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The Structure and Unitary Representations of SU(2,1)

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An Honors Paper for the Department of Mathematics

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Contents

1. Introduction

In this paper, we are concerned with studying the representations of the Lie group $G = SU(2,1)$. In particular, we want to classify its **unitary dual**, denoted $\widehat{G}$, consisting of all equivalence classes of irreducible unitary group representations. Our main motivation for this is to give an explicit Plancherel formula for $SU(2,1)$, generalizing the notion of the classical Fourier transform on $\mathbb{R}$. To motivate the study of unitary representations in this context, we consider the theory of Fourier series which decomposes an arbitrary function in $L^2(S^1)$ into a discrete sum of imaginary exponentials $e^{2\pi int}$, where S^1 is the circle group and

$$L^2(S^1) = \{f : S^1 \to \mathbb{C} \mid \int_0^1 |f(e^{2\pi it})|^2 dt < \infty\}.$$

The Fourier transform $\hat{f}(n)$ of $f \in L^2(S^1)$ is given by

$$\hat{f}(n) = \int_0^1 f(t)e^{-2\pi int}dt$$

Then by the Fourier Inversion formula, $f \in L^2(S^1)$ can be expressed as a convergent Fourier series, with coefficients $\hat{f}(n) \in \mathbb{C}$ defined as above:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int}.$$

Moreover,

$$\int_0^1 |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty.$$

This last line applied to S^1 is known as the Plancherel theorem. We can interpret this in terms of group representations for S^1 acting on itself by rotation. Elements $g \in S^1$ are given by $g = e^{2\pi it}$, and since the group is abelian, all of its irreducible representations are known to be one-dimensional. They are, in fact, unitary and indexed by the integers $n \in \mathbb{Z}$, given by $\pi_n(e^{2\pi it}) = (e^{2\pi it})^n$. These unitary representations are precisely the imaginary exponentials that decompose $f \in L^2(S^1)$ into a discrete sum.

Now consider the analogous theory of the Fourier transform on $\mathbb{R}$. We note that the noncompactness of the real line forces the decomposition of an element of $L^2(\mathbb{R})$ to no longer be discrete (Pg. 5 in [**8**]). The classical Fourier transform $\hat{f}$ of a function $f \in L^2(\mathbb{R})$ is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$$

Then by the Fourier Inversion formula, $f \in L^2(\mathbb{R})$ can be expressed as a convergent Fourier integral, with "coefficients" $\hat{f}(\xi) \in \mathbb{C}$ defined as above:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi}d\xi.$$

Moreover,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi < \infty.$$

The imaginary exponentials $(e^{-2\pi ix})^\xi$ here indexed by $\xi \in \mathbb{R}$, are all the irreducible unitary representations of $\mathbb{R}$. Loosely speaking, we are decomposing an element $f \in L^2(\mathbb{R})$ into a "continuous sum", i.e., an integral, of irreducible unitary representations which are weighted by the terms $\hat{f}(\xi)$.

It is natural to ask whether this sort of Fourier analysis can generalize to a theory on more general Lie groups G that are not necessarily compact or abelian. In essence, we want to express an element of $L^2(G)$ by decomposing it into a weighted sum (discrete or continuous) of unitary representations where

$$L^2(G) = \{f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty\} \text{ for } dg \text{ a left-invariant Haar measure on } G.$$

Of course, there are an immense number of technical issues when making this generalization. Determining the set of irreducible unitary representations is the most daunting task, and remains an unsolved problem for general G. Also of particular importance is the need for a measure on $\widehat{G}$ which makes sense for the group in question. For the case of S^1 which is compact, the measure only needed to be discrete, while for $\mathbb{R}$, the measure was continuous. It turns out that $SU(2,1)$ falls into a class of groups, which requires a measure defined both on a continuous series of representations as well as on a separate discrete collection.

Our main goal in this paper is to determine $\widehat{G}$ for $G = SU(2,1)$. Working towards constructing the desired unitary representations of $SU(2,1)$, we will primarily be focused on investigating the group structure of $SU(2,1)$.

2. Definitions and Notation

2.1. Representation Theory.

Definition 2.1. Let G be a set and $*$ a binary operation on G. The set G is a **group** if and only if the following hold for all $g_1, g_2, g_3 \in G$

(i) Closure: $g_1 * g_2 \in G$
(ii) Associativity: $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$
(iii) Identity: There exists $e \in G$ such that for all $g \in G$, $e * g = g * e = g$
(iv) Inverses: For all $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$

Groups often arise in mathematics as the symmetries of some object we wish to study. We can think of the group as consisting of the symmetry transformations acting on the object under study. The above definition allows for many varieties of groups with potentially exotic group actions. One important technique for obtaining more concrete realizations of an abstract group is to consider its **representations** on a vector space. As the definition below indicates, group elements are realized as linear transformations acting on a vector space, with composition replacing the group action. When the vector space is finite-dimensional, we can realize the group elements as matrices and the group action as matrix multiplication.

