1-1-2014

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Ann. Comb. 18 (2014) 515–531
DOI 10.1007/s00026-014-0237-6
Published online July 2, 2014

Annals of Combinatorics

Sign Under the omino Robinson-Schensted Maps

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Received May 23 2013

Mathematics Subject Classi cation: 05E10

Abstract We generalize a formula obtained independently by Reifegerste and Sjöstrand for the sign of a permutation under the classical Robinson-Schensted map to a family of domino Robinson-Schensted algorithms.

Keywords: Robinson-Schensted domino tableaux

1 Introduction

In their work verifying Stanley's sign imbalan e formula, Reifegerste and Sjöstrand independently obtained the following remarkable formula for reading the sign of a permutation from its image under the lassi al Robinson-S hensted map. It is based on two tableaux statisti s *e* and *sign*:

Theorem 1 1 ([18, 20]) *Consider* $w \in S_n$ *and let* $RS\ w) = \ P, Q)$ *be its image under the classical Robinson-Schensted map Then*

$$sign\ w) = \ -1)\ \cdot sign\ P) \cdot sign\ Q)$$

In [6], building on the work of Barbas h and Vogan in [1], Garfinkle introdu ed a generalization of the Robinson-S hensted algorithm relating elements of the other lassi al Weyl groups and same-shape pairs of domino tableaux. This algorithm was further extended to a one-parameter family of maps G_r by van Leeuwen using domino tableaux with non-empty ore. For large values of r, van Leeuwen's maps re over yet another generalization of the Robinson-S hensted maps in this setting introdu ed by Stanley [21, §6]. The aim of this paper is to address the natural question whether it is again possible to re over the parity of the length fun tion, whi h we all its sign, from the tableaux images of these maps.

In fa t, this has already been done for Stanley's map in [15] in the more general ontext of omplex refle tion groups. For the family of maps G_r, our main result relies on three domino tableaux statisti s *d*, *spin*, and *sign* defined in Se tion 2.3.

Let H_n be the Weyl group of type B_n.

We would like to thank Skidmore College for its hospitality during the writing of this manuscript

Theorem 1 2 *Consider* $w \in H_n$, *and let* $G_r\ w) =\ P, Q)$ *be its image among the same-shape standard domino tableaux of rank r Then*

$$sign\ w) = \ -1)^d \cdot \ -1)^{spin\ P)+spin\ Q)} \cdot sign\ P) \cdot sign\ Q)$$

The proof involves three steps. We first verify the equation for large r by translating between Stanley's map and G_r and appealing to the formula established in [15]. It is then possible to extend the result to involutions in H_n for arbitrary r by using a relation between onse utive maps G_r des ribed in [17], and finally to all signed permutations by tra king the behavior of the established sign formula under Taşkın's pla ti relations introdu ed in [23].

The domino tableaux Robinson-S hensted algorithms appear in the work lassifying Kazhdan-Lusztig ells in unequal parameter Iwahori-He ke algebras of type B, see [3]. At least onje turally, for ertain values of the parameter, one-sided ells orrespond to pla ti and opla ti lasses for the maps G_r. As orollary to the above sign formula, we note that the Möbius fun tion for the Bruhat order, ubiquitous in Kazhdan-Lusztig theory, is well-behaved with respe t to these ells. First des ribed by Verma in [24], the Möbius fun tion μ takes the form

$$\mu\ v, w) = \ -1)^{\ v)+\ w)},$$

where ℓ is the length fun tion on the Weyl group. In type B, the values of μ an be readily read off from the tableaux of v and w arising from the maps G_r. Further, we have the following result.

Corollary 1 3 *Consider* $x, x', y, y' \in H_n$ *and fix a map* G_r *Suppose that the left tableaux of the pair x and y as well as* x' *and* y' *are the same, and the right tableaux of the pairs* x, x' *and* y, y' *similarly agree Then, we have*

$$\mu\ x, y) = \mu\ x', y')$$

2 Preliminaries

We define the notions of standard and domino tableaux, des ribe a family of Robinson-S hensted maps, and detail several tableaux statisti s whi h will be ne essary for our work.

2.1. Partitions and Tableaux

Our first obje tive is to define the notions of standard Young, bi-, and domino tableaux. A non-in reasing sequen e of positive integers $\ =\ \ ,\ _2, \ \ , \ _t)$ is alled a *partition* of the integer $n = \ _i\ _i$. We will write $\ \vdash n$ and $|\ | = n$. Partition notation an often be abbreviated by using exponents to denote multipli ity; for instan e, $4, 4, 3, 3, 3, 1)$ an be written as $(4^2, 3^3, 1$. We will identify a partition with its Young diagram [ ], or a left-justified array of squares ontaining $\ _i$ squares in row i.

If the rightmost square s of a row in a Young diagram [ ] an be removed while leaving another Young diagram $[\ ]_s$, then it will be alled a *corner of* [ ]. Beginning with [ ], one an start su essively removing orners until this pro ess inevitably

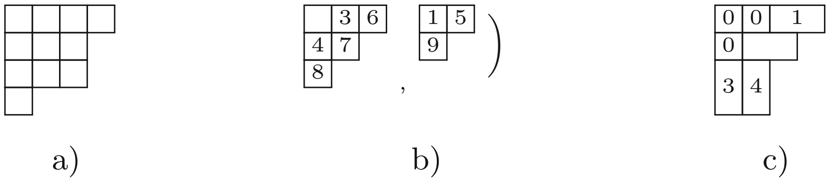


Figure 1: (a) The Young diagram of the partition $(4, 3^2, 1$, (b) a standard bitableau of shape 3, 2, 1), 2, 1)), and ( ) a standard domino tableau of rank 2.

terminates after n steps with the empty partition. If the square removed at the ith step of this pro edure is labeled with the number $n-i+1$, then the result is alled a *standard Young tableau*. For a tableau T, we will write sh $T)$ for the underlying partition, $|T|$ for $|sh$ $T)|$, SYT) for the set of all standard Young tableaux of shape , and SYT $n)$ for the set of standard Young tableaux with n boxes.

