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Aba Mbirika
University of Wisconsin-Eau Claire

Thomas Pietraho
Bowdoin College

William Silver
Bowdoin College

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ORIGINAL PAPER

On the sign representations for the complex reflection groups $G(r, p, n)$

Aba Mbirika[1] **Thomas Pietraho**[2]
William Silver[3]

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Abstract We present a formula for the values of the sign representations of a complex reflection group $G(r, p, n)$ in terms of its image under a generalized Robinson–Schensted algorithm.

Keywords Complex reflection groups · Robinson-Schensted map

Mathematics Subject Classification 05E10

1 Introduction

The classical Robinson-Schensted algorithm establishes a bijection between permutations $w \in S_n$ and ordered pairs of same-shape standard Young tableaux of size n. This map has proven particularly well-suited to certain questions in the representation

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Aba Mbirika
mbirika@uwec.edu

Thomas Pietraho
tpietrah@bowdoin.edu

William Silver
wsilver@bowdoin.edu

1 Department of Mathematics, University of Wisconsin-Eau Claire, Eau Claire, WI, USA

2 Department of Mathematics, Bowdoin College, Brunswick, ME, USA

3 Cognex Corporation and Department of Computer Science, Bowdoin College, Brunswick, ME, USA

theory of both S_n and the semisimple Lie groups of type A. For instance, Kazhdan-Lusztig cells as well as the primitive spectra of semisimple Lie algebras can be readily described in terms of images of this correspondence.

Other sometimes more elementary representation-theoretic information requires more work to extract from standard Young tableaux. For instance, in independent work, Reifegerste (2004) and Sjöstrand (2005) developed a method for reading the value of the sign representation of a permutation $w \in S_n$ based on two tableaux statistics. Let $w \in S_n$ and write $RS(w) = (P, Q)$ for its image under the classical Robinson-Schensted map. If we write e for the number of squares in the even-indexed rows of P, let $sign(T)$ be the sign of a tableau T derived from its inversion number, and let sgn be the usual sign representation on S_n, then

$$sgn(w) = (-1)^e \cdot sign(P) \cdot sign(Q). \tag{1.1}$$

The focus of this note is to extend this result to the complex reflection groups $G(r, p, n)$. Its two main ingredients generalize readily to this setting. First, the classical Robinson-Schensted algorithm admits a straightforward extension mapping each element $w \in G(r, p, n)$ to a same-shape pair of r-multitableaux, see Stanley (1982, Sect. 6) and Iancu (2003). At the same time, the sign of a permutation in S_n extends to a family of r one-dimensional representations of $G(r, p, n)$. After defining new *spin* and *sign* statistics on r-multitableaux, we offer a short proof of the following:

Theorem *Let $w \in G(r, p, n)$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity ζ and the associated family $\{sgn_i\}_{i=0}^{r-1}$ of representations of $G(r, p, n)$ we have*

$$sgn_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}),$$

where $e(\mathbf{P})$ is the total sum of the lengths of the even-indexed rows of the component tableaux of $\mathbf{P}$.

A weaker version of this theorem has been used to verify a formula for the sign representation of the classical Weyl groups in type B for a family of domino tableaux Robinson-Schensted maps, see (Pietraho 2014). For classical Weyl groups, all of which appear among complex reflection groups, values of the sign representation can be used to compute the Möbius function of Bruhat order (Verma 1971). The above sign formulas show that the Möbius function is well-behaved with respect to the characterization of Kazhdan-Lusztig cells by equivalence classes of tableaux and multitableaux as in Joseph (1977), Ariki (2000) and Bonnafé and Iancu (2003).

2 Preliminaries

After defining the family of complex reflection groups and their one-dimensional representations, we define multipartitions, a generalization of the Robinson-Schensted algorithm, and tableaux statistics that we will use to describe these representations.

2.1 Sign representations

Consider positive integers r, p, and n with p dividing r and let be the primitive root of unity $\exp(2\pi\sqrt{-1})/r)$. We define the complex reflection groups $G(r, p, n)$ as subgroups of $GL_n(\ )$ consisting of matrices such that

- The entries are either 0 or powers of ,
- There is exactly one nonzero entry in each row and column,
- The (r/p)-th power of the product of all nonzero entries is 1.

