $c_2 = \{f\}$, as required in the on lusion of 1(b)(ii). ∎

Lemma 3 8 *The set* $\gamma\ T\ \ k+1))$ *is the union of the open cycles that correspond to cycles in* $\gamma\ T\ \ k))$ *and the cycles through* $\delta\ T\ \ k+1))$.

Proof. Let's write $\tilde{\gamma}\ T\ )$ for the set of open y les in T that orrespond to open y les in $\gamma\ T\ )$. We may take $k > 1$, otherwise this is trivial. First assume that $\ \ _{k\ \ 1} = e$. Then $P_e = \{S_{1\ s}\ \ S_{1\ s\ \ 1}\}$ and ould be in situations 1(a), 1(b), 2(a)(i), or 2(a)(ii) of Lemma 3.7. In the first and third ases, let c be the y le in T through the square $S_{1\ s-1}$. Then $c = c\ \cup \{e\}$ is the open y le in T orresponding to c , $OC\ T\ ) \setminus \{c\ \} = OC\ T) \setminus \{c\}$, and $c \in \gamma\ T)$ if and only if $c\ \ \in \gamma\ T\ )$. Sin e $\Delta\ T) \subset \tilde{\gamma}\ T\ )$, the result follows. If P_e is in situation 1(b) of Lemma 3.7, then $OC\ T\ ) = OC\ T) \setminus \{e\}$. Sin e $\{k+1\}$ is a y le in S, $\{e\}$ must be an extended y le implying that $\{e\} \notin \gamma\ T)$. Again, $\Delta\ T) \subset \tilde{\gamma}\ T\ )$ and the result follows. If P_e is in situation 2(a)(ii) of Lemma 3.7, then $\{k+1\}$ is a y le in S, $\{e\}$ must be an extended y le and sin e $\{e\} \in \Delta\ T)$, the result follows.

The rest of the proof is by indu tion on the size of the tableau. We will assume that $\gamma\left(\overline{T}\right) = \tilde{\gamma}\left(\overline{T}\ \right) \cup \Delta\ T)$. We treat ases A–C and L from the proof of [4, Theorem 2.2.3] in orporating the additional possibilities that arise in higher rank tableaux. Remaining ases are handled along similar lines.

Case A. Suppose $\overline{P}$ is in situation 1(a) and $\overline{P} = P_{\underline{e}}$. Then $P = P_e$ and they both equal to the set $\{S_{i\ \ 1\ s}\ \ S_{i\ \ 1\ s\ \ 1}\}$ for some s. The squares of P may be in situations 1(a), 1(b), 2(a)(i), or 2(a)(ii) of Lemma 3.6. In the first ase, onsider c as in Lemma 3.6 1(a) . The y le c orresponds to $c = c\ \ e\ \ T)$ sin e $S_f\ \ c) = S_f\ \ c\ \ )$. Examining the position of $D\ \ k+1\ \ S)$, we find that the rest of the extended y le stru ture of T is the same as in T . Hen e if c is any y le in T that orresponds to a y le c in T , then $c \in \gamma\ T)$ iff $c\ \ \in \gamma\ T\ )$. If P lies in situation 2(a)(ii), then $S_{i\ \ 1\ s} \in \delta\ T)$, $c\ \ e\ \ T)$ is a y le through $\delta\ T)$ and lies in $\gamma\ T)$. Similar arguments work for the remaining two ases.

Case . Here $\overline{P}$ is in situation 1(a) and $P_{\underline{e}} = \ \ S_{i\ \ 1\ j-1}\ \ S_{i\ \ 2\ j-1}\}$, implying that $P = \overline{P}$ and $P_e = P_{\underline{e}}$. First onsider the y le $c = c\ \ e\ \ T) = \{e\}$. Note that c orresponds to $c\ \ = c\ \ e\ \ T\ )$ sin e $S_b\ \ c) = S_b\ \ c\ \ )$. Let $\overline{c}\ \ = c\ \ \setminus \{e\} \in OC\left(\overline{T}\ \right)$. Let $f = label\ \ S_{i\ j\ \ 1}\ \ T)$ and note that the squares of P form a domino in S with label $k+1$. Then $S_b\ \ k+1\ \ S) = S_b\ \ f\ \ T)$ and $S_f\ \ k+1\ \ S) = S_f\ \ e\ \ T)$, so that e and f are both in the same extended y le of T relative to S. Hen e $e \in \gamma\ T)$ iff $f \in \gamma\ T)$ iff

$f \in \gamma(\overline{T})$ iff $\overline{c} \in \gamma T$) iff $c \in \gamma T$), as desired. For any open y le c not ontaining e in T, the result follows by indu tion.