We will all an ordered pair of partitions $, \mu)$ a *bipartition* of n if $|\ \ |+|\mu|=n$. A square of $[\ \ ], [\mu])$ is a orner if it is a orner of either $[\ \ ]$ or $[\mu]$. Su essive removal of orners starting with $[\ \ ], [\mu])$ terminates after n steps, and if the square removed at the ith step is labeled with $n-i+1$, then the result will be alled a *standard bitableau*. The shape of a bitableau is the pair of its underlying partitions; we will write SBT $, \mu)$ for the set of all standard bitableaux of shape $, \mu)$, and SBT $n)$ for the set of standard bitableaux with n boxes.

Two squares of a Young diagram are *adjacent* if they share a ommon side. Adja ent squares s, t in $[\ \ ]$ form a *domino corner* if s is a orner for $[\ \ ]$ and t is a orner for $[\ \ ]_s$. Beginning with $[\ \ ]$, one an start su essively removing domino orners and ontinue until this is no longer possible, say after n steps. The resulting shape is a stair ase partition $\delta_r =$ $r, r-1, r-2,$ $, 1)$ for some $r \geq 0$ and is independent of the order of removal of domino orners, see [14]. The partition δ_r is known as the 2-*core* of . If the squares of $[\ \ ]$ orresponding to the 2- ore are labeled with 0 and the domino removed at the ith step is labeled with $n-i+1$, then the result is a *standard domino tableau* of rank r. The set of all standard domino tableaux of shape with 2- ore δ_r will be denoted by SDT_r) while SDT_r $n)$ will denote the set of all standard domino tableaux onsisting of the 2- ore δ_r and n dominos. We will all the set of squares in a domino tableau T labeled with 0 the *core* of T.

2.2. Robinson-S hensted Maps

Consider a permutation $w \in S_n$. We will write it in one-line notation as w $w_2 \cdots w_n$ with ea h entry $w_i \in$ $_n$. The lassi al Robinson-S hensted map establishes a bije - tion between permutations in S_n and same-shape pairs of standard Young tableaux in SYT $n) \times SYT$ $n)$ via an insertion and a re ording algorithm. We assume the reader is familiar with the basi s; details an be found in [5] or [22]. We will write RS $w) =$ P $w), Q$ $w))$ for the image of a permutation under this map.

A *signed permutation* is a permutation together with a hoi e of sign for ea h of its entries. We will again use one-line notation, using a bar over a letter to denote the hoi e of a negative sign. The set of signed permutations on n letters forms a group under omposition and multipli ation of signs; it is isomorphi to the hypero tahedral group $H_n = \mathbb{Z}_2 \wr S_n$ and is generated by

$$s_i = 1\,2 \cdots i+1 \quad i \cdots n \qquad \text{and} \qquad t = \overline{1}\,2 \cdots n,$$

for $1 \leq i < n$. Let ℓ be the length fun tion on H_n defined in terms of this generating set and write $sign\ \ w) = \ \ -1) \ \ ^{w)}$

We are interested in two generalizations of the Robinson-S hensted map in this setting. The first establishes a map

$$G \ \ : H_n \longrightarrow SBT \ \ n) \times SBT \ \ n),$$

whi h is a bije tion onto same-shape pairs of bitableaux. Given a signed permutation, the insertion bitableau for $w \in H_n$ is onstru ted by a variant of the lassi al insertion algorithm. Positive letters are inserted into the first tableau and negative into the se - ond following their order of appearan e in the one-line notation for w. The re ording bitableau tra ks the shape of the insertion bitableau at ea h step. See [21].

Example 2 1 Consider the signed permutation $w = \ \ \overline{4}\overline{3}21) \in H_4$ The sequen e of bitableaux onstru ted by su essive insertion of the letters of w is:

$$\ \ ) \qquad \boxed{4}\,) \qquad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\Bigg) \qquad \boxed{2}\ \ \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\Bigg) \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\ \ \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\Bigg).$$

Keeping tra k of the shapes appearing in this sequen e, we an onstru t another bitableau of the same shape:

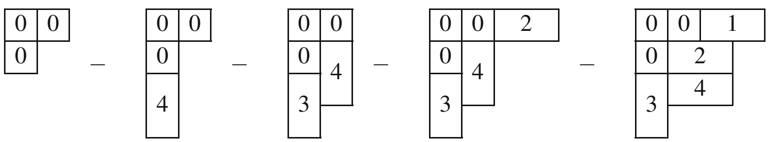
$$G \ \ \ w) = \left(\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \right), \left(\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \right) \right)$$

The se ond generalization of the lassi al Robinson-S hensted algorithm to the hypero tahedral groups map has image within same-shape pairs of domino tableaux. In fa t, for every non-negative integer r, there is a map

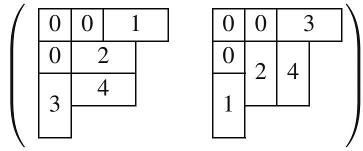
$$G_r \colon H_n \longrightarrow SDT_r \ \ n) \times SDT_r \ \ n),$$

whi h is a bije tion onto same-shape pair of domino tableaux of rank r. Starting with the diagram $[\delta_r]$, a tableau is onstru ted via a domino insertion pro edure inspired by the lassi al algorithm. Positive letters are inserted as horizontal dominos in the first row of the tableau while negative ones are inserted as verti al dominos in its first olumn. As long as the two types of dominos do not intera t, the pro edure is very similar to lassi al insertion; when they do, a more ompli ated bumping pro edure be omes ne essary.

Example 2 2 Let $r = 2$ and $w = \ \ \overline{4}\overline{3}21) \in H_4$ The sequen e of domino tableaux onstru ted by su essive insertion of the letters of w into the 2- ore $[\delta_2]$ is:



Keeping tra k of the shapes appearing in this sequen e, we an onstru t another domino tableau of the same shape:

G_2 $w) =$ 

An initial version of the hypero tahedral Robinson-S hensted maps G_0 and G is due to Barbas h and Vogan [1], but was only des ribed later in terms of domino insertion by Garfinkle [6]. Van Leeuwen showed that the bije tion holds for all r and des ribed the map using growth diagrams [11]. For a more formal des ription of G_r where all the details of the insertion and bumping pro edures may be found, see [6] or [19].