Definition 2.2. Suppose $(G, *)$ is a group. A **finite-dimensional representation** (π, V) of G on a finite dimensional complex vector space V of dimension n is a group homomorphism

$$\pi : G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ represents the group of invertible linear transformations on V under composition, or equivalently all $n \times n$ matrices M such that $\det(M) \neq 0$ under matrix multiplication. Thus we can think of $\pi(g)$ as an invertible matrix acting on the vector space V and we call V the representation space of π.

We note that the representation π of G assigns a linear action to each element $g \in G$, namely $\pi(g)$. Now it is possible to study the structure of the the vector space V as determined by the actions of all elements $\pi(g)$ on V. Representation theory is a powerful tool in the study of groups as it allows us to linearize the action of group elements and use techniques of linear algebra to study otherwise nonlinear objects.

Note that in the previous definition of representation, we required that V be finite-dimensional. While it is a more straight-forward definition to grasp, we will spend much of our time later discussing infinite-dimensional representations, so we need to define the notion of a representation acting on a Hilbert space.

Definition 2.3. A Hilbert space H is a real or complex, inner product space (not-necessarily finite-dimensional) that is also a complete metric space with respect to the distance function induced by the inner product. The inner product on H will be denoted $\langle v, w\rangle$ for all $v, w \in H$. It must be sesquilinear, conjugate symmetric, and positive-definite. The norm of v is given by

$$|v| = \sqrt{\langle v, v\rangle}$$

We say that a linear operator $T : H \to H$ is bounded if and only if there exists $c < \infty$ such that for all $v \in H$, we have $|T(v)| \leqslant c|v|$. A linear transformation $T : H \to H$ is bounded if and only if it is continuous under the metric induced by the inner product.

Definition 2.4. Suppose $(G, *)$ is a group, H is a complex Hilbert space, and $B(H)$ is the group of bounded linear operators on H with bounded inverses under composition. A **representation** (π, H) of G on H is a group homomorphism

$$\pi : G \to B(H)$$

such that the map $G \times H \to H$ given by $(g, v) \mapsto \pi(g)v$ is continuous. In this case $\pi(g)$ is no longer necessarily represented by a matrix, but more generally as a bounded linear operator on H.

When studying representations, we will be interested in questions about the linearized action of group elements acting on H, so we will first introduce some relevant definitions. We know that $\pi(g)$ is a linear transformation, so we may think of it as represented by a matrix (though this is only in an abstract sense for infinite-dimensional π). Thus we may think of the coefficients of this matrix as functions $\Phi : G \to \mathbb{C}$ where

$$\Phi : g \mapsto \langle \pi(g)v, w\rangle \text{ for } v, w \in H$$

where H is the representation space of π and $\langle \cdot, \cdot\rangle$ is the inner product on H. Choosing a suitable basis $\{e_i\}_{i=1}^{\dim H}$ for H, the canonical matrix coefficients of the representation π are denoted $\Phi_{i,j}$ for all $1 \leqslant i, j \leqslant \dim H$ where $\Phi_{i,j} : g \mapsto \langle \pi(g)e_i, e_j\rangle$.

Definition 2.5. If (π, H) is a finite-dimensional representation on a Hilbert space H, its **character** is the function

$$\chi_\pi(g) = \operatorname{tr} \pi(g) = \sum_i \langle \pi(g) v_i, v_i \rangle$$

where v_i is an orthonormal basis for H. Note that there is a more sophisticated character theory for representations of infinite dimension that, while important for discussion of the Plancherel formula, is beyond the scope of this paper.

Definition 2.6. An **invariant subspace** of a representation (π, H) is a vector subspace $U \subseteq H$ such that $\pi(g)U \subseteq U$ for all $g \in G$. A representation (π, H) is said to be **irreducible** if it has no closed invariant subspaces other than 0 and H.

Definition 2.7. Let π and $\tilde{\pi}$ be representations of a group G acting on a representation spaces H and $\tilde{H}$. We say π and $\tilde{\pi}$ are **equivalent** if and only if there exists a bounded linear operator $T : H \to \tilde{H}$ with a bounded inverse such that

$$(T \circ \pi(g))v = (\tilde{\pi}(g) \circ T)v$$

for all $v \in H$. Such an operator T is called an **intertwining operator**.

For the purpose of generalizing the Fourier transform as was discussed in the introduction, we will be mostly concerned with classifying the unitary representations, so we need to define unitary operators.