Case C. Here $\overline{P}$ is in situation 1(a) and $\overline{P}_e =$ $S_{i\ 1\ j-2}\ S_{i\ 1\ j-1}\}$ is in situation 2(b)(i). Then $P = \overline{P}$ and $P_e = \overline{P}_e$. Let $c = c\ e\ T)$ and by the on lusion of Lemma 3.7 we find $S_f\ c) = S_{i\ 1\ j}$ and $S_b\ c) = S_{i\ j\ 1}$. Note that c orresponds to no open y les in T . Sin e $S_f\ c\ T) = S_f\ k+1\ S)$ and $S_b\ c\ T) = S_b\ k+1\ S)$, the extended y le of e is just c. Hen e $c \in \gamma\ T)$ iff c passes through $\delta\ T)$. For any open y le c not ontaining e in T, the result follows by indu tion.

Case L. Consider $\overline{P}$ in situation 2(a)(ii) and $\overline{P}_e =$ $S_{i\ j}\ S_{i\ 1\ j}\}$, so that $\overline{P}_e$ is in situation 2(a)(ii) as well. We then have $P =$ $S_{i\ j\ 1}\ S_{i\ 1\ j\ 1}\}$ and $P_e =$ $S_{i\ 1\ j}\ S_{i\ 1\ j\ 1}\}$. First onsider the y le $c = c\ e\ T) = \{e\}$. Note that $\overline{P}$ is a domino in T, say, with label f, and $\overline{P}_e$ is a domino in S, say, with label l. Then $S_b\ c\ l\ S)\ S) = S_{i\ j} = S_b\ c\ f\ T)\ T)$ and $S_f\ c\ l\ S)\ S) = S_{i\ 2\ j} = S_f\ c\ e\ T)\ T)$. Hen e $\{e\}$ lies in the extended y le through $c\ f\ T)$. Sin e $c\ f\ T) \in \Delta\ T)$, we must have $\{e\} \in \gamma\ T)$. If we let $c = c\ e\ T$), then $S_f\ c) = S_f\ c$), whi h means that c orresponds to c . In other words, $\{e\}$ lies in $\gamma\ T)$ and $\tilde{\gamma}\ T\ ) \cup \Delta\ T)$. Finally, onsider any open y le c not ontaining e in T. Then c is also an open y le in $\overline{T}$, and the rest follows by indu tion. We omit the argument when $\overline{P}$ and $\overline{P}_e$ are in situation 2(a)(i) instead. ∎

If we abuse notation and write $MMT\ T)$ for $MT\ T\ \gamma\ T))$, then we an state the following version of Garfinkle's [4, Theorem 2.2.9], whi h verifies Lemma 3.2 for left tableaux.

Lemma 3 9 *Consider $\in H$ and write $T\ m)$ for the left tableau of G_r^m). Then*

$$\alpha_{k\ 1}\ MMT\ T\ k))) = MMT\ T\ k+1)).$$

Proof. Using Lemma 3.8, we have to show that

$$\alpha_{k\ 1}\ MMT\ T\ k))) = MT\ T\ k+1)\ \tilde{\gamma}\ T\ k)) \cup \Delta\ T\ k+1))).$$

whi h is an adaptation of [4, Theorem 2.2.9]. However, we annot adapt the proof of [4, Theorem 2.2.9] verbatim, as it uses indu tion on the number of open y les in the extended y le defining the moving through operation. In our situation, moving through a set of y les smaller than $\gamma\ T\ k))$ may leave us with a domino tableau on whi h α is undefined. Nevertheless, sin e only one pair P of squares is added to $T\ k)$ with domino insertion, and moving through open y les an be done independently, we an essentially follow the original proof and examine the relationship of P with the y les in $\gamma\ T\ k))$ individually.