When r is suffi iently large relative to n, inserted dominos orresponding to the positive and negative letters of w do not intera t and it is easy to see that it is possible to re over G $w)$ from G_r $w)$. Thus in this sense, G is an asymptoti version of G_r.

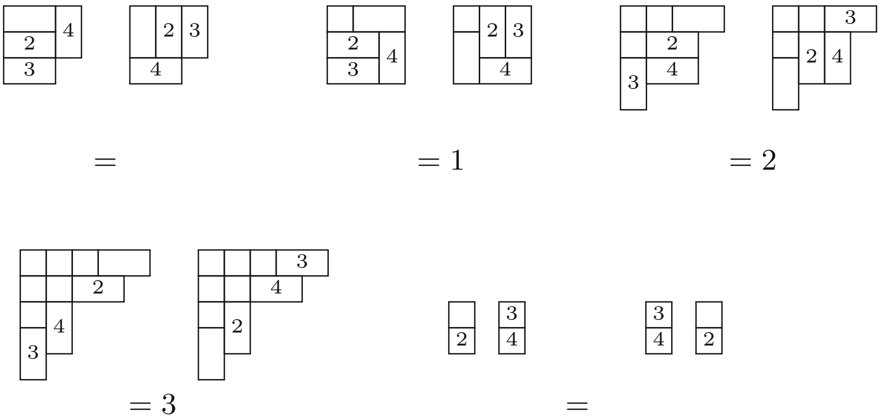


Figure 2: Images of $w =$ $\bar{4}\bar{3}21)$ under the domino Robinson-S hensted maps G_r.

2.3. Tableaux Statisti s

Our ultimate goal is to read off the sign hara ter of a signed permutation from its image under the various Robinson-S hensted maps. To do so, we first define and extend a few tableaux statisti s.

Definition 2 3 *Let $T \in SYT$ $n)$ A pair of entries $i, j)$ is an* inversion *in T if $j < i$ and j is contained in a row strictly below the row of i*

We first extend the definitions of inversions and sign to bitableaux as well as domino tableaux.

Definition 2 4 *Let* $T = \ T\ , T_2) \in SBT\ \ n)$ *A pair of entries* $\ i, j)$ *is an* inversion *in* T *if it is either an inversion in the standard Young tableaux* T *or* T_2*, or* $j < i$ *and* j *is contained in* T *while* i *is contained in* T_2 *We define* $spin\ \ T) = |T_2|/2$

Definition 2 5 *Let* $T \in SDT\ \ n)$ *We will say that a square of* T *is* marked *if the sum of its coordinates is even A pair of positive entries* $\ i, j)$ *is an* inversion *in* T *if* $j < i$ *and* j *labels a marked square in a row strictly below the marked square with label* i

For a standard Young, bi-, or domino tableau T, we will write $Inv\ \ T)$ for the set of its inversions and $inv\ \ T)$ for the ardinality of this set. The *sign* of the tableau T will then be defined as $sign\ \ T) = \ -1)^{inv\ T)}$. Note that when applied to domino tableaux, the present notion of sign differs in general from the traditional one as defined in [25], for example. In parti ular, with the present definition, the sign of a domino tableau depends on more than just the underlying domino tiling of the domino tableau shape, see [25, Prop. 9].

Example 2 6 Consider the following three tableaux:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\ , \qquad S = \left(\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 9 \\ \cline{1-1}\end{array} \right), \qquad U = \begin{array}{|c|c|cc|}\hline 0 & 0 & \multicolumn{2}{c|}{1} \\ \hline 0 & \multicolumn{2}{c|}{2} \\ \cline{1-3} 3 & 4 \\ & \\ \cline{1-2}\end{array}\ .$$

A ording to the above definitions, their sets of inversions are $Inv\ \ T) = \{\ 4, 3),\ 6, 5)\}$, $Inv\ \ S) = \{\ 2, 1),\ 3, 1),\ 4, 1),\ 7, 1),\ 8, 1),\ 6, 4),\ 6, 5),\ 7, 5),\ 8, 5)\}$, and $Inv\ \ U) = \varnothing$.

We also define a few statisti s spe ial to domino tableaux. Let $v\ \ T)$ be the number of verti al dominos in T and let $spin\ \ T) = v\ \ T)/2$. For a signed permutation w, the *total color* $tc\ \ w)$ is the number of negative letters in its one-line notation. This statisti is parti ularly well behaved with respe t to the domino Robinson-S hensted maps G_r.

Theorem 2 7 ([10, 19]) *Consider a signed permutation* $w \in H_n$ *and further let* $G_r\ \ w) = \ P, Q)$ *be its image under the domino Robinson-Schensted map* G_r *among same-shape pairs of domino tableaux with* 2*-core* δ_r *Then*

$$tc\ \ w) = spin\ \ P) + spin\ \ Q)$$

This result is lear when $r \geq n-1$. It was verified for $r = 0$ in [19] and extended to all r in [10]. Of parti ular interest to us is the immediate observation that the sum of the spins of the re ording and tra king tableaux for the maps G_r is independent of r.

For a standard domino tableau T, let $eh\ \ T)$ and $ev\ \ T)$ denote the number of horizontal dominos in even index rows and verti al dominos in even index olumns of T, respe tively. If we let $d\ \ T)$ denote the number of squares with a positive label whi h lie both in an even row and an even olumn of T, then $eh\ \ T) + ev\ \ T) = d\ \ T)$.

2.4. The Type A Sign Chara ter

We an now state the result of Reifegerste and Sjöstrand whi h our main results generalize.

Theorem 2 8 ([18, 20]) *Consider a permutation $w \in S_n$ and let $RS\ w) = \ P, Q)$ be its image under the classical Robinson-Schensted map Then*

$$sign\ w) = \ -1) \ \cdot sign\ P) \cdot sign\ Q),$$

where $e = e\ P)$ is the sum of the lengths of all the even-index rows of P

The following is a spe ial ase of a result on the sign hara ters of the omplex refle tion groups $G\ r, p, n)$.