Together with the thirty-four exceptional groups, the groups $G(r, p, n)$ account for all finite groups generated by complex reflections (Shephard and Todd 1954), and include among them all the classical Weyl groups. In our work the parameter r will generally be fixed allowing us to write simply W_n for the group $G(r, 1, n)$. In order to establish succinct notation, we will write

$$[\ ^{a_1}\sigma_1,\ ^{a_2}\sigma_2, \ldots,\ ^{a_n}\sigma_n]$$

for the matrix whose nonzero entry in the ith column is $\ ^{a_i}$ and appears in row σ_i. Utilizing this notation, define the set $S = \{s_0, \ldots, s_{n-1}\}$ where

$$s_0 = [\ \cdot 1, 2, 3, \ldots, n], \quad \text{and}$$
$$s_i = [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n].$$

Furthermore, let $S' = \{s_0^p, s_0s_1s_0, s_i \mid 1 \leq i \leq n-1\}$. The set S generates W_n with presentation given as

$$W_n = \ s_i \mid s_0^r, s_m^2, (s_js_k)^2, (s_0s_1)^4, (s_ls_{l+1})^3, m \geq 1, |j-k| > 1, l \quad [1, n-2]\Big\rangle.$$

Subject to similar relations, S' generates a subgroup $G(r, p, n)$ of W_n of index p, see Ariki (1995). Let $\sigma = [\sigma_1, \ldots \sigma_n] \quad S_n$, and define $Inv(\sigma)$ to be the set of pairs (σ_i, σ_j) with $i < j$ and $\sigma_i > \sigma_j$.

There are exactly $2r$ one-dimensional representations of W_n; they divide naturally into two families.

Definition 2.1 For each integer i between 0 and $r-1$, we define representations ς_i and sgn_i of W_n by specifying their values on the generating set S. Let

$$\tau_i^\epsilon(s_j) = \begin{cases} \ ^i & \text{if } j = 0, \quad \text{and} \\ (-1)^\epsilon & \text{if } j = 1, \ldots, n-1 \end{cases}$$

and define $\varsigma_i = \tau_i^0$ and $sgn_i = \tau_i^1$. Each becomes a representation of the subgroup $G(r, p, n)$ by restriction.

2.2 Multitableaux

We write a partition λ of an integer m as a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and de ne its rank as $|\lambda| = m$. A *Young diagram* $[\lambda]$ of λ is a left-justi ed array of boxes containing λ_i boxes in its ith row. The shape of a Young diagram will refer to its underlying partition. With the integer r xed, a *multipartition of rank n* is an r-tuple

$$= \left(\lambda^0, \lambda^1, \ldots, \lambda^{r-1}\right)$$

of partitions the sum of whose individual ranks equals n. The *Young diagram* $[\ ]$ of is the r-tuple $([\lambda^0], \ldots, [\lambda^{r-1}])$. We refer to as the *shape* of the diagram $[\ ]$ and de ne $|\ | = n$. We will follow a convention of denoting objects derived from multipartitions in boldface while writing those derived from single partitions using a normal weight font.

A *standard Young tableaux of shape* is the Young diagram $[\ ]$ of rank n together with a labeling of each of its boxes with the elements of $\mathbb{N}_n := \{1, 2, \ldots, n\}$ in such a way that each number is used exactly once, and the labels of the boxes within each component Young diagram $[\lambda^i]$ increase along its rows and down its columns. Remembering that r is xed, we will write $\mathbf{SYT}_n$ for the set of all standard Young tableaux of rank n whose shape is a multipartition with r components.

Example 2.1 Take $r = 3$. The following standard Young tableau $\mathbf{T}$ is of rank 11 and has the shape $= ((2, 1), (1, 1), (3, 3))$:

$$\mathbf{T} = \left(\begin{array}{|c|c|}\hline 6 & 11 \\ \hline 8 & \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 2 & 4 & 9 \\ \hline 3 & 5 & 10 \\ \hline \end{array} \right)$$

