11-15-2011

The validity of the kdv approximation in case of resonances arising from periodic media

Christopher Chong
Universität Stuttgart

Guido Schneider
Universität Stuttgart

J. Math. Anal. Appl. 383 (2011) 330–336

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The validity of the KdV approximation in case of resonances arising from periodic media

Christopher Chong, Guido Schneider *

Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

ARTICLE INFO

Article history:
Received 19 October 2010
Available online 19 May 2011
Submitted by P. Broadbridge

Keywords:
KdV approximation
Error estimates
Resonances

ABSTRACT

It is the purpose of this short note to discuss some aspects of the validity question concerning the Korteweg–de Vries (KdV) approximation for periodic media. For a homogeneous model possessing the same resonance structure as it arises in periodic media we prove the validity of the KdV approximation with the help of energy estimates.

1. Introduction

There are various papers proving that a number of systems, such as the Fermi–Pasta–Ulam (FPU) system or the water wave problem, can be approximately described in the long wave limit by solutions of a formally derived Korteweg–de Vries (KdV) equation,

$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 A \partial_X A$$

with $\nu_1, \nu_2 \in$, $T \in$, $X \in$ and $A(X, T) \in$. See for instance [4,11–13,1,7]. Proving the validity of such approximations is a nontrivial task since the solutions, which are of order $\mathcal{O}(\varepsilon^2)$, have to be shown to exist on time scales of order $\mathcal{O}(\varepsilon^{-3})$ where $0 < \varepsilon \ll 1$ is the small perturbation parameter used for the description of the long wave limit, see e.g. Eq. (5).

One encounters new difficulties when trying to prove the validity of the KdV equation for systems with some kind of periodicity, such as polyatomic FPU systems or the water wave problem with a periodic bottom. A rst effort to address this issue can be found in Ref. [3], where the water wave problem with a long wave periodic bottom is addressed. In investigating the validity question for general periodic bottoms and the polyatomic FPU system one sees that the proof in the homogeneous case cannot be transferred line for line to the periodic case. The reason is the occurrence of a resonance which has not been handled before. It is the purpose of this work to address this issue in a model with the same resonance structure, but without additional technical difficulties arising from the periodicity of the system.

By validity, we mean that the error between the approximation based on the formally derived equation (KdV in our case) and the actual solution of the original problem is bounded over a long time interval ($\mathcal{O}(\epsilon^{-3})$ in our case), see e.g. Theorem 1. One method to bound this error is with use of energy estimates and Gronwall's inequality [8]. There are some systems where this method of energy estimates cannot be directly applied, such as systems with quadratic nonlinearities. A common strategy in such situations is to transform the problem, by means of a normal form transform, to an equivalent one where energy estimates and Gronwall's inequality can be applied. In performing such normal form transforms, the so-called non-resonance conditions arise, which are restrictions on the wavenumbers, see e.g. Eq. (4). In some instances, it

* Corresponding author.
E-mail address: guido.schneider@mathematik.uni-stuttgart.de (G. Schneider).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.05.028

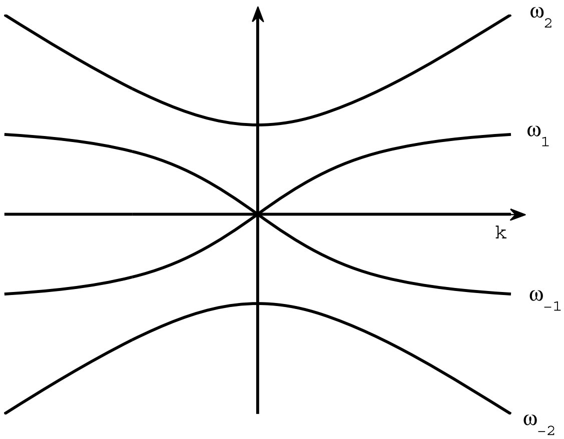

Fig. 1. Spectral situation corresponding to a coupled Boussinesq and Klein–Gordon system.

turns out that the validity result can still be shown *in spite* of resonances. How to deal with these resonances is a nontrivial issue, and each type must be handled separately. The type of resonance we deal with in this note follows from the spectral situation depicted in Fig. 1.