The ase when $_{k\ 1} = e$ is simple, and we assume that $_{k\ 1} \neq e$. We pro eed by indu tion on n, noting that the ase $n = 1$ orresponds to $_{k\ 1} = e$. Following the original proof of [4, Theorem 2.2.9], we show that ea h domino in $\alpha_{k\ 1}\ MMT\ T\ ))$ lies in the same position in $MMT\ T)$. For dominos with labels less than e, this will follow by indu tion; for the domino with label e, it will follow by inspe tion of ea h of the ases below.

Let $\overline{P}_1$ be the squares in $\alpha_{k\ 1}(\overline{T}\ )$ that are not in $\overline{T}$, $\overline{P}_2$ be the squares in $\alpha_{k\ 1}(MMT\ (\overline{T}\ ))$ that are not in $MMT\ (\overline{T}\ )$. Write T_1 for T, $\overline{T}_1$ for $\overline{T}$, T_2 for

$MMT\ T)$, T_2 for $MMT\ T\ )$, and T_3 for $\alpha_{k\ \ 1}\ MMT\ T\ ))$. Hen e we are verifying that $T_2 = T_3$.

Case A. Assume that $\overline{P}_1 = P_e = \ \ S_{i\ j}\ S_{i\ j\ \ 1}\}$, and $\overline{P}_1$ is in situation 1(a). Then $P_e = P = \{S_{i\ \ 1\ s}\ S_{i\ \ 1\ s\ \ 1}\}$ for some s. Suppose first that $S_{i\ \ 1\ s}$ is variable and that no y le $c\ \in \gamma\ T_1)$ has $S_f\ c\ ) = S_{i\ \ 1\ s}$. If $S_{i\ \ 1\ s} \in \delta$, then $\{e\} \in \gamma\ T_1)$ and $P_e\ T_2) = \{S_{i\ \ 1\ s\ \ 1}\ S_{i\ \ 1\ s\ \ 2}\} = P_e\ T_3)$. When $S_{i\ \ 1\ s} \notin \delta$, we have $P_e\ T_2) = P_e\ T_1) = P_e\ T_3)$. Suppose next that there is a y le $c\ \in \gamma\ T_1)$ with $S_f\ c\ ) = S_{i\ \ 1\ s}$, then e lies in a y le in $\gamma\ T_1)$ and $P_e\ T_2) = \{S_{i\ \ 1\ s\ \ 1}\ S_{i\ \ 1\ s\ \ 2}\} = P_e\ T_3)$. If $S_{i\ \ 1\ s}$ is fixed, then $P_e\ T_2) = \{S_{i\ \ 1\ s-1}\ S_{i\ \ 1\ s}\} = P_e\ T_3)$ if $S_{i\ \ 1\ s-1}$ lies in some y le of $\gamma\ T_1)$, and $P_e\ T_2) = \{S_{i\ \ 1\ s}\ S_{i\ \ 1\ s\ \ 1}\} = P_e\ T_3)$ if it does not.

Case K . Here $\overline{P}_1 = P_e$ are in situation 2(a)(ii). Then $P_e = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 1\ j\ \ 1}\}$. Note that $c = \{e\}$ is a y le in T_1 and $d = \{k+1\}$ is a y le in $S\ k+1)$ with $S_f\ c\ T_1) = S_{i\ \ 2\ j} = S_f\ d\ S\ k+1))$ and $S_b\ c\ T_1) = S_{i\ \ 1\ j\ \ 1} = S_b\ d\ S\ k+1))$. Hen e $c = \{e\}$ is an extended y le not ontained in $\gamma\ T_1)$ and onsequently $D\ e\ T_2) = D\ e\ T_1) = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 1\ j\ \ 1}\}$. Now note that $\overline{P}_2 = P\ e\ T_2)$ and by a similar argument, we obtain $D\ e\ T_3) = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 1\ j\ \ 1}\}$, as desired.