Following Stanley (1982, Sect. 6) and Iancu (2003), we de ne a map from W_n to same-shape pairs of r-tuples of standard Young tableaux. Consider an element

$$w = [\ ^{a_1}\sigma_1,\ ^{a_2}\sigma_2, \ldots,\ ^{a_n}\sigma_n] \quad W_n$$

and de ne the ordered sets $w^{(k)} = (\sigma_i \mid a_i = k)$ for $0 \le k < r$. Let $Inv_P(w^{(k)}, w^{(l)})$ consist of (i, j) $Inv(\sigma)$ with i $w^{(l)}$ and j $w^{(k)}$, and let $Inv_Q(w^{(k)}, w^{(l)})$ consist of (i, j) $Inv(\sigma)$ with i $w^{(k)}$ and j $w^{(l)}$. Moreover, write $inv_P(w^{(k)}, w^{(l)})$ and $inv_Q(w^{(k)}, w^{(l)})$ for the respective cardinalities of these sets.

Let $RS(w^{(k)}) = (P_k, Q_k)$ be the image of the sequence $w^{(k)}$ under the classical Robinson-Schensted map, labeling squares of Q_k according to the relative positions of i $w^{(k)}$ within w, and de ne

$$\mathbf{P} := \mathbf{P}(w) = (P_0, P_1, \ldots, P_{r-1}) \quad \text{and} \quad \mathbf{Q} := \mathbf{Q}(w) = (Q_0, Q_1, \ldots, Q_{r-1}).$$

The multitableaux Robinson-Schensted map is de ned by $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$. It maps W_n onto the set of same-shape pairs of elements of $\mathbf{SYT}_n$ and is in fact a bijection.

2.3 Tableaux and multitableaux statistics

Our goal is to describe values of the sign representations on W_n under the above generalization of the Robinson-Schensted map. To do so, we rely on a few statistics that can be readily computed from multitableaux.

Definition 2.2 An *inversion* in a Young tableau T is a pair (i, j) with $j > i$ for which the box labeled by i is contained in a row strictly below the box labeled j. Let $Inv(T)$ be the set of inversions in T, and write $inv(T)$ for its cardinality. If $\mathbf{T} = (T_0, T_1, \dots T_{r-1})$ is a multitableau, we extend this notion and de ne:

$$Inv(\mathbf{T}) = \bigsqcup_{k} Inv(T_k) \sqcup \bigsqcup_{k<l} Inv(T_k, T_l)$$

where $Inv(T_k, T_l) = \{(j, i) \mid j > i, j \text{ is a label in } T_k, i \text{is a label in } T_l\}$. We will be interested mainly in the parity of the size of this set and de ne

$$sign(\mathbf{T}) = (-1)^{inv(\mathbf{T})}.$$

Definition 2.3 For a Young tableau T, write $e(T)$ for the total number of boxes in its rows of even index. For a multitableau $\mathbf{T} = (T_0, T_1, \dots T_{r-1})$, we write $\mathrm{sh(T_k)}$ for the shape of the Young diagram underlying T_k and de ne the statistics e and *spin* as follows:

$$e(\mathbf{T}) = \sum_{k=0}^{r-1} e(T_k) \quad \text{and} \quad spin(\mathbf{T}) = \frac{1}{2} \sum_{k=0}^{r-1} k \cdot |\mathrm{sh(T_k)}|.$$

The *spin* statistic provides a simple description of the image of the subgroup $G(r, p, n)$ under the r-multitableaux Robinson-Schensted map. The following is easy to verify:

Proposition 2.1 $(\mathbf{P}, \mathbf{Q}) \quad \mathbf{RS}(G(r, p, n))$ *if and only if* $2\, spin(\mathbf{P}) \equiv 0 \pmod{p}$.

2.4 A set of functions and an example

We de ne a family of functions on W_n. In the next section we will show that they coincide with the sign representations on W_n. Again, for $w \quad W_n$, let $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$. For $0 \leq i < r$, we will write

$$\pi_i(w) = (-1)^{e(\mathbf{P})} \cdot (\ ^{i})^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}).$$

Example 2.2 Consider $w = [\ ^{1}5, 1,\ ^{2}3, 6,\ ^{2}7,\ ^{1}4, 2, 8]$ in $G(4, 1, 8)$. Recalling the notation in Sect. 2.2, we have $w^{(0)} = (1, 6, 2, 8)$, $w^{(1)} = (5, 4)$, $w^{(2)} = (3, 7)$, and $w^{(3)} = \emptyset$. Furthermore,

$$RS(w^{(0)}) = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 6 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|} \hline 2 & 4 & 8 \\ \hline 7 \\ \cline{1-1} \end{array} \right) \qquad RS(w^{(1)}) = \left(\begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array} \right)$$

$$RS(w^{(2)}) = \left(\begin{array}{|c|c|} \hline 3 & 7 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & 5 \\ \hline \end{array} \right) \qquad RS(w^{(3)}) = (\emptyset, \emptyset) .$$