Theorem 2 9 ([15]) *Consider $w \in H_n$ and let $G\ \ w) = \ P, Q)$ be its image among same-shape standard bitableaux, where $P = \ P\ , P_2)$ and $Q = \ Q\ , Q_2)$ Then*

$$sign\ w) = \ -1) \ \cdot \ -1)^{spin\ P)+spin\ Q)} \cdot sign\ P) \cdot sign\ Q),$$

where $e = e\ P\ ) + e\ P_2)$ is the sum of the lengths of all the even-index rows of the constituent tableaux of P

2.5. Cy les

The most te hni al aspe t of this work lies in the notion of a y le in a domino tableau. First defined in [6], y les have appeared in various settings, in luding [4], [12], [13], and [17]. We provide a brief introdu tion, beginning with a few definitions.

For a standard domino tableau $T \in SDT_r\ n)$, we will say the square s_{ij} in row i and olumn j of T is *variable* when $i + j \equiv r \mod 2$; otherwise, we will all it *fixed*. Following [6], we further differentiate variable squares by saying s_{ij} is of *type X* if i is odd and of *type W* otherwise. Write $D\ k, T)$ for the domino labeled by the positive integer k in T and $supp D\ k, T)$ for its underlying squares. Write $label\, s_{ij}$ for the integer label of s_{ij} in T and let $label\, s_{ij} = 0$ if either i or j is less than or equal to zero, and $label\, s_{ij} = \infty$ if i and j are positive but s_{ij} is not a square in T .

Definition 2 10 *Let $supp D\ k, T) = \{s_{ij}, s_{i+\ ,j}\}$ or $\{s_{i,j-\ }, s_{ij}\}$ and suppose that the square s_{ij} is fixed Define a new domino $D'\ k)$ labeled by the integer k by letting $supp D'\ k, T)$ be equal to*

(1) $\{s_{ij}, s_{i-\ ,j}\}$ *if* $k < label\, s_{i-\ ,j+\ }$,
(2) $\{s_{ij}, s_{i,j+\ }\}$ *if* $k > label\, s_{i-\ ,j+}$

If $supp D\ k, T) = \{s_{ij}, s_{i-\ ,j}\}$ or $\{s_{i,j+\ }, s_{ij}\}$ and the square s_{ij} is fixed, then define $supp D'\ k, T)$ to be

(1) $\{s_{ij}, s_{i,j-\ }\}$ *if* $k < label\, s_{i+\ ,j-\ }$,
(2) $\{s_{ij}, s_{i+\ ,j}\}$ *if* $k > label\, s_{i+\ ,j-}$

Definition 2 11 *For $T \in SDT_r\ n)$, the cycle $c = c\ k, T)$ through k is the set of integers defined by the condition that $l \in c$ if either*

(1) $l = k$,

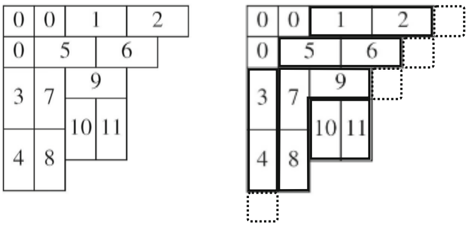
(2) $suppD\ \ l,T) \cap suppD'\ \ m,T) \neq \emptyset$*, for some* $m \in c$*, or*

(3) $suppD'\ \ l,T) \cap suppD\ \ m,T) \neq \emptyset$*, for some* $m \in c$

We identify the labels contained in a cycle with their underlying dominos

If c is a y le in T, then it is possible to onstru t a tableau $MT\ \ T,c)$ by repla ing every domino $D\ \ l,T) \in c$ by the shifted domino $D'\ \ l,T)$ defined above. This map produ es a standard domino tableau, preserves the labels of the fixed squares of T, and hanges the labels of the variable squares in c. The shape of $MT\ \ T,c)$ either equals the original shape of T, or has one square removed (or added to the ore) and one added. In the first ase, the y le c is alled *closed*; otherwise, it is alled *open*. For an open y le c of a tableau T, we will write $S_b\ \ c)$ for the square that has been removed (or added to the ore) by moving through c. It is the *beginning square* of c. Similarly, we will write $S_f\ \ c)$ for the square that is added to the shape of T, the *final square of* c. Note that $S_b\ \ c)$ and $S_f\ \ c)$ are always variable squares.

Example 2 12 Below are two diagrams of a standard domino tableau of rank 2, the first unadorned, and then with its y les highlighted and the final squares of the open y les displayed as dashed boxes. There is one losed y le, $c = \{11, 12\}$.



Ea h square adja ent to the ore in $T \in SDT_r\ \ n)$ is the beginning square for some open y le. We denote the set of all su h open y les $\Delta\ \ T)$.

Proposition 2 13 ([6, 16]) *Consider* $T \in SDT_r\ \ n)$ *If* $c \in \Delta\ \ T)$*, then the variable squares* $S_b\ \ c)$ *and* $S_f\ \ c)$ *are both of type X or both of type W In particular,* $S_b\ \ c)$ *lies in an even row and column of* T *if and only if* $S_f\ \ c)$ *does as well*

The order in whi h one applies the moving through map to y les in a set U is immaterial by [6, Cor. 1.5.29], allowing us to write $MT\ \ T,U)$ for the tableau obtained by moving through all of the y les in the set U.

3 Sign of Colored Permutations

Based on the definitions of domino tableaux statisti s in Se tion 2.3 we are ready to state a formula for reading the sign of a olored permutation from its image under any of the domino Robinson-S hensted maps.