The model we consider is a Boussinesq equation coupled with a Klein–Gordon equation, namely

$$\partial_t^2 v = \partial_x^2 v - v + u^2 + 2uv + v^2, \tag{1}$$

$$\partial_t^2 u = \partial_x^2 u + \partial_t^2 \partial_x^2 u + \partial_x^2 \big(u^2 + 2uv + v^2\big), \tag{2}$$

where $u = u(x,t)$, $v = v(x,t)$, $x, t \in$. The curves of eigenvalues are given by

$$_{\pm 1}(k)^2 = k^2/\big(1+k^2\big) \quad \text{and} \quad _{\pm 2}(k)^2 = k^2 + 1. \tag{3}$$

We will derive KdV equation for the u-variable and validate the corresponding approximation. The v-variable is the counterpart of the additional modes of the periodic FPU and water wave problem (see Section 3) that introduce the new difficulty. Namely, when trying to eliminate terms in the error equation resulting from the v-variable, one obtains the following non-resonance condition, see Section 2.2,

$$\inf_{j,n \in \{-2,-1,1,2\}, k \in} \big| \; _j(k) - \; _1(0) - \; _n(k)\big| > 0, \tag{4}$$

which does not hold with the spectral situation considered here. In this note, we will show how to overcome this seemingly troublesome issue.

Notation. The many possible constants that are independent of $0 < \varepsilon \ll 1$ are denoted by C. The space $H^s(m)$ consists of s-times weakly differentiable functions for which $\|u\|_{H^s(m)} = \|u\rho^m\|_{H^s} = (\; _{j=0}^{s} \int |\partial_x^j (u\rho^m)|^2 dx)^{1/2}$ with $\rho(x) = \sqrt{1+x^2}$ is nite, where we do not distinguish between scalar and vector-valued functions or real- and complex-valued functions. We use H^s as an abbreviation for $H^s(0)$. Fourier transform of a function u is denoted with

$$(\mathcal{F}u)(k) = \hat{u}(k) = \frac{1}{2\pi} \int u(x) e^{-ikx}\, dx$$

and is an isomorphism between $H^s(m)$ and $H^m(s)$. The point-wise multiplication $(uv)(x) = u(x)v(x)$ in the x-space corresponds to the convolution

$$(\hat{u} * \hat{v})(k) = \int_{-} \hat{u}(k-l)\hat{v}(l)\, dl$$

in Fourier space. The pseudo-differential operator $(i\partial_x)$ in the x-space is de ned in Fourier space,

$$(i\partial_x)u(x) = \mathcal{F}^{-1}\big(\; (k)\hat{u}(k)\big)(x),$$

where (k) is a piece-wise analytic function.

2. The validity of the KdV approximation

In order to derive the KdV equation we make the ansatz

$$\epsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A\big(\varepsilon(x-t), \varepsilon^3 t\big) \quad \text{and} \quad \epsilon^2 \psi_v^{\mathrm{KdV}} = 0. \tag{5}$$

Inserting this ansatz in the u-equation yields

$$\begin{aligned}\mathrm{Res}_u &= -\partial_t^2 u + \partial_x^2 u + \partial_t^2 \partial_x^2 u + \partial_x^2\big(u^2 + 2uv + v^2\big) \\ &= \varepsilon^6\big(-2\partial_T \partial_X A - \partial_X^4 A + \partial_X^2\big(A^2\big)\big) + \mathcal{O}\big(\varepsilon^8\big),\end{aligned}$$

where $T = \varepsilon^2 t$ and $X = \varepsilon(x-t)$. Hence equating the coefficient of ε^6 to zero yields the KdV equation

$$-2\partial_T A - \partial_X^3 A + \partial_X\big(A^2\big) = 0. \tag{6}$$

The remainder of this work is dedicated to proving the following approximation theorem, which is also sketched at the end of the section.

Theorem 1. *Let $A \in C([0,T_0], H^8(,))$ be a solution of the KdV equation* (6)*. Then there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions (u,v) of* (1)–(2) *such that*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \sup_{x\in} \big|(u,v)(x,t) - \big(\epsilon^2 \psi_u^{\mathrm{KdV}}(x,t), 0\big)\big| \quad C\varepsilon^{7/2}.$$

Remark 2. As already said the proof of such an approximation theorem is a nontrivial task since solutions of order $\mathcal{O}(\varepsilon^2)$ have shown to exist on time scales of order $\mathcal{O}(\varepsilon^{-3})$. There are counterexamples where formally derived amplitude equations make incorrect predictions, cf. [9].