Definition 2.8. A bounded linear operator $U : H \to H$ on a Hilbert space H is called a **unitary operator** if it satisfies $U^*U = UU^* = I$ where U^* is the adjoint of U and I is the identity operator. The adjoint U^* is defined by the map $U^* : H \to H$ satisfying

$$\langle Ux, y \rangle = \langle x, U^* y \rangle \quad \text{for all } x, y \in H$$

where $\langle *, * \rangle$ is the inner product on H. In the case where $H = \mathbb{C}^n$ and $\langle x, y \rangle = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$, then U will be an $n \times n$ matrix, and $U^* = \overline{U}^t$, the conjugate transpose of U.

Definition 2.9. A representation π of a group G is called a **unitary representation** if $\pi(g)$ is unitary for all $g \in G$, i.e.,

$$\pi(g)^*\pi(g) = \pi(g)\pi(g)^* = I$$

For any group G, we will denote the set of equivalence classes of irreducible unitary representations by $\widehat{G}$.

2.2. Matrix Lie Groups.

A Lie group is a differentiable manifold G which is also a group and such that the group product

$$G \times G \to G \quad \text{defined by } (g, h) \mapsto gh$$

and the inverse map $G \to G$ defined by $g \to g^{-1}$ are differentiable. In this way, a Lie group is simultaneously a smooth manifold and a group. Since we do not wish to discuss manifold theory in this paper, and because we are only concerned with Lie groups consisting of matrices, we introduce the notion of a matrix Lie group.

Definition 2.10. The **general linear group** of degree n over the real numbers, denoted $\mathrm{GL}(n, \mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. The general linear group of degree n over the complex numbers, denoted $\mathrm{GL}(n, \mathbb{C})$, is the group of all invertible $n \times n$ matrices with complex entries. Both are groups under matrix multiplication.

To proceed, we must first establish a notion of convergence for a sequence of matrices. Thus let A_m be a sequence of $n \times n$ matrices with complex entries (or real without loss of generality). We say that A_m converges to a matrix A if each entry of A_m converges to the corresponding entry of A.

Definition 2.11 (Definition 1.4 in [**2**]). A matrix Lie group is any subgroup G of $\mathrm{GL}(n, \mathbb{C})$ with the following property: If A_m is any sequence of matrices G, and A_m converges to some matrix A, then either $A \in G$ or A is not invertible, i.e., $A \notin \mathrm{GL}(n, \mathbb{C})$. In other words, a **matrix Lie group** is a closed subgroup G of $\mathrm{GL}(n, \mathbb{C})$ using the notion of convergence described above.

Definition 2.12 (Definition 1.6 in [**2**]). A matrix Lie group G is said to be **compact** if the following are satisfied:

(1) If A_m is a sequence of matrices in G that converges to a matrix A, then $A \in G$.
(2) There exists a constant C such that for all $A \in G$, $|A_{ij}| \leqslant C$ for all $1 \leqslant i, j \leqslant n$ where A_{ij} is the ijth entry of A.

This definition essentially amounts to claiming that G is compact if it is a closed and bounded subset of $M_n(\mathbb{C})$, the set of 3×3 matrices with complex entries, which can be thought of as $\mathbb{C}^{n^2}$. As is often the case, compactness is a favorable property, and the representation theory of compact groups is well understood. In particular, it is known that all irreducible unitary representations of a compact group G are finite-dimensional, i.e., they act on finite-dimensional representation spaces.

Since it is our goal to determine the irreducible unitary representations of $SU(2, 1)$, it certainly would be a nice simplifying convenience to be able to limit our search to finite-dimensional representations. Unfortunately, we are not so lucky to have such convenience since $SU(2, 1)$ happens to be non-compact because it contains a noncompact subgroup which will be discussed later. In fact, any nontrivial irreducible unitary representation of noncompact G is of infinite dimension (CH 11.1, [**5**]). Left with the daunting task of determining the infinite-dimensional representations of $SU(2, 1)$ that are unitary, we will turn our attention to studying well-behaved subgroups of $SU(2, 1)$, in particular those subgroups that are compact. This will involve inducing unitary representations of these nice subgroups to representations of the whole group, a process that will be discussed in detail when we introduce the principal series representations.

3. Relationship Between Lie Groups and Lie Algebras

An important technique in the study of a Lie group is being able to translate between the Lie group and its associated Lie algebra, a non-associative algebra built on the tangent space to the group's identity element.

Definition 3.1. A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ (or $\mathbb{C}$) is a finite dimensional real (or complex) vector space endowed with a Lie bracket operation, $[\cdot,\cdot]$ defined to have the following properties:

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]$$

where

(a) $[x, y]$ is linear in x and y
(b) $[x, x] = 0$ for all $x \in \mathfrak{g}$
(c) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$ (the Jacobi identity)

Let $\mathfrak{gl}(n, \mathbb{C})$ be the vector space of $n \times n$ matrices with complex entries. We define the Lie bracket on $\mathfrak{gl}(n, \mathbb{C})$ by $[x, y] = xy - yx$. Note that with this definition of the Lie bracket, the properties above hold for any $x, y, z \in \mathfrak{gl}(n, \mathbb{C})$.