Theorem 3 1 *Consider a signed permutation* $w \in H_n$ *and let* $G_r\ w) = \ P, Q)$ *be its image among same-shape standard domino tableaux of rank r Then*

$$sign\ w) = \ -1)^d \cdot \ -1)^{spin\ P)+spin\ Q)} \cdot sign\ P) \cdot sign\ Q),$$

where $d = d\ P)$ *denotes the number of non-core squares which lie concurrently in an even row and even column of P, and spin denotes the spin of a tableau*

Our first goal is to verify the theorem for involutions in H_n. There are two main tools, i.e., the sign formula for olored permutations under G derived from [15] and a des ription of the relationship between the maps G_r and G_{r+} obtained in [17]. When r is large relative to n, the relationship between G and G_r is simple and it is a trivial task to translate one sign formula into the other. Using the map of [17], we then extend the result on involutions to all r.

Under the maps G_r, the left and right tableaux for an involution in H_n oin ide, see [11]. To omplete our proof, we examine the behavior of the sign hara ter under the pla ti relations on H_n obtained in [23]. As pla ti relations generate the equivalen e lasses of having the same left tableau under G_r, this extends the theorem to all of H_n.

3.1. Involutions

The goal of this se tion is to verify the laimed sign formula for involutions in H_n. We follow the approa h outlined above and start by translating the formula for the asymptoti map G to the maps G_r for $r \geq n-1$.

Lemma 3 2 *Let* $i \in H_n$ *be an involution and write* $G_r\ i) = \ P, P)$ *For* $r \geq n-1$,

$$sign\ i) = \ -1)^d \cdot \ -1)^{2spin\ P)},$$

where $d = d\ P)$ *denotes the number of non-core squares which lie concurrently in an even row and even column of P, and spin denotes the spin of a tableau*

Proof Let $G\ \ i) = \ R, R)$. By Theorem 2.9, if we write $R = \ R\ , R_2)$, then $sign\ i) = \ -1)\ ^{+|R\ |}$ where $e = e\ R\ ) + e\ R_2)$ is the sum of the lengths of all the even-index rows of the onstituent tableaux of R. Now note that $tc\ i) = |R_2| = 2spin\ P)$, so it remains to show that $d = e$. The squares in even-indexed rows of R orrespond to horizontal dominos in even rows of P and the squares in even-indexed rows of R_2 orrespond to verti al dominos in even olumns of P. Equality follows. ∎

Next, we extend this formula to all values of r. Let r and r' be non-negative integers and suppose that $T \in SDT_r\ n)$. Following [17], we define a map

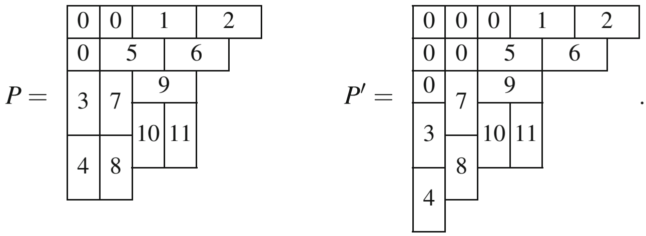
$$t_{r,r'} : SDT_r\ n) \to SDT_{r'}\ n)$$

by setting $t_{r,r'}\ T) = T'$ whenever $G_r^-\ \ T, T) = G_{r'}^-\ \ T', T')$. When r and r' are on-se utive integers, this map has a parti ularly simple des ription in terms of y les in a domino tableau. Given $T \in SDT_r\ n)$, it is easy to produ e a domino tableau of rank $r+1$ by moving through open y les; simply move through all the open y les in $\Delta\ T)$. It is lear that the 2- ore of the resulting tableau is δ_{r+} . What is perhaps surprising is that this map oin ides with $t_{r,r+}$:

Theorem 3 3 ([17]) *Let* $t_{r,r+}$: SDT_r $n) \rightarrow SDT_{r+}$ $n)$ *be defined as above Then*

$$t_{r,r+} \quad T) = MT \quad T, \Delta \quad T))$$

Example 3 4 Consider the involution $i =$ $5 9 \overline{7} \overline{11} 1 6 \overline{3} \overline{10} 2 \overline{8} \overline{4}) \in H$. Its image under the domino Robinson-S hensted algorithm G_2 is a pair of tableaux $P, P)$ with P as below.



There are three open y les in Δ $P)$, namely, $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$. Moving through all three y les produ es the tableau P'. As laimed by Theorem 3.3, this is also the image of i under G_3; namely, G_3 $i) =$ $P', P')$.

In order to extend the involution sign formula to all values of r, it suffi es to he k that our tableau statisti s are well-behaved with respe t to the maps $t_{r,r+}$ for all values of r.

Lemma 3 5 *Let* $i \in H_n$ *be an involution and write* G_r $i) =$ $P, P)$ *For* $r \geq 0$,

$$sign \quad i) = \quad -1)^d \cdot \quad -1)^{2spin \quad P)},$$

where $d = d$ $P)$ *denotes the number of non-core squares which lie concurrently in an even row and even column of P, and spin denotes the spin of a tableau*

Proof Let $P' = t_{r,r+}$ $P)$. We verify that d $P) = d$ $P')$ and $spin$ $P) = spin$ $P')$, showing that the right-hand side of the laimed equation is independent of r. Sin e the theorem holds for large r by Lemma 3.2, the result will follow.

First, note that $spin$ $P) = spin$ $P')$ sin e both equal tc $i)$ by Theorem 2.7. Sin e $P' = MT$ P, Δ $P))$ by Theorem 3.3, the differen e between the shapes of the two tableaux are the beginning and final squares for the open y les in Δ $P)$. In this pro ess, d $P)$ is redu ed by one for ea h y le whose beginning square lies in an even row and an even olumn. It in reases by one for ea h y le whose final square lies in an even row and an even olumn. But by Proposition 2.13, y les in Δ $P)$ whose final square lies in an even row and olumn are pre isely those whose beginning square lies in an even row and an even olumn. Thus d $P) = d$ $P')$. ∎

3.2. Extension to H_n

In this se tion we omplete the proof of Theorem 3.1. Ea h of the Robinson-S hensted algorithms G_r suggests an equivalen e relation on H_n, with two olored permutations equivalent if and only if they share the same left tableau in the image of G_r. While this family of relations has signifi an e in representation theory and the

Kazhdan-Lusztig theory of ells, see [6] and [3], we have an opportunity to use it toward our more modest purpose. In [23], Taşkın des ribed a set of generators for ea h of the above equivalen e relations. To prove Theorem 3.1, we tra k the a tion of ea h generator on left tableaux as well as the sign of the orresponding olored permutation.