From these we construct the Robinson-Schensted image of w:

$$\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q}) = \left(\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 6 \\ \cline{1-1} \end{array}, \ \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \ \begin{array}{|c|c|} \hline 3 & 7 \\ \hline \end{array}, \ \emptyset \right), \left(\begin{array}{|c|c|c|} \hline 2 & 4 & 8 \\ \hline 7 \\ \cline{1-1} \end{array}, \ \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array}, \ \begin{array}{|c|c|} \hline 3 & 5 \\ \hline \end{array}, \ \emptyset \right) \right).$$

We read off $inv(\mathbf{P}) = 10$, $inv(\mathbf{Q}) = 14$, $e(\mathbf{P}) = 2$, and $spin(\mathbf{P}) = spin(\mathbf{Q}) = 3$. Hence $\pi_i(w) = (\ ^i)^2$ which coincides with $sgn_i(w)$.

3 Sign under the Robinson Schensted map

With the appropriate de nitions of the tableaux statistics in place, we can now verify the claimed formulas for the family of sign representations $\{sgn_i\}$. Recall our notation $w = [\ ^{a_1}\sigma_1, \ ^{a_2}\sigma_2, \ldots, \ ^{a_n}\sigma_n] \quad W_n$ where $\sigma = \sigma_1 \ldots \sigma_n \quad S_n$. Directly from the de nitions of Inv_P and Inv_Q, we obtain the following partition:

$$Inv(\sigma) = \bigsqcup_{k=0}^{r-1} Inv(w^{(k)}) \bigsqcup_{k<l} Inv_P(w^{(k)}, w^{(l)}) \bigsqcup_{k<l} Inv_Q(w^{(k)}, w^{(l)}). \tag{3.1}$$

Applying this to sgn_i, we have:

$$sgn_i(w) = (\ ^i)^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv_P(w^{(k)},w^{(l)})+inv_Q(w^{(k)},w^{(l)})}. \tag{3.2}$$

Write the reverse of $\sigma \quad S_n$ as $\sigma^{\text{rev}} := \sigma_n\sigma_{n-1}\ldots\sigma_1 \quad S_n$ and for xed integers k, l such that $0 \leq k < l \leq r-1$, let

$$\begin{aligned} _1 &= Inv_P(w^{(k)}, w^{(l)}) = \left\{ (i, j) \quad Inv(\sigma) | i \quad w^{(l)} \quad \text{and} \quad j \quad w^{(k)} \right\}, \\ _2 &= Inv_Q(w^{(k)}, w^{(l)}) = \left\{ (i, j) \quad Inv(\sigma) | i \quad w^{(k)} \quad \text{and} \quad j \quad w^{(l)} \right\}, \quad \text{and} \\ _3 &= \left\{ (i, j) \quad Inv(\sigma^{\text{rev}}) | i \quad w^{(k)} \quad \text{and} \quad j \quad w^{(l)} \right\}. \end{aligned}$$

Lemma 3.1 *For fixed integers k and l as above we have*

$$Inv(P_k, P_l) = \quad _2 \sqcup \quad _3 \quad \text{and} \quad |Inv(Q_k, Q_l)| = | \quad _1 \sqcup \quad _3|.$$

Proof To prove the rst claim, let $(\sigma_i, \sigma_j) \quad Inv(P_k, P_l)$. Then $\sigma_i > \sigma_j$, $\sigma_i \quad w^{(k)}$, and $\sigma_j \quad w^{(l)}$. If $i < j$ then $(\sigma_i, \sigma_j) \quad Inv(\sigma)$ and hence lies in $\ _2$. On the other hand if $i > j$, then $(\sigma_i, \sigma_j) \quad Inv(\sigma^{\text{rev}})$ and hence lies in $\ _3$. To prove the second claim, let $(j, i) \quad Inv(Q_k, Q_l)$. Then $j > i$, $\sigma_j \quad w^{(k)}$, and $\sigma_i \quad w^{(l)}$. If $\sigma_i > \sigma_j$, then $(\sigma_i, \sigma_j) \quad Inv(\sigma)$ and hence lies in $\ _1$. On the other hand if $\sigma_i < \sigma_j$, then $(\sigma_j, \sigma_i) \quad Inv(\sigma^{\text{rev}})$ and hence lies in $\ _3$. Thus each $(j, i) \quad Inv(Q_k, Q_l)$ corresponds to either a $(\sigma_i, \sigma_j) \quad\ _1$ or a $(\sigma_j, \sigma_i) \quad\ _3$.