The most concrete relationship between the Lie group and its Lie algebra is achieved via the exponential map, which for a Lie group G and its Lie algebra $\mathfrak{g}$ is denoted by $\exp : \mathfrak{g} \to G$. For a matrix Lie group, such as $SU(2, 1)$, the exponential map is given by the matrix exponential, which for an $n \times n$ matrix A, is given by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \ldots$$

It can be shown that the series defining $\exp(A)$ converges for any $n \times n$ matrix A. Thus $\exp : A \mapsto \exp(A)$ takes an element, A, of the Lie algebra $\mathfrak{g}$ to an element, $\exp(A)$, of the matrix Lie group G. While in general, $\exp : \mathfrak{g} \to G$ is neither surjective nor injective, there are open neighborhoods U and W of the identities 0 in $\mathfrak{g}$ and e in G, respectively, such that $\exp : U \to W$ is actually a bijection. For more on the construction of a Lie algebra as the tangent space to the identity of a Lie group, see Section I in [**7**]

In much of this paper, we will concentrate on Lie algebras, whose natural vector space structure makes them easier to work with than the associated Lie groups. We are primarily motivated by questions about the Lie group $SU(2, 1)$ and thus its Lie algebra $\mathfrak{su}(2, 1)$, but first, we will give some general facts about Lie algebras.

Definition 3.2. Suppose $\mathfrak{h}$ is a vector subspace of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ is a Lie subalgebra if and only if $\mathfrak{h}$ is closed under the Lie bracket. That is:

$$[x, y] \in \mathfrak{h} \quad \text{for all } x, y \in \mathfrak{h}$$

Definition 3.3. An **ideal** of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ and all $y \in \mathfrak{g}$.

Since $[x, y] = -[y, x]$, there is no distinction between left and right ideals in a Lie algebra. We say that a non-zero Lie algebra $\mathfrak{g}$ is **simple** if it is not abelian, i.e., $[\mathfrak{g}, \mathfrak{g}] \neq 0$, and if it has no ideals other than $\mathfrak{g}$ or 0. A more general class of Lie algebras built on simple Lie algebras are called semisimple. We say that a Lie algebra is **semisimple** if it is the direct sum of simple Lie algebras. Note that most of the Lie algebras that we will work with for the remainder of the paper fall under the classification of semisimple unless otherwise stated.

Before we define the various Lie groups and Lie algebras of interest to us, it is worth discussing the concept of complexification, which allows us to study complex Lie algebras even when our underlying object is a real Lie algebra or real Lie group.

Definition 3.4. If V is a finite-dimensional real vector space, then the complexification of V, denoted $V_{\mathbb{C}}$ is the space of formal linear combinations

$$v_1 + iv_2 \quad \text{with } v_1, v_2 \in V$$

This becomes a real vector space in the obvious way and becomes a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1$$

Proposition 3.5 (Proposition 2.44 in [**2**])**.** Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification as a real vector space. Then, the bracket operation on $\mathfrak{g}$ has a unique extension to $\mathfrak{g}_{\mathbb{C}}$ which makes $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra. We say the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is the **complexification** of the real Lie algebra $\mathfrak{g}$.

The extension of the bracket is done in the predictable way, since we require the new bracket operation on $\mathfrak{g}_{\mathbb{C}}$ to be bilinear, giving

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1])$$

It is shown following Proposition 2.44 in [**2**] that this bracket operation in fact satisfies the properties listed in Definition 3.1, implying that $\mathfrak{g}_{\mathbb{C}}$ is in fact a complex Lie algebra. We say a real Lie algebra $\mathfrak{g}$ is a **real form** of a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ if $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$. Note that while a real Lie algebra has only one unique complexification, a complex Lie algebra will generally have multiple non-isomorphic real forms.

Definition 3.6 (Definition 3.12 in [**2**])**.** If G is a matrix Lie group with Lie algebra $\mathfrak{g}$ and H is a Lie subgroup of G, then H is a **connected Lie subgroup** of G if the Lie algebra $\mathfrak{h}$ of H is a subspace of $\mathfrak{g}$ and every element $h \in H$ can be written in the form

$$h = \exp(X_1)\exp(X_2)\dots\exp(X_m)$$

with $X_1, X_2, \dots, X_m \in \mathfrak{h}$. Connected Lie subgroups are also called **analytic subgroups**.

Definition 3.7 (Pg. 437 [**7**])**.** The **complexification** of a Lie group G over $\mathbb{R}$ is a complex Lie group $G_{\mathbb{C}}$, containing G as an analytic subgroup such that the Lie algebra $\mathfrak{g}$ of G is a real form of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$. We refer to G as a real form of the Lie group $G_{\mathbb{C}}$.