Case L . Here $\overline{P}_1$ is in situation 2(a)(ii) and $P_e = \ \ S_{i\ j}\ S_{i\ \ 1\ j}\}$, so it is in situation in 2(a)(ii) as well. Then $P_e = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 1\ j\ \ 1}\}$ and $P = \ \ S_{i\ j\ \ 1}\ S_{i\ \ 1\ j\ \ 1}\}$. Note that $c\ = \{e\}$ is a y le in T_1 with $S_f\ c) = S_{i\ \ 2\ j}$ and that the squares P_e form a domino in $S\ k)$, say with label f. Let $d = c\ f\ S\ k+1))$ and note that $d \in \gamma\ S\ k+1))$. Furthermore, $S_f\ d) = S_{i\ \ 2\ j}$ implying that $c\ e\ T_1) \in \gamma\ T_1)$, and $D\ e\ T_2) = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 2\ j}\}$. Now observe that $D\ e\ T_2) = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 2\ j}\}$ and $\overline{P}_2 = \ \ S_{i\ j\ \ 1}\ S_{i\ j\ \ 2}\}$. This means $D\ e\ T_3) = \ \ S_{i\ \ 1\ j}\ S_{i\ \ 2\ j}\}$ and $D\ e\ T_2) = D\ e\ T_3)$, as desired. ∎

3.3. Domino Insertion and Moving Through

Armed with the te hni al results of the previous se tion, we an now address the main lemma of the paper. We prove Lemma 3.2, verifying that domino insertion on tableau pairs ommutes with the minimal moving through map. Write $T_1\ S_1) = T\ k)\ S\ k))$, $T_1\ S_1) = T\ k+1)\ S\ k+1))$, $T_2\ S_2) = MMT\ T_1\ S_1)$, $T_2\ S_2) = \alpha_{k\ \ 1}\ T_2\ S_2))$, and $T_3\ S_3) = MMT\ T_1\ S_1)$. Expressed in this notation, we would like to prove that $T_2\ S_2) = T_3\ S_3)$. Lemma 3.9 says that $T_2 = T_3$, and it remains to show that $S_2 = S_3$.

Proof of Lemma 3.2. Write P_1 for the squares in T_1 that are not in T_1 and P_2 for the squares in T_2 that are not in T_2. Note that P_1 forms a domino in S_1 and P_2 forms a domino in S_2, both with label $k+1$. Assume that $P_1 = \ \ S_{i\ j}\ S_{i\ j\ \ 1}\}$. We will examine the ases when P_1 is in situations 1(a), 1(b)(ii), and 2(b). The others follow along similar lines.

So suppose that P_1 is in situation 1(a) of Lemma 3.6 and c is the open y le with $S_b\ c) = S_{i\ j\ \ 1}$ des ribed therein. Then $D\ k+1\ S_1)$ is in situation 1(a) of Lemma 3.7 and there is an open y le $\overline{d}$ in S_1 with $S_b\left(\overline{d}\right) = S_{i\ j-1}$ su h that $d = \overline{d} \cup \{k+1\}$ is an open y le in S_1. Note that $c \in \gamma\ T_1)$ iff $d \in \gamma\ S_1)$. If $c \in \gamma\ T_1)$, then by Lemma 3.9, $P_2 = \ \ S_{i\ j-1}\ S_{i\ j}\}$, whi h implies that $D\ k+1\ S_3) = D\ k+1\ S_2)$. Sin e the rest of the y le stru ture in T_1 remains the same as in T_1, the rest of the y les in $\gamma\ S_1)$ are the same as in $\gamma\ S_1)$ and onsequently, $S_2 = S_3$. If $c \notin \gamma\ T_1)$, the result is lear.

If P_1 is in situation 1(b)(ii) of Lemma 3.6, then $D\ k+1\ S_1)$ is in situation 1(b) of Lemma 3.7. Let $c\ \ c_1$ and c_2 be as des ribed in Lemma 3.6 1(b)(ii) and let $d = c\ k+1\ S_1)$. Sin e $S_b\ c_1) = S_b\ d)$ and $S_f\ c_2) = S_f\ d)$, c_1 and c_2 lie in the same extended y le relative to d, so $c_1\ \ c_2 \in \gamma\ T_1)$ iff $d \in \gamma\ S_1)$. If $c_1\ \ c_2 \in \gamma\ T_1)$ then by Lemma 3.9, $P_2 = \ \ S_{i\ j}\ \ S_{i\ \ 1\ j}\}$. Sin e $d \in \gamma\ S_1)$, this means $D\ k+1\ S_2) = D\ k+1\ S_3)$. Sin e the rest of the y le stru ture in T_1 remains the same as in T_1, the rest of the y les in $\gamma\ S_1)$ are the same as in $\gamma\ \ S_1)$ and we an on lude that $S_2 = S_3$. If $c_1\ \ c_2 \notin \gamma\ T_1)$ the on lusion is the same.