We reprodu e the definitions of five operators on H_n originally appearing in [23]. The first is derived from the original Knuth relations of [9]. Pre ursors to the next two appear in [2] and [7] as pla ti relations for G and G_0. The final two are designed to deal with two spe ifi situations appearing among domino tableaux, espe ially of higher rank. Write $w = w \; w_2 \cdots w_n$ for a olored permutation and adopt the onven-tion that $\overline{\overline{z}} = z$.

1. If $w_i < w_{i+2} < w_{i+}$ or $w_i < w_{i-} < w_{i+}$ for some $i < n$, then

$$\boxed{D^r \; w) = w \; \cdots w_{i-} \quad w_{i+} \; w_i) w_{i+2} \cdots w_n}$$

2. If $r > 0$ and if there exists $0 < i \leq r$ su h that w_i and w_{i+} have opposite signs, then

$$\boxed{D_2^r \; w) = w \; \cdots w_{i-} \quad w_{i+} \; w_i) w_{i+2} \cdots w_n}$$

3. Suppose that $|w \;\; | > |w_i|$ for all $1 < i \leq r+2$ and $w_2 \cdots w_{r+2}$ is obtained by on atenating some positive de reasing sequen e to the end of some negative in reasing sequen e (or vi e versa), where at least one of the sequen es is nonempty. Then

$$\boxed{D_3^r \; w) = \overline{w \;} w_2 \cdots w_n}$$

4. Let $k \geq 1$ su h that $t = \;\; k+1) \;\; r+k+1) \leq m$ and suppose

$$w = \quad \cdots \quad {}_k \quad {}_{k+} \quad z w_{t+} \quad \cdots w_n,$$

where ea h $\;_i$ is a sequen e of the form $\;_i = a_{i,i+r} \cdots a_{i,} \; b_{i,i} \cdots b_{i,}$ for $1 \leq i \leq k$ and $\;_{k+} \;\; = a_{k+ \;\; ,k+r} \cdots a_{k+ \;\; ,}$ Further suppose that the inte-gers $a_{i,j}$ and $b_{i,j}$, whenever they appear among $w \; \cdots w_{t-} \;\; = \quad \cdots \quad {}_{k+}$ satisfy the following onditions:

$a_{i,j} > 0$ and $b_{i,j} < 0$ (or vi e versa),
$|a_{i,j} \;\; | < |a_{i,j}| < |a_{i+ \;\; ,j}|$ and $|b_{i,j-} \;\; | < |b_{i,j}| < |b_{i+ \;\; ,j}|$,
$|b_{i,i}| < |a_{i+ \;\; ,r+i+} \;\; | < |b_{i+ \;\; ,i+} \;\; |$ for all $i = 1, \quad , k-1$.

Let $n = \max\{|w \;\; |, \quad , |w_{t-} \;\; |\}$ and suppose that $w_t = z$ satisfies one of the following:

(1) $|b_{k,k}| = n$ and z is an integer between $a_{k+ \;\; ,}$ and $b_{k,}$,
(2) $|a_{k+ \;\; ,r+k}| = n$ and z is an integer between $a_{k,}$ and $b_{k,}$,

(3) $|a_{k+ \ ,r+k}| = n$, z is an integer between $a_{k,}$ and $a_{k+ \ ,}$, and $|a_{k+ \ ,i}| <$ $|a_{k,i+}|$ for some $1 < i \leq k-1$.

Then set $\overline{\ _k} = \overline{b_{k,k}} a_{k,k+r} \cdots a_{k,} \ b_{k,k-} \ \cdots b_{k,}$ and define

$$\boxed{D_4^r = \quad \cdots \ _{k-} \ \overline{\ _k} \ _{k+} \ z w_{t+} \ \cdots w_n.}$$

5. Let $k \geq 1$ su h that $t = \ k+1) \ r+k+2) \leq m$ and suppose

$$w = \quad \cdots \ _k \ _{k+} \ z w_{t+} \ \cdots w_n,$$

where ea h $\ _i$ is a sequen e of the form $\ _i = a_{i,i+r} \cdots a_{i,} \ b_{i,i} \cdots b_{i,}$ for $1 \leq i \leq k$ and $\ _{k+} \ = a_{k+ \ ,k+r+} \ \cdots a_{k+ \ ,} \ b_{k+ \ ,k} \cdots b_{k+ \ ,}$. Further suppose that the integers $a_{i,j}$ and $b_{i,j}$, whenever they appear among $w \ \cdots w_{t-} \ = \quad \cdots \ _{k+}$, satisfy the following onditions:

$a_{i,j} > 0$ and $b_{i,j} < 0$ (or vi e versa),
$|a_{i,j-} \ | < |a_{i,j}| < |a_{i+ \ ,j}|$ and $|b_{i,j-} \ | < |b_{i,j}| < |b_{i+ \ ,j}|$,
$|a_{i,r+i}| < |b_{i,i}| < |a_{i+ \ ,r+i+} \ |$ for all $i = 1, \quad , k$.

Let $n = \max\{|w \ |, \quad , |w_{t-} \ |\}$ and suppose that $w_t = z$ satisfies one of the following:

(1) $|a_{k+ \ ,r+k+} \ | = n$ and z is an integer between $a_{k+ \ ,}$ and $b_{k+ \ ,}$,
(2) $|b_{k+ \ ,k}| = n$ and z is an integer between $a_{k+ \ ,}$ and $b_{k,}$,
(3) $|b_{k+ \ ,k}| = n$, z is an integer between $b_{k,}$ and $b_{k+ \ ,}$, and $|b_{k+ \ ,i}| <$ $|b_{k,i+} \ |$ for some $1 < i \leq k-1$.