2.1. Estimates for the residual

For the proof of the approximation theorem we need the formal error, i.e. the residual, to be made small. Thus, besides Res_u we require that

$$\mathrm{Res}_v = -\partial_t^2 v + \partial_x^2 v - v + u^2 + 2uv + v^2$$

is sufficiently small. Using ansatz (5) we have

$$\mathrm{Res}_v = \varepsilon^4 A^2\big(\varepsilon(x-t), \varepsilon^3 t\big).$$

Since this is too large for our purposes we modify our previous approximation by adding higher order terms, namely we consider

$$\varepsilon^2 \psi_u = \varepsilon^2 A\big(\varepsilon(x-t), \varepsilon^3 t\big), \qquad \varepsilon^4 \psi_v = \varepsilon^4 B_1\big(\varepsilon(x-t), \varepsilon^3 t\big) + \varepsilon^6 B_2\big(\varepsilon(x-t), \varepsilon^3 t\big). \tag{7}$$

Throughout the remainder of this note we will work with ansatz (7). The result in Theorem 1 follows since the modi ed ansatz is sufficiently close to ansatz (5). By choosing

$$B_1 = A^2 \quad \text{and} \quad B_2 = 2AB_1$$

we nd

$$\begin{aligned}\mathrm{Res}_u &= \varepsilon^8\big(-\partial_T^2 A + 2\partial_X^2(AB_1) - 2\partial_T \partial_X^3 A\big) + \mathcal{O}\big(\varepsilon^{10}\big), \\ \mathrm{Res}_v &= \varepsilon^8\big(2\partial_X \partial_T B_1 + 2AB_2 + B_1^2\big) + \mathcal{O}\big(\varepsilon^{10}\big).\end{aligned}$$

Lemma 3. *Fix $s \geqslant 2$ and let $A \in C([0,T_0], H^{s+6}(,))$ be a solution of the KdV equation* (6)*. Then there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \big(\|\mathrm{Res}_u\|_{H^s} + \|\mathrm{Res}_v\|_{H^s}\big) \quad C\varepsilon^{15/2}.$$

Proof. The assumption $A(\cdot,T) \in H^{s+6}(,)$ is necessary to estimate $\partial_T^2 B_2 \in H^s(,)$ via $B_2 = \mathcal{O}(B_1)$, $B_1 = \mathcal{O}(A^2)$, and $\partial_T A = \mathcal{O}(\partial_X^3 A)$. For presentation purposes, we did not state all terms of Res_u and Res_v explicitly (we wrote instead $\mathcal{O}(\varepsilon^{10})$). They can be computed in a straight-forward way, and we have sufficient regularity such that the necessary estimates can be obtained. The loss of $\varepsilon^{-1/2}$ comes from the scaling properties of the L^2-norm,

$$\int \big|A(\varepsilon x)\big|^2\,dx = \varepsilon^{-1} \int \big|A(X)\big|^2\,dX.$$

2.2. The error estimates

We de ne the error functions R_u and R_v through $\varepsilon^\beta R_u = u - \varepsilon^2\psi_u$ and $\varepsilon^\beta R_v = v - \varepsilon^4\psi_v$ with $\beta = 7/2$. They satisfy

$$\partial_t^2 R_v = \partial_x^2 R_v - R_v + \underline{2\varepsilon^2\psi_u R_u} + \underline{\underline{2\varepsilon^2\psi_u R_v}} + \varepsilon^3 g_v, \tag{8}$$

$$\partial_t^2 R_u = \partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + \partial_x^2\big(2\varepsilon^2\psi_u R_u + \underline{2\varepsilon^2\psi_u R_v}\big) + \varepsilon^3 g_u, \tag{9}$$

where the terms g_v and g_u provide high enough orders w.r.t. ε such that they cause no difficulties in arriving at the $\mathcal{O}(\varepsilon^{-3})$ time scale. In detail, we have that their H^s-norm can be estimated by

$$C\big(\varepsilon\big(\|R_u\|_{H^s} + \|R_v\|_{H^s}\big) + \varepsilon^{1/2}\big(\|R_u\|_{H^s} + \|R_v\|_{H^s}\big)^2 + \varepsilon\big).$$

The residual terms Res_u and Res_v are contained in g_u and g_v respectively, which is why we require Lemma 3. Throughout this note, several other higher order terms will arise which we will abbreviate in a similar way. The terms that *do* cause difficulties are those in Eqs. (8)–(9) with an ε^2 coefficient. The term which is not underlined is already present in the classical case and can be controlled with a suitable chosen energy due to the presence of the second spatial derivative. See the estimate of the term s_2 de ned in (13), for example.

We start by trying to eliminate the terms that are underlined once. In order to do so we write (8)–(9) as a rst order system, which in Fourier space has the form

$$\begin{aligned}
\partial_t\widehat{R}_v &= \;_2\widehat{W}_v,\\
\partial_t\widehat{W}_v &= -\;_2\widehat{R}_v + \;_2^{-1}\big(\underline{2\varepsilon^2\hat{\psi}_u * \widehat{R}_u} + \underline{\underline{2\varepsilon^2\hat{\psi}_u * \widehat{R}_v}}\big) + \varepsilon^3\check{g}_v,\\
\partial_t\widehat{R}_u &= \;_1\widehat{W}_u,\\
\partial_t\widehat{W}_u &= -\;_1\widehat{R}_u + \;_1\big(2\varepsilon^2\hat{\psi}_u * \widehat{R}_u + \underline{2\varepsilon^2\hat{\psi}_u * \widehat{R}_v}\big) + \varepsilon^3\check{g}_u,
\end{aligned}$$