At this point, let us define the Lie groups and Lie algebras of interest to us. These include the real Lie group $SU(2,1)$ and its real Lie algebra $\mathfrak{su}(2,1)$:

$$SU(2,1) = \left\{A \in GL(3,\mathbb{C}) : A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \det A = 1\right\}$$

$$\mathfrak{su}(2,1) = \left\{X \in M(3,\mathbb{C}) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} X^* + X \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0 \text{ and } \operatorname{tr} X = 0\right\}$$

where A^* denotes the conjugate transpose of A. The theory when working with a complex Lie algebra is often more elegant, so it useful to translate to the complexification of $\mathfrak{su}(2,1)$, the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. Thus we will define the complexifications of $SU(2,1)$ and $\mathfrak{su}(2,1)$, denoted by $SL(3,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$, respectively:

$$SL(3,\mathbb{C}) = \{A \in GL(3,\mathbb{C}) : \det(A) = 1\}$$
$$\mathfrak{sl}(3,\mathbb{C}) = \{A \in M(3,\mathbb{C}) : \mathrm{tr}\,(A) = 0\}$$

Any other groups and algebras that we discuss will be introduced as needed.

4. Root Space Decomposition

In this section we will discuss the general theory behind the root space decomposition of a complex Lie algebra $\mathfrak{g}$. After a discussion of the theory for general semisimple Lie algebras, we will focus on the example case of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra. A **representation** of $\mathfrak{g}$ on a complex vector space $V \neq 0$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, the Lie algebra of all linear transformations of V into itself with the bracket product $[A, B] = AB - BA$ for $A, B \in \mathfrak{gl}(V)$. In particular, π must preserve the Lie bracket, meaning

$$\pi[x, y] = \pi(x)\pi(y) - \pi(y)\pi(x) = [\pi(x), \pi(y)]$$

Let $\mathrm{End}\,\mathfrak{g}$ denote all linear maps $f : \mathfrak{g} \to \mathfrak{g}$. Suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and consider the adjoint map $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}\,\mathfrak{g}$ where for $x \in \mathfrak{h}$,

$$\mathrm{ad}(x) : y \mapsto [x, y] \quad \text{for all } y \in \mathfrak{g}$$

Note that the notation $\mathrm{ad}(x)y$ and $[x, y]$ can be used interchangeably when we deal with $x \in \mathfrak{h}$, however we think of these expressions in subtly different ways. The adjoint representation of the element x acting on y is denoted $\mathrm{ad}(x)y$, while $[x, y] = xy - yx$ describes the action of the Lie bracket on the two elements $x, y \in \mathfrak{g}$.

Of particular importance in studying the structure of a complex Lie algebra $\mathfrak{g}$ are its Cartan subalgebras. We give the conditions for a Cartan subalgebra for semisimple Lie algebras below. Note that the definition of a Cartan subalgebra for a general Lie algebra is technical and not particularly illuminating in our case where all of the primary Lie algebras we consider are semisimple, so we do not include it.

Proposition 4.2 (Corollary 2.13 in [**7**])**.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a **Cartan subalgebra** if and only if $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ such that for all $H \in \mathfrak{h}$, the maps $\mathrm{ad}\big|_H$ are diagonalizable.

An astute reader will notice that ad restricts to a representation of the Cartan subalgebra $\mathfrak{h}$ acting on the entire Lie algebra $\mathfrak{g}$. While a Lie algebra does not necessarily have a unique Cartan subalgebra, all Cartan subalgebras of a complex Lie algebra are conjugate under automorphisms of the Lie algebra by Theorem 2.15 in [**7**].

Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. By Proposition 4.2 we know that $\{\mathrm{ad}(x) : x \in \mathfrak{h}\}$ is simultaneously diagonalizable. Since these maps act on $\mathfrak{g}$, this fact allows us to find a simultaneous eigenspace decomposition of $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the eigenspace with eigenvalue 0, and each $\mathfrak{g}_\alpha$ is the eigenspace with eigenvalue α. Each α is in the dual space $\mathfrak{h}^*$, i.e., $\alpha : \mathfrak{h} \to \mathbb{C}$ is a linear transformation. The eigenspace $\mathfrak{g}_\alpha$ is the set of all $y \in \mathfrak{g}$ such that

$$\mathrm{ad}(x)y = \alpha(x)y$$

for all $x \in \mathfrak{h}$. We refer to α as a **root** of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}_\alpha \neq \{0\}$ and α is not identically zero. Let Φ denote the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. We note that in the case where $\mathfrak{g}$ is a complex Lie algebra, each $\mathfrak{g}_\alpha$ is a 1-dimensional complex eigenspace spanned by some vector x_α so that $\mathfrak{g}_\alpha = \mathbb{C}x_\alpha$. Thus we can rewrite the decomposition of $\mathfrak{g}$ above as:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C}x_\alpha$$

This is the **root space decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Note that $\dim \mathfrak{g} = \dim \mathfrak{h} + |\Phi|$.