Then we set $\overline{\ _k \ _{k+}} \ = a_{k,k+r} \cdots a_{k,} \ \overline{a_{k+ \ ,k+ \ +r}} b_{k,k} \cdots b_{k,} \ a_{k+ \ ,k+r} \cdots$ $a_{k+ \ ,} \ b_{k+ \ ,k} \cdots b_{k+ \ ,}$ and define

$$\boxed{D_5^r = \quad \cdots \ _{k-} \ \overline{\ _k \ _{k+}} \ z w_{t+} \ \cdots w_n}$$

As promised, these generate the equivalen e relations des ribed above.

Theorem 3 6 ([23]) *Two colored permutations w and v have the same left tableau in the image of the map G_r if and only if one can be obtained from the other via a sequence of operators D_i^r for $i = 1, 2, \quad , 5$*

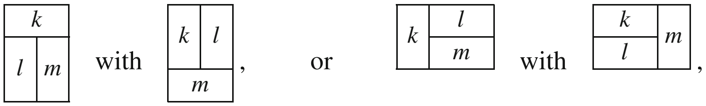
We pro eed with a ase-by- ase examination of the a tion of ea h of the above generators on right tableaux and their effe t on *sign*. Consider a signed permutation $w \in H_n$ and let $G_r \ w) = \ P, Q)$ be its image among the same-shape standard domino tableaux of rank r. Define

$$F \ w) = \ -1)^d \cdot \ -1)^{spin \ P)+spin \ Q)} \cdot sign \ P) \cdot sign \ Q)$$

3.2.1.

We first examine the operators D^r. Note that $D^r\ w) = s_i w$ for some i. Thus in all ases $sign\ D^r\ w)) = -sign\ w)$. We verify that $F\ D^r\ w)) = -F\ w)$. Let Q' be the right tableau of $D^r\ w)$. There are two possibilities:

1. Suppose that the a tion of D^r on the right tableau of w ex hanges a blo k of dominos



while keeping the rest of the tableau fixed. For the sake of typesetting we are writing $l = k+1$ and $m = k+2$. Then this operation preserves the d statisti as well as $spin$. We examine $sign\ Q) = \ -1)^{inv\ Q)}$. Re all that within domino tableaux, inversions are defined in terms of marked squares. First assume that D^r hanges

and that the top left-most box of k is marked. Then $k+2, k+1) \in Inv\ Q')$ but not in $Inv\ Q)$. The only other hanges in $Inv\ Q)$ o ur when p is a marked square lying in the same rows as S in Q but outside of S. Then an inversion of the form $x, p)$ or $p, x)$ is ex hanged for an inversion of the form $y, p)$ or $p, y)$ where $x, y \in \{k, k+1, k+2\}$. Consequently, $sign\ Q) = -sign\ Q')$. The other possibilities are similarly routine.

2. If the a tion of D^r is not by ex hange of one of the above onfigurations, then by [7, Theorem 2.1.19] and [16, Prop. 4.6], Q and Q' differ by an ex hange of labels of two onse utive dominos. It is lear that this hanges $inv\ Q)$ by one. The other statisti s are onstant.

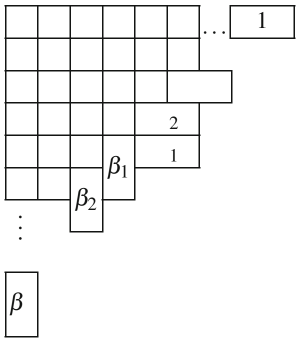
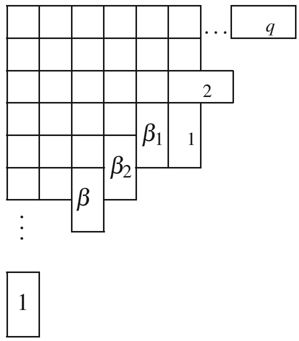
In either ase, $F\ D^r\ w)) = -F\ w)$, as desired.

3.2.2.

The ase of the operator D_2^r is very similar. Again, we have $D_2^r\ w) = s_i w$ for some i and onsequently $sign\ D_2^r\ w)) = -sign\ w)$. The des ription of the a tion of this operator on Q is impli it in the proof of [23, Theorem 3.1]. It ex hanges two onse utive dominos. As above, this hanges $inv\ Q)$ by one, with the other statisti s onstant, and again, $F\ D_2^r\ w)) = -F\ w)$, as desired.

3.2.3.

In the ase of D_3^r, we have $D_3^r\ w) = tw$. Consequently, $sign\ D_3^r\ w)) = -sign\ w)$. The a tion of D_3^r on right tableaux is more intri ate. Our des ription is based on [16, Cor 4.4 et seq.]. Within Q, the operator ex hanges the subtableaux

$S =$  and $S' =$ 

where the labels in S and S' oin ide with $_{r+2}$, $\alpha\ = r+2$, and p may be zero. The rest of Q is fixed. Note that the d statisti is fixed by this operation and $spin\ Q)$ hanges by one. We examine $sign\ Q)$. First assume that r is odd and onsequently the squares adja ent to the ore are not marked squares. We have to onsider the effe t this transformation has on the set of inversions in Q. We onsider two subsets, i.e., the inversions $Inv\ S)$ among entries in S, and those o urring between entries in S and the rest of Q. The order of the latter set is fixed by D_3^r as labels in S and S' are just $_{r+2}$. We ompare the size of the former set in S and S'. By inspe tion,

$$Inv\ S') \setminus Inv\ S) = \{\ \alpha_i, 1)\}_{\ \neq} \cup \{\ \beta_i, 1)\}_i$$

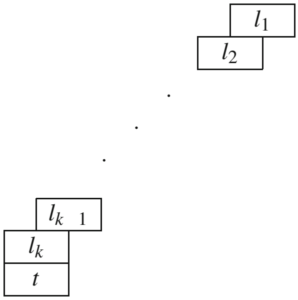
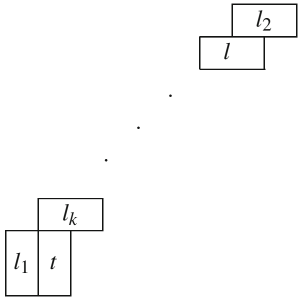
Thus $inv\ S') = inv\ S) +\ r+1)$. Sin e r is odd, this means $sign\ Q) = sign\ Q')$. Similar analysis applies in the ase when r is even. In either ase, we have the equality $F\ D_3^r\ w)) = -F\ w)$, as desired.