where $\widehat{W}_u = \;_2^{-1}\partial_t\widehat{R}_u$ and $\widehat{W}_v = \;_1^{-1}\partial_t\widehat{R}_v$, and $\;_{\pm1}$ and $\;_{\pm2}$ are de ned in Eq. (3). The $H^0(s)$-norm of the terms $\check{g}_u(l,t) = \;_1(l)^{-1}\hat{g}_u(l,t)$ and $\check{g}_v(l,t) = \;_1(l)^{-1}\hat{g}_v(l,t)$, where $\hat{g}_u$ and $\hat{g}_v$ are the Fourier transform of g_u and g_v, can be estimated by

$$C\big(\varepsilon\big(\|\widehat{R}_u\|_{H^0(s)} + \|\widehat{R}_v\|_{H^0(s)}\big) + \varepsilon^{1/2}\big(\|\widehat{R}_u\|_{H^0(s)} + \|\widehat{R}_v\|_{H^0(s)}\big)^2 + 1\big).$$

The reasons are as follows. Since the nonlinear terms in (2) have two spatial derivatives in front, in Fourier space they are $\mathcal{O}(l^2)$, and so the application of $\;_1(l)^{-1}$ is well-de ned for all the terms containing $\widehat{R}_u$ and $\widehat{R}_v$ and for most terms from the residual. The terms which remain in the residual are time derivatives. They can be expressed via the right hand side of the KdV equation as terms with spatial derivatives in front. Hence, in Fourier space all terms in $\hat{g}_u(l,t)$ have at least an l factor in front and so the application of $\;_1(l)^{-1}$ to these terms is well-de ned. However, in the residual there is a loss of $\mathcal{O}(\varepsilon^{-1})$ since one derivative is canceled by the application of $\;_1(l)^{-1}$. Such a loss does not occur in the linear and nonlinear terms w.r.t. $\widehat{R}_u$ and $\widehat{R}_v$ since their order w.r.t. ε purely comes from the amplitude and not from the long wave character of the ansatz.

We diagonalize this system with

$$\begin{pmatrix}\widehat{R}_v\\ \widehat{W}_v\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}\widehat{R}_2\\ \widehat{R}_{-2}\end{pmatrix}, \qquad \begin{pmatrix}\widehat{R}_u\\ \widehat{W}_u\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}\widehat{R}_1\\ \widehat{R}_{-1}\end{pmatrix},$$

and nd

$$\begin{aligned}
\partial_t\widehat{R}_2 &= i\;_2 R_2 - i\;_2^{-1}\big(\underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_1 + \widehat{R}_{-1})} + \underline{\underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_2 + \widehat{R}_{-2})}}\big) + \varepsilon^3 g_2,\\
\partial_t\widehat{R}_{-2} &= -i\;_2\widehat{R}_{-2} + i\;_2^{-1}\big(\underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_1 + \widehat{R}_{-1})} + \underline{\underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_2 + \widehat{R}_{-2})}}\big) + \varepsilon^3 g_{-2},\\
\partial_t\widehat{R}_1 &= i\;_1\widehat{R}_1 - i\;_1\big(\varepsilon^2\hat{\psi}_u * (\widehat{R}_1 + \widehat{R}_{-1}) + \underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_2 + \widehat{R}_{-2})}\big) + \varepsilon^3 g_1,\\
\partial_t\widehat{R}_{-1} &= -i\;_1\widehat{R}_{-1} + i\;_1\big(\varepsilon^2\hat{\psi}_u * (\widehat{R}_1 + \widehat{R}_{-1}) + \underline{\varepsilon^2\hat{\psi}_u * (\widehat{R}_2 + \widehat{R}_{-2})}\big) + \varepsilon^3 g_{-1}.
\end{aligned}$$

The $H^0(s)$-norm of the terms $g_{-2},\dots,g_2$ can be estimated by

$$C\big(\varepsilon\big(\|\widehat{R}_{-2}\|_{H^0(s)} + \cdots + \|\widehat{R}_2\|_{H^0(s)}\big) + \varepsilon^{1/2}\big(\|\widehat{R}_{-2}\|_{H^0(s)} + \cdots + \|\widehat{R}_2\|_{H^0(s)}\big)^2 + 1\big).$$