The set of roots Φ spans $\mathfrak{h}^*$, however Φ is not a linearly independent set. Now we introduce an notion of positivity in $\mathfrak{h}^*$ by the following properties:

(1) for any non-zero $\lambda \in \mathfrak{h}^*$, exactly one of λ and $-\lambda$ is positive
(2) the sum of positive elements is positive, and an positive multiple of a positive element is positive

This determines a subset $\Phi^+ \subset \Phi$ we we call the **positive roots**. Note that such a set Φ^+ always exists by Theorem 6.36 in [**2**]. Within Φ^+ there exists a subset of "simple roots" Π: a positive root α is a **simple root** if and only if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots. While the choice of Φ^+ is not unique, once this subset is chosen, we can write any positive root as a non-negative combination of simple roots, and further results show that the particular choice of the positive roots does not matter in an important way for our purposes.

Now let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. One Cartan subalgebra of $\mathfrak{g}$ consists of the 3 x 3 diagonal matrices with trace zero with the following basis

$$\mathfrak{h} = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} = \mathrm{Span}\{H_1, H_2\} \tag{1}$$

Let E_{ij} denote the 3×3 matrix of all 0s except for a 1 at the ij-th entry, e.g.,

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and so on. Also note that any $x \in \mathfrak{h}$ can be written

$$x = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{where } \lambda_1 + \lambda_2 + \lambda_3 = 0$$

For any E_{ij}, we compute $\mathrm{ad}(x)E_{ij} = [x, E_{ij}] = xE_{ij} - E_{ij}x = (\lambda_i - \lambda_j)E_{ij}$. Thus the eigenspace spanned by E_{ij} has eigenvalue $\lambda_i - \lambda_j$. In order to describe this eigenvalue in terms of roots, we define $e_i \in \mathfrak{h}^*$ as follows:

$$e_i : \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mapsto \lambda_i$$

It therefore follows that the set of roots Φ equals $\{e_i - e_j\}_{i,j}$ for $1 \leqslant i \neq j \leqslant 3$. We choose $\Phi^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$.

Therefore we have simple roots $e_1 - e_2$ and $e_2 - e_3$, which we will denote by α_{12} and α_{23}, respectively. In this notation

$$\Phi = \{\alpha_{12},\ \alpha_{23},\ \alpha_{12} + \alpha_{23},\ -\alpha_{12},\ -\alpha_{23},\ -\alpha_{12} - \alpha_{23}\}. \tag{2}$$

With the choice

$$\Phi^+ = \{\alpha_{12},\ \alpha_{23},\ \alpha_{12} + \alpha_{23}\}, \tag{3}$$

we have

$$\Pi = \{\alpha_{12},\ \alpha_{23}\}. \tag{4}$$

With the set of roots determined, we can write the explicit root space decomposition of $\mathfrak{sl}(3, \mathbb{C})$ by finding a vector $X \in \mathfrak{sl}(3, \mathbb{C})$ to span each 1-dimensional $\mathfrak{g}_\alpha$. In other words, for each α, we want to find X such that for all $H \in \mathfrak{h}$, we have

$$[H, X] = \alpha(H)X$$

Such a vector X spans the space $\mathfrak{g}_\alpha$. These spaces are calculated as follows:

$$\begin{aligned} \mathfrak{g}_{\alpha_{12}} &= \mathrm{Span}\{E_{12}\} & \mathfrak{g}_{-\alpha_{12}} &= \mathrm{Span}\{E_{21}\} & \mathfrak{g}_{\alpha_{12}+\alpha_{23}} &= \mathrm{Span}\{E_{13}\} \\ \mathfrak{g}_{\alpha_{23}} &= \mathrm{Span}\{E_{23}\} & \mathfrak{g}_{-\alpha_{23}} &= \mathrm{Span}\{E_{32}\} & \mathfrak{g}_{-\alpha_{12}-\alpha_{23}} &= \mathrm{Span}\{E_{31}\} \end{aligned}$$

Thus we have the full explicit root space decomposition as follows:

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{h} \oplus \mathbb{C}E_{12} \oplus \mathbb{C}E_{23} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{32} \oplus \mathbb{C}E_{13} \oplus \mathbb{C}E_{31}$$

5. Classification of Finite-Dimensional Representations of $\mathfrak{sl}(3, \mathbb{C})$

In this section, we will explore the general notion of weight spaces of a Lie algebra and how weights can be used to classify the finite-dimensional representations of a Lie algebra. To develop this theory, we will again focus on $\mathfrak{sl}(3, \mathbb{C})$. We note that the classification of the finite-dimensional representations of $\mathfrak{sl}(3, \mathbb{C})$ is well known. One method of proof is based on a theorem of Hermann Weyl, often referred to as Weyl's unitary trick, which we give below specific to our case.