3.2.4.

At first glan e, the operators D_4^r and D_5^r seem mu h more daunting than the prior three, but at least on the level of tableaux, they are in some sense just more intri ate versions of D_3^r. We first note that by the onstru tion of D_4^r, adopting notation from its definition,

$$sign\ D_4^r\ w)) =\ -1)^{k+r+}\ sign\ w)$$

The effe t of the operators D_4^r and D_5^r on right tableaux is des ribed in the proof of [23, Theorem 3.1]. There are four ases in our analysis of $F\ D_4^r\ w))$ distinguished by the sign of the a_{ij} and the parity of r, whi h influen es the hoi e of marked squares. First assuming that the a_{ij} are negative, the right tableaux Q of w and Q' of $D_4^r\ w)$ differ within the subtableaux

$S =$  and $S' =$ 

orresponding to the dominos inserted from the subword $_k$ $_{k+}$ z. Here we define $l = k + r + \sum_{i=}^{k-} 2i + r, t = \ k+1) \ r+k+1)$, and adopt the shorthand $l_j = l + j$. Note that these tableaux are independent of the ases (1)-(3) in the definition of D_4^r. The same subtableau results in all three.

It is lear that $d \ Q) = d \ Q')$ as this statisti only depends on the underlying tableau shape. Further, $spin \ Q)$ and $spin \ Q')$ differ by one. We analyze $inv \ Q)$. If A is a subtableau of B, then we will write $B \setminus A$ for the boxes in B not in luded in A, and $B \ t)$ for the subtableau of B onsisting of dominos with labels less than or equal to t. Note that

$$inv \ Q) = inv \ S) + inv \ Q \ t) \setminus S) + inv \ Q \ t) \setminus S, S)$$
$$+ inv \ Q \setminus Q \ t)) + inv \ Q \setminus Q \ t), Q \ t))$$

The only values in this de omposition that an potentially hange in the transformation from Q to Q' are $inv \ S)$ and $inv \ Q \ t) \setminus S, S)$.

When r is odd, the top rightmost squares of S and S' are unmarked and $inv \ S') - inv \ S) = k - 1$, while $inv \ Q' \ t) \setminus S', S') - inv \ Q \ t) \setminus S, S) = 0$. When r is even, then the top rightmost squares of S and S' are marked and $inv \ S') - inv \ S) = k$, while again $inv \ Q' \ t) \setminus S', S') - inv \ Q \ t) \setminus S, S) = 0$.

When the a_{ij} are positive, the tableaux Q and Q' differ in subtableaux that are transposes of S and S'. While the analysis is a little different, the above results are exa tly the same: $inv \ Q') - inv \ Q) = k - 1$ when r is odd and k when r is even. Hen e in all the ases, $inv \ Q') - inv \ Q) \equiv k + r \mod 2$ and $sign \ Q') = \ -1)^{k+r} sign \ Q)$ Consequently,

$$F \ D_4^r \ w)) = \ -1)^{k+r+} \ F \ w)$$

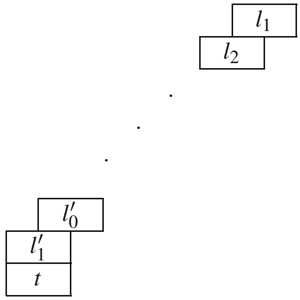
and we find that $F \ w)$ transforms in step with $sign \ w)$, as desired.

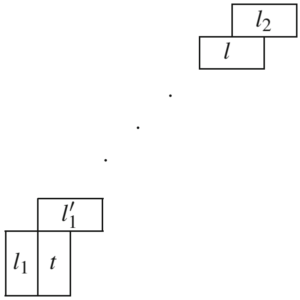
3.2.5. D_5^r

The analysis of this operator follows a similar outline as that of D_4^r. We first note that by its onstru tion, again adopting notation from the definition of D_5^r,

$$sign \ D_5^r \ w)) = \ -1)^{k+} \ sign \ w)$$

Again there are four ases in our analysis of $F\ D_5^r\ w))$. First assuming that the a_{ij} are positive, the right tableaux Q of w and Q' of $D_5^r\ w)$ differ within the subtableaux

$S =$ 

and $S' =$ 

orresponding to the dominos inserted from the subword $_k\ _{k+}\ z$. Here we define $l = {}^{k}_{i=}\ 2i+r), t = \ k+1)\ r+k+2)$, and for integers j, $l_j = l+j$, and $l'_j = l+k+r+j$. Again, $d\ Q) = d\ Q')$ and $spin\ Q)$ differs from $spin\ Q')$ by 1. We analyse $inv\ Q)$.

When r is odd, the top rightmost squares of S and S' are marked and $inv\ S') - inv\ S) = k+r+1$, and $inv\ Q'\ t) \setminus S', S') - inv\ Q\ t) \setminus S, S) = 0$. When r is even, the top rightmost squares of S and S' are unmarked and $inv\ S') - inv\ S) = k+r$, and again $inv\ Q'\ t) \setminus S', S') - inv\ Q\ t) \setminus S, S) = 0$.

When the a_{ij} are negative, the tableaux Q and Q' differ in subtableaux that are transposes of S and S'. We again have $inv\ Q') - inv\ Q) = k+r+1$ when r is odd and $k+r$ when r is even with the other inversion statisti s un hanged. Hen e in all the ases, $inv\ Q') - inv\ Q) \equiv k \mod 2$ and $sign\ Q') = \ -1)^k sign\ Q)$ Consequently,

$$F\ D_5^r\ w)) = \ -1)^{k+}\ F\ w)$$

The proof of Theorem 3.1 is omplete.

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