The terms which are underlined once can be eliminated by a near identity change of coordinates

$$\widetilde{\mathcal{R}} = \mathcal{R} + \varepsilon^2\mathcal{M}(\psi_u, \mathcal{R}) \tag{10}$$

with $\widetilde{R} = (R_2, R_{-2}, R_1, R_{-1})$, $\mathcal{R} = (\mathcal{R}_2, \mathcal{R}_{-2}, \mathcal{R}_1, \mathcal{R}_{-1})$, and $\mathcal{M}$ a suitably chosen bilinear mapping which in Fourier space has the form

$$\widehat{\mathcal{M}}_j(\hat{\psi}_u, \widehat{\mathcal{R}}) = \sum_{j_1 \in \{-2,-1,1,2\}} \int m_{jj_1}(k) \hat{\psi}_u(k-m) \widehat{\mathcal{R}}_{j_1}(m)\, dm$$

for $j \in \{-2,-1,1,2\}$. Direct computation leads to $m_{22} = m_{2-2} = m_{-2-2} = m_{-22} = m_{11} = m_{1-1} = m_{-1-1} = m_{-11} = 0$ and

$$\begin{aligned}
m_{21}(k) &= i \ {}_2^{-1}(k)/\big(\ {}_2(k) - \ {}_1(0) - \ {}_1(k)\big),\\
m_{2-1}(k) &= i \ {}_2^{-1}(k)/\big(\ {}_2(k) - \ {}_1(0) - \ {}_{-1}(k)\big),\\
m_{-2-1}(k) &= -i \ {}_2^{-1}(k)/\big(\ {}_{-2}(k) - \ {}_1(0) - \ {}_{-1}(k)\big),\\
m_{-21}(k) &= -i \ {}_2^{-1}(k)/\big(\ {}_{-2}(k) - \ {}_1(0) - \ {}_1(k)\big),\\
m_{12}(k) &= i \ {}_1(k)/\big(\ {}_1(k) - \ {}_1(0) - \ {}_2(k)\big),\\
m_{1-2}(k) &= i \ {}_1(k)/\big(\ {}_1(k) - \ {}_1(0) - \ {}_{-2}(k)\big),\\
m_{-1-2}(k) &= -i \ {}_1(k)/\big(\ {}_{-1}(k) - \ {}_1(0) - \ {}_{-2}(k)\big),\\
m_{-12}(k) &= -i \ {}_1(k)/\big(\ {}_{-1}(k) - \ {}_1(0) - \ {}_2(k)\big),
\end{aligned}$$

where we used the fact that ψ_u is strongly concentrated close to the wavenumber $k = 0$. A more detailed description of the above normal form transform can be found in several other works, see e.g. [10, Section 2.2]. By avoiding the terms that are underlined twice in the normal form transform, we arrive at the less restrictive condition,

$$\begin{aligned}
&\inf_{k\in} \big| \ {}_{\pm 2}(k) - \ {}_1(k) - \ {}_{\pm 1}(k)\big| > 0,\\
&\inf_{k\in} \big| \ {}_{\pm 1}(k) - \ {}_1(k) - \ {}_{\pm 2}(k)\big| > 0,
\end{aligned}$$

which is satis ed with spectral situation considered here, see Fig. 1. Since all m_{jj_1} are uniformly bounded the transformation $I + \varepsilon^2 \mathcal{M}$ is a smooth linear mapping from H^s to H^s for every $s \geqslant 0$. Therefore, after the normal form transform we have successfully eliminated the terms with a single underline, resulting in the system

$$\begin{aligned}
\partial_t \widehat{\mathcal{R}}_2 &= i \ {}_2 \widehat{\mathcal{R}}_2 - i \ {}_2^{-1} \underline{\underline{\big(\varepsilon^2 \hat{\psi}_u * (\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_{-2})\big)}} + \varepsilon^3 \widetilde{g}_2,\\
\partial_t \widehat{\mathcal{R}}_{-2} &= -i \ {}_2 \widehat{\mathcal{R}}_{-2} + i \ {}_2^{-1} \underline{\underline{\big(\varepsilon^2 \hat{\psi}_u * (\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_{-2})\big)}} + \varepsilon^3 \widetilde{g}_{-2},\\
\partial_t \widehat{\mathcal{R}}_1 &= i \ {}_1 \widehat{\mathcal{R}}_1 - i \ {}_1 \big(\varepsilon^2 \hat{\psi}_u * (\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_{-1})\big) + \varepsilon^3 \widetilde{g}_1,\\
\partial_t \widehat{\mathcal{R}}_{-1} &= -i \ {}_1 \widehat{\mathcal{R}}_{-1} + i \ {}_1 \big(\varepsilon^2 \hat{\psi}_u * (\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_{-1})\big) + \varepsilon^3 \widetilde{g}_{-1}.
\end{aligned}$$