Theorem 5.1 (Weyl's Unitary Trick- Proposition 7.15 in [**7**]).

Let $G_{\mathbb{C}} = SL(3, \mathbb{C})$. We note that its Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ has real forms $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{su}(2, 1)$, and $\mathfrak{su}(3)$. For a finite dimensional complex vector space V, a representation of any of the following kinds on V leads, via the formula

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{sl}(3, \mathbb{R}) \oplus i\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{su}(3) \oplus i\mathfrak{su}(3) = \mathfrak{su}(2, 1) \oplus i\mathfrak{su}(2, 1)$$

to a representation of each of the other kinds:

(i) a representation of $SL(3, \mathbb{R})$ on V
(ii) a representation of $SU(3)$ on V
(iii) a representation of $SU(2, 1)$ on V
(iv) a holomorphic representation of $SL(3, \mathbb{C})$ on V
(v) a representation of $\mathfrak{sl}(3, \mathbb{R})$ on V
(vi) a representation of $\mathfrak{su}(3)$ on V
(vii) a representation of $\mathfrak{su}(2, 1)$ on V
(viii) a complex-linear representation of $\mathfrak{sl}(3, \mathbb{C})$ on V.

Note that holomorphic representation in (iv) means where the associated representation of $\mathfrak{sl}(3, \mathbb{C})$ is complex-linear. Moreover, under this correspondence, invariant subspaces and equivalences are preserved. Thus corresponding representations of all of the above types share the same reducibility properties. Most frequently, this "unitary" trick uses knowledge the unitary representations of the compact group, in this case $SU(3)$, to understand unitary representations of the other groups and algebras in the list. More generally, if we can classify and understand all finite-dimensional irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$, we also therefore understand the finite-dimensional irreducible representations of $SU(2, 1)$, (and $SL(3, \mathbb{R})$, and $SU(3)$) and their respective Lie algebras.

The approach we take to classify the representations of $\mathfrak{sl}(3, \mathbb{C})$ involves simultaneously diagonalizing the linear transformations $\pi(H_1)$ and $\pi(H_2)$ where H_1 and H_2 are as defined in (1) in the previous section.

Definition 5.2. Suppose (π, V) is a representation of $\mathfrak{sl}(3, \mathbb{C})$. An ordered pair $\mu = (k_1, k_2) \in \mathbb{C}^2$ is called a **weight** for π if there exists $v \neq 0$ in V such that

$$\pi(H_1)v = k_1 v$$
$$\pi(H_2)v = k_2 v$$

Such a vector v is called a **weight vector**, and for a particular μ, the space spanned by all such v is the **weight space** corresponding to μ.

Proposition 5.3. Every representation of $\mathfrak{sl}(3,\mathbb{C})$ has at least one weight.

PROOF. Our proof is adapted from Proposition 5.4 in [**2**]. Since $\mathfrak{sl}(3,\mathbb{C})$ is a vector space over $\mathbb{C}$, we know that $\pi(H_1)$ has at least one eigenvalue $k_1 \in \mathbb{C}$. Let $W \subset V$ be the eigenspace for $\pi(H_1)$ with eigenvalue k_1. Since $[H_1, H_2] = 0$ and π must preserve the Lie bracket, we have that $[\pi(H_1), \pi(H_2)] = 0$, i.e., $\pi(H_1)$ and $\pi(H_2)$ commute. Note that this implies that $\pi(H_2)w \in W$ for all $w \in W$ because

$$\pi(H_1)(\pi(H_2)w) = \pi(H_2)(\pi(H_1)w) = \pi(H_2)k_1 w = k_1(\pi(H_2)w)$$

Therefore $\pi(H_2)$ restricts to an operator on W, and it must therefore have at least one eigenvector w with eigenvalue k_2. We therefore have that w is a simultaneous eigenvector for $\pi(H_1)$ and $\pi(H_2)$ with eigenvalues k_1, k_2, respectively, meaning (k_1, k_2) is a weight for the representation π. □

Proposition 5.4 (Corollary 5.5 in [**2**])**.** If π is a representation of $\mathfrak{sl}(3,\mathbb{C})$, then all of the weights of π are of the form $\mu = (k_1, k_2)$ where $k_1, k_2 \in \mathbb{Z}$

PROOF. The proof is based on well-known results for $\mathfrak{sl}(2,\mathbb{C})$. Let $\mathfrak{g}_1 = \text{Span}\{H_1, E_{12}, E_{21}\}$. It is straightforward to check that $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ that is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The representation π of $\mathfrak{sl}(3,\mathbb{C})$ restricts to a representation π_1 of $\mathfrak{g}_1$, which by Theorem 4.12 in [**2**] guarantees that the eigenvalue k_1 of $\pi(H_1)$ is an integer. The same argument applied to $\mathfrak{g}_2 = \text{Span}\{H_2, E_{23}, E_{32}\}$ shows that the eigenvalue k_2 of $\pi(H_2)$ is also an integer. □

A weight $\mu = (k_1, k_2)$ is called a **dominant integral** element when k_1 and k_2 are non-negative integers. We now turn our attention to a particular important kind of weight.