The most troublesome ase is when P_1 is in situation 2(b) of Lemma 3.6. Then $D\ k+1\ S_1)$ is either in situation 2(b)(i) or 2(b)(ii) of Lemma 3.7. So suppose first that $D\ k+1\ S_1)$ is in situation 2(b)(i). Let $\overline{d}$ be the y le in S_1 with $S_f\left(\overline{d}\right) = S_{i\ j}$ and $S_b\left(\overline{d}\right) = S_{i-1\ j\ \ 1}$. Then $d = \overline{d} \cup \{k+1\}$ is a losed y le in S_1 and onsequently does not lie in $\gamma\ S_1)$. Let c be the y le in T_1 with $S_f\ c\ ) = S_{i\ j}$ and $S_b\ c\ ) = S_{i-1\ j\ \ 1}$. Then c is the entire extended y le in T_1 that orresponds to $\overline{d}$ in S_1; in parti ular, this means that $c\ \ \notin \gamma\ \ T_1)$ and $\overline{d} \notin \gamma\ S_1)$. Consequently, $S_2 = S_3$.

Finally, onsider $D\ k+1\ S_1)$ in situation 2(b)(ii). Let d_1 and d_2 be the y les in S_1 with $S_b\ d_1) = S_{i-1\ j\ \ 1}$ and $S_f\ d_2) = S_{i\ j}$. Then $d_1 \cup d_2 \cup \{k+1\}$ is an open y le in S_1. Let c be as in Lemma 3.6 2(b) and note that $c\ \ \in \gamma\ T_1)$ iff $d_1\ \ d_2 \in \gamma\ S_1)$. If $c\ \ \in \gamma\ T_1)$, then $P_2 = \ \ S_{i-1\ j\ \ 1}\ \ S_{i\ j\ \ 1}\}$ by Lemma 3.9 and we again on lude that $S_2 = S_3$. If $c\ \ \notin \gamma\ \ T_1)$, the result is lear. ∎

3.4. Restri tion to Involutions

We follow van Leeuwen in the next definition, whi h onstru ts a map between domino tableaux of unequal rank [7].

Definition 3 10 *Let r and r be non-negative integers and suppose that* $T \in SDT_r\ n)$. *We define the map* $t_{r\ r}: SDT_r\ n) \to SDT_{r}\ \ n)$ *by setting* $t_{r\ r}\ \ T) = T$ *whenever* $G_r^{-1}\ T\ T) = G_r^{-1}\ \ T\ \ \ T\ )$.

Armed with Theorem 3.1, the maps $t_{r\ r\ \ 1}$ take a parti ularly simple form. The domino tableau $t_{r\ r\ \ 1}\ T)$ in $SDT_{r\ \ 1}\ n)$ is simply the image of T after all the y les in $\Delta\ T)$ have been moved through.

Corollary 3 11 $t_{r\ r\ \ 1}\ T) = MT\ \ T\ \ \Delta\ T))$

Proof. If is an involution and $G_r\ \ \ \ ) = \ \ T\ \ S)$, then S must equal T. The definition of extended y les implies that every extended y le in T relative to S onsists of a unique y le. In our setting, this implies $\gamma = \ \ \Delta\ T)\ \ \Delta\ T))$. Using Theorem 3.1 and the definition of moving through extended y les, we now have

$$t_{r\ r\ \ 1}\ T)\ \ t_{r\ r\ \ 1}\ T)) = MT\ \ \ T\ \ T)\ \ \gamma) = \ \ MT\ \ T\ \ \Delta\ T))\ \ MT\ \ T\ \ \Delta\ T)))$$

as desired. ∎

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