The $H^0(s)$-norm of the terms $\widetilde{g}_{-2}, \ldots, \widetilde{g}_2$ can be estimated by

$$C\Big(\big(\|\widehat{\mathcal{R}}_{-2}\|_{H^0(s)} + \cdots + \|\widehat{\mathcal{R}}_2\|_{H^0(s)}\big) + \varepsilon^{1/2}\big(\|\widehat{\mathcal{R}}_{-2}\|_{H^0(s)} + \cdots + \|\widehat{\mathcal{R}}_2\|_{H^0(s)}\big)^2 + 1\Big).$$

If we try to proceed in the same way to eliminate the term which is underlined twice, one sees that the non-resonance condition (4) is violated since $\ {}_2(k) = \ {}_1(0) + \ {}_2(k)$. Moreover, there is no spatial derivative present in this term, and so partial integration w.r.t. x cannot be applied to gain higher powers of the small perturbation parameter ε. Therefore, a very serious difficulty seems to be present in the problem. Surprisingly, simple energy estimates are sufficient to estimate this term. We start by unapplying the diagonalization and returning to the x-space,

$$\begin{aligned}
\partial_t \mathcal{R}_v &= \ {}_2(i\partial_x) \mathcal{W}_v + \varepsilon^3 \grave{g}_{-2},\\
\partial_t \mathcal{W}_v &= - \ {}_2(i\partial_x) \mathcal{R}_v - \ {}_2^{-1}(i\partial_x) \underline{\underline{\big(2\varepsilon^2 \psi_u \mathcal{R}_v\big)}} + \varepsilon^3 \grave{g}_{-1},\\
\partial_t \mathcal{R}_u &= \ {}_1(i\partial_x) \mathcal{W}_u + \varepsilon^3 \grave{g}_1,\\
\partial_t \mathcal{W}_u &= - \ {}_1(i\partial_x) \mathcal{R}_u - \ {}_1(i\partial_x) \big(2\varepsilon^2 \psi_u \mathcal{R}_u\big) + \varepsilon^3 \grave{g}_2.
\end{aligned} \tag{11}$$

The H^s-norm of the terms $\grave{g}_{-2}, \ldots, \grave{g}_2$ can be estimated by

$$C\Big(\big(\|\mathcal{R}_v\|_{H^s} + \cdots + \|\mathcal{W}_u\|_{H^s}\big) + \varepsilon^{1/2}\big(\|\mathcal{R}_v\|_{H^s} + \cdots + \|\mathcal{W}_u\|_{H^s}\big)^2 + 1\Big).$$

We consider the energy

$$\frac{1}{2} \ {}_0 = \sum_{j=0}^{s-1} \bigg(\int \big|\partial_x^j \ {}_2 \mathcal{R}_v\big|^2 dx + \int \big|\partial_x^j \ {}_2 \mathcal{W}_v\big|^2 dx \bigg) + \sum_{j=0}^{s} \bigg(\int \big|\partial_x^j \mathcal{R}_u\big|^2 dx + \int \big|\partial_x^j \mathcal{W}_u\big|^2 dx \bigg),$$

where we have written $\ _j$ instead of $\ _j(i\partial_x)$ for notational simplicity. We compute $\partial_t \ _0$ and nd that the autonomous linear terms cancel so that

$$\partial_t \ _0 = -\varepsilon^2 s_1 - \varepsilon^2 s_2 + \varepsilon^3 g$$

where

$$s_1 = 2\sum_{j=0}^{s-1} \int \big(\partial_x^j \ \ _2 \mathcal{W}_v\big)\psi_u\big(\partial_x^j \mathcal{R}_v\big)\,dx, \tag{12}$$

$$s_2 = 2\sum_{j=0}^{s} \int \big(\partial_x^j \mathcal{W}_u\big) \ \ _1\big(\psi_u\big(\partial_x^j \mathcal{R}_u\big)\big)\,dx, \tag{13}$$

where we used $\partial_x\psi_u = \mathcal{O}(\varepsilon)$. We have the estimate

$$|g \ \ | \quad C\big(\ _0 + \varepsilon^{1/2} \ _0^{3/2} + 1\big).$$

Making use of (11) the terms s_1 and s_2 can be rewritten as

$$\begin{aligned} s_1 &= 2\sum_{j=0}^{s-1} \int \big(\partial_t\partial_x^j \mathcal{R}_v\big)\psi_u\big(\partial_x^j \mathcal{R}_v\big)\,dx + \varepsilon^2 g_{s_1} \\ &= \partial_t \sum_{j=0}^{s-1} \int \big(\partial_x^j \mathcal{R}_v\big)\psi_u\big(\partial_x^j \mathcal{R}_v\big)\,dx + \varepsilon g_{s_1,2}, \\ s_2 &= 2\sum_{j=0}^{s} \int \big(\partial_t\partial_x^j \mathcal{R}_u\big)\psi_u\big(\partial_x^j \mathcal{R}_u\big)\,dx + \varepsilon^2 g_{s_2} \\ &= \partial_t \sum_{j=0}^{s} \int \big(\partial_x^j \mathcal{R}_u\big)\psi_u\big(\partial_x^j \mathcal{R}_u\big)\,dx + \varepsilon g_{s_2,2}, \end{aligned}$$