Immediately, we obtain:

Corollary 3.1 *For any $k < l$*

$$inv_P(w^{(k)}, w^{(l)}) + inv_Q(w^{(k)}, w^{(l)}) \equiv inv(P_k, P_l) + inv(Q_k, Q_l) \pmod 2.$$

We are now ready to prove that the functions π_i de ned in Sect. 2.4 coincide with sgn_i, hence proving our main theorem.

Theorem 3.1 *Let $w \quad W_n$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity and the associated family $\{sgn_i\}_{i=0}^{r-1}$ of representations of W_n we have*

$$sgn_i(w) = (-1)^{e(\mathbf{P})} \cdot (\ ^{i})^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}).$$

Proof Observe that the functions π_i can be decomposed as follows

$$\begin{aligned}
\pi_i(w) &= (-1)^{e(\mathbf{P})} \cdot (\ ^{i})^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}) \\
&= (-1)^{\sum_{k=0}^{r-1} e(P_k)} (\ ^{i})^{\sum_{k=1}^{n} a_k} \prod_{k=0}^{r-1} sign(P_k) \prod_{k=0}^{r-1} sign(Q_k) \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)} \\
&= (\ ^{i})^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} \Big((-1)^{e(P_k)} sign(P_k) sign(Q_k)\Big) \cdot \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)}
\end{aligned}$$

Since each $(-1)^{e(P_k)} sign(P_k) sign(Q_k)$ coincides with $sgn(w^{(k)})$ by Eq. (1.1), we have

$$\begin{aligned}
\pi_i(w) &= (\ ^{i})^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)} \\
&= (\ ^{i})^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv_P(w^{(k)},w^{(l)})+inv_Q(w^{(k)},w^{(l)})} \\
&= sgn_i(w)
\end{aligned}$$

where the second equality holds as a consequence of Corollary 3.1 and the nal equality holds by Eq. (3.2).

References

Ariki, S.: Representation theory of a Hecke algebra of $G(r, p, n)$. J. Algebra **77**(1), 164–185 (1995)

Ariki, S.: Robinson-Schensted correspondence and left cells. Combinatorial methods in representation theory (Kyoto, 1998). Adv. Stud. Pure Math. **28**, 1–20 (2000) (Kinokuniya, Tokyo)

Bonnafé, C., Iancu, L.: Left cells in type B_n with unequal parameters. Represent. Theory **7**, 587–609 (2003)

Iancu, L.: Cellules de Kazhdan-Lusztig et correspondance de Robinson-Schensted. C. R. Math. Acad. Sci. Paris **336**(10), 791–794 (2003)

Joseph, A.: A characteristic variety for the primitive spectrum of a semisimple Lie algebra. In: Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1976), Lecture Notes in Math., vol. 587, pp. 102–118. Springer, Berlin (1977)

Pietraho, T.: Sign under the domino Robinson-Schensted maps. Ann. Comb. **18**(3), 515–531 (2014)

Reifegerste, A.: Permutation sign under the Robinson-Schensted correspondence. Ann. Comb. **8**(1), 103–112 (2004)

Shephard, G.C., Todd, T.A.: Finite unitary reflection groups. Can. J. Math. **6**, 274–304 (1954)

Sjöstrand, J.: On the sign-imbalance of partition shapes. J. Combin. Theory Ser. A **111**(2), 190–203 (2005)

Stanley, R.: Some aspects of groups acting on nite posets. J. Combin. Theory Ser. A **32**(2), 132–161 (1982)

Verma, D.-N.: Möbius inversion for the Bruhat ordering on a Weyl group. Ann. Sci. École Norm. Sup. **4**(4), 393–398 (1971)