where we used $\partial_t\psi_u = \mathcal{O}(\varepsilon)$. As a consequence we nd

$$\partial_t \ _1 = \varepsilon^3(g \ \ + g_{s_1,2} + g_{s_2,2}) = \varepsilon^3 g \ _1 \tag{14}$$

where

$$_1 = \ _0 + \varepsilon^2 \sum_{j=0}^{s-1} \int \big(\partial_x^j \mathcal{R}_v\big)\psi_u\big(\partial_x^j \mathcal{R}_v\big)\,dx + \varepsilon^2 \sum_{j=0}^{s} \int \big(\partial_x^j \mathcal{R}_u\big)\psi_u\big(\partial_x^j \mathcal{R}_u\big)\,dx$$

and where

$$|g \ _1| \quad C\big(\ _1 + \varepsilon^{1/2} \ _1^{3/2} + 1\big).$$

Hence, a simple application of Gronwall's inequality yields the $\mathcal{O}(1)$-boundedness of $\ _1$ for all $t \in [0, T_0/\varepsilon^3]$ for $\varepsilon > 0$ sufficiently small. Since $\|R_u + W_u + R_v + W_v\|_{H^s} \quad \sqrt{\ _1}$ for $\varepsilon > 0$ sufficiently small the result follows. A more detailed account of these last arguments can be found in various other works, see e.g. Ref. [10].

In order to increase the readability of the paper and to illustrate the robustness of the scheme w.r.t. future applications we provide a short sketch of what we have done.

Sketch of the proof. We write a true solution of (1)–(2) as approximation plus error, i.e., $u = \varepsilon^2\psi_u + \varepsilon^\beta R_u$ and $v = \varepsilon^4\psi_v + \varepsilon^\beta R_v$ with $\beta = 7/2$. The error satis es

$$\begin{aligned} \partial_t^2 R_v &= \partial_x^2 R_v - R_v + 2\varepsilon^2\psi_u R_u + 2\varepsilon^2\psi_u R_v + \mathcal{O}\big(\varepsilon^3\big), \\ \partial_t^2 R_u &= \partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + \partial_x^2\big(2\varepsilon^2\psi_u R_u + 2\varepsilon^2\psi_u R_v\big) + \mathcal{O}\big(\varepsilon^3\big). \end{aligned}$$

After elimination of the nonresonant terms the system decouples up to order $\mathcal{O}(\varepsilon^3)$, namely

$$\begin{aligned} \partial_t^2 R_v &= \partial_x^2 R_v - R_v + 2\varepsilon^2\psi_u R_v + \mathcal{O}\big(\varepsilon^3\big), \\ \partial_t^2 R_u &= \partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + \partial_x^2\big(2\varepsilon^2\psi_u R_u\big) + \mathcal{O}\big(\varepsilon^3\big). \end{aligned}$$

Then multiplying the first equation with $\partial_t R_v$ and the second equation with $\partial_t R_u$ gives after integration w.r.t. x the energy estimates

$$\partial_t \int \big((\partial_t R_v)^2 + (\partial_x R_v)^2 + (R_v)^2 - 2\varepsilon^2 \psi_u (R_v)^2 \\ + (\partial_t R_u)^2 + (\partial_x R_u)^2 + (\partial_t \partial_x R_u)^2 - 2\varepsilon^2 \psi_u (\partial_x R_u)^2\big)\, dx = \mathcal{O}\big(\varepsilon^3\big),$$

where we used partial integration, $\partial_t \psi_u = \mathcal{O}(\varepsilon)$, and $\partial_x \psi_u = \mathcal{O}(\varepsilon)$. Hence the integral stays $\mathcal{O}(1)$ bounded on an $\mathcal{O}(\varepsilon^{-3})$ time scale. Multiplication of the second equation with $\partial_t \partial_x^{-2} R_u$ and integration w.r.t. x yields estimates for the L^2-norm of R_u and $\partial_t \partial_x^{-1} R_u$.

3. Outlook

It is the goal of future research to transfer the method developed in this short note to the water wave problem with periodic bottom and to polyatomic FPU models. In the periodic setting, the solutions of the linearized equations are given by oscillations of Bloch waves $e^{i\omega_n(l)t} e^{ilx} w_n(l,x)$, with $w_n(l,x) = w_n(l, x + X_p)$ where X_p is the underlying periodicity of the system and l is the spectral variable. The curves of eigenvalues ω_1 and ω_{-1} satisfy $\omega_1(0) = \omega_{-1}(0) = 0$, where all other curves of eigenvalues $\omega_n(l)$, which are not present in the corresponding spatially homogeneous case, are bounded away from zero. In this setting, the KdV equation can also be derived for the modes u_1 and u_{-1}. The modes u_n belonging to the new curves of eigenvalues $\omega_n(l)$, along with u_1 and u_{-1}, are resonant with themselves, just as we had here with v playing the role of u_n. A consequence is that the $u_n u_1$ appearing in the equation for the error cannot be eliminated with the usual normal form transform in the equation corresponding to u_n. However, we showed in this article that these terms can be handled with the use of energy estimates. For polyatomic FPU models only finitely many curves of eigenvalues occur and thus, we believe the only remaining difficulties in proving an approximation result in this periodic setting are of a technical nature since the Bloch wave transform must be used instead of the Fourier transform. However, normal form transforms in Bloch space are an involved task, as can be seen for instance in [2], due to the infinitely many curves of eigenvalues which occur in general. Moreover, in general normal form transforms of quasi-linear systems leads to a loss of regularity. This loss occurs for the water problem and has not been solved so far in case of finite depth.

Interestingly, the same resonance structure occurs in case of modulations of periodic wave trains in dispersive systems. With a different scaling Whitham's equations can be derived for the same modes u_1 and u_{-1}. An approximation theorem has been proved in [6] in case that the other curves ω_n are not present. In case of lattice equations approximation results can be found in [5] in the limit of linear equations and in the hard sphere limit. Again such approximation proofs are a nontrivial task since solutions of order $\mathcal{O}(1)$ have shown to exist on time scales of order $\mathcal{O}(\varepsilon^{-1})$ where ε is defined via a spatial scaling. The validity of Whitham's equations is still one of the major open problems in the theory of modulation equations. It will be the subject of future research to investigate how the above ideas can be used to solve, at least partially, this important problem.

Acknowledgments

This work is partially supported by the Deutsche Forschungsgemeinschaft (DFG) grant SCHN 520/8-1.

References

[1] Jerry L. Bona, Thierry Colin, David Lannes, Long wave approximations for water waves, Arch. Ration. Mech. Anal. 178 (3) (2005) 373–410.
[2] K. Busch, G. Schneider, L. Tkeshelashvili, H. Uecker, Justification of the nonlinear Schrödinger equation in spatially periodic media, Z. Angew. Math. Phys. 57 (2006) 1–35.
[3] Florent Chazel, Influence of bottom topography on long water waves, M2AN Math. Model. Numer. Anal. 41 (4) (2007) 771–799.
[4] Walter Craig, An existence theory for water waves and the Boussinesq and Korteweg–de Vries scaling limits, Comm. Partial Differential Equations 10 (8) (1985) 787–1003.
[5] W. Dreyer, M. Herrmann, A. Mielke, Micro–macro transition in the atomic chain via Whitham's modulation equation, Nonlinearity 19 (2) (2006) 471–500.
[6] Wolf-Patrick Düll, Guido Schneider, Validity of Whitham's equations for the modulation of periodic traveling waves in the NLS equation, J. Nonlinear Sci. 19 (5) (2009) 453–466.
[7] Tatsuo Iguchi, A mathematical justification of the forced Korteweg–de Vries equation for capillary-gravity waves, Kyushu J. Math. 60 (2) (2006) 267–303.
[8] Pius Kirrmann, Guido Schneider, Alexander Mielke, The validity of modulation equations for extended systems with cubic nonlinearities, Proc. Roy. Soc. Edinburgh Sect. A 122 (1–2) (1992) 85–91.
[9] Guido Schneider, Validity and limitation of the Newell–Whitehead equation, Math. Nachr. 176 (1995) 249–263.
[10] Guido Schneider, Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances, J. Differential Equations 216 (2) (2005) 354–386.
[11] Guido Schneider, C. Eugene Wayne, Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi–Pasta–Ulam model, in: International Conference on Differential Equations, vols. 1, 2, Berlin, 1999, World Sci. Publ., River Edge, NJ, 2000, pp. 390–404.
[12] Guido Schneider, C. Eugene Wayne, The long-wave limit for the water wave problem. I. The case of zero surface tension, Comm. Pure Appl. Math. 53 (12) (2000) 1475–1535.
[13] Guido Schneider, C. Eugene Wayne, The rigorous approximation of long-wavelength capillary-gravity waves, Arch. Ration. Mech. Anal. 162 (3) (2002) 247–285.