Definition 5.5. An ordered pair $\alpha = (a_1, a_2) \in \mathbb{C}^2$ is called a **root** if

(1) a_1 and a_2 are both non-zero,
(2) there exists a nonzero vector $v \in \mathfrak{sl}(3,\mathbb{C})$ such that

$$[H_1, v] = a_1 v$$
$$[H_2, v] = a_2 v$$

Such a vector v is called a **root vector** corresponding to the root α.

Therefore a root is the pair of eigenvalues corresponding to the simultaneous eigenvector v for $\text{ad}(H_1)$ and $\text{ad}(H_2)$. In this way we note that roots are a special case of weights where the representation π in question is the adjoint representation of $\mathfrak{h}$ acting on $\mathfrak{g}$. We note that since all the weights of a particular representation π are pairs of integers, it follows that $\alpha = (a_1, a_2) \in \mathbb{Z} \times \mathbb{Z}$. We would like to establish a correspondence between roots as we have defined them here versus the context they were introduced in the last section. In the previous section in (2), we found that the roots of $\mathfrak{sl}(3,\mathbb{C})$ with respect to the diagonal matrices $\mathfrak{h}$ were

$$\Phi = \{\alpha_{12},\ \alpha_{23},\ \alpha_{12} + \alpha_{23},\ -\alpha_{12},\ -\alpha_{23},\ -\alpha_{12} - \alpha_{23}\}.$$

We can calculate:

$$\begin{aligned}
[H_1, E_{12}] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 2E_{12} = \alpha_{12}(H_1)E_{12}
\end{aligned}$$

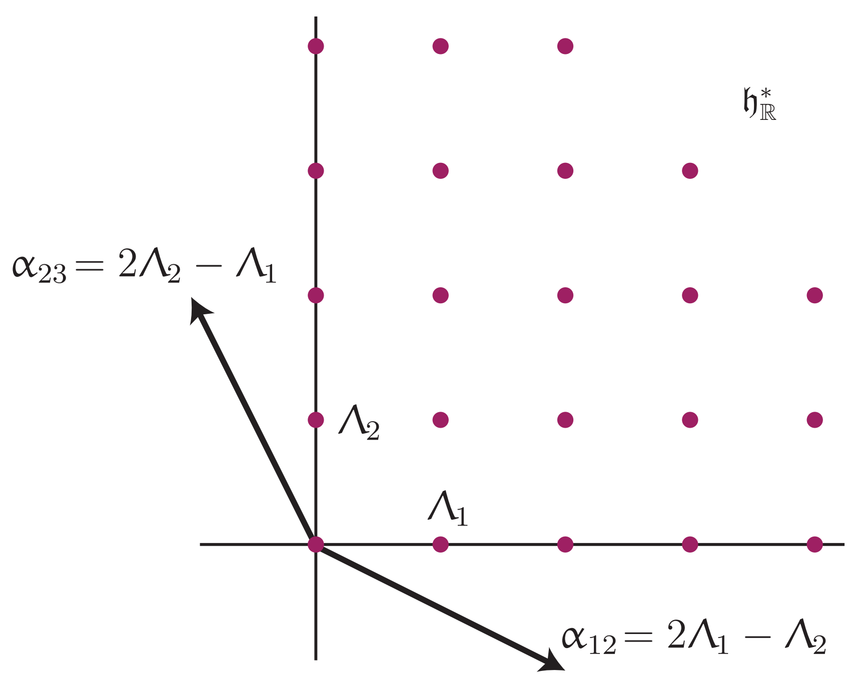
And similarly for other root vectors, we have

$$\begin{aligned}
&[H_1, E_{12}] = \alpha_{12}(H_1)E_{12} = 2E_{12} \qquad && [H_1, E_{23}] = \alpha_{23}(H_1)E_{23} = -1E_{23} \\
&[H_2, E_{12}] = \alpha_{12}(H_2)E_{12} = -1E_{12} \qquad && [H_2, E_{23}] = \alpha_{23}(H_2)E_{23} = 2E_{23}
\end{aligned}$$

By considering the eigenvalue of each eigenvector E_{12}, E_{23} for each simple root under the adjoint representation of $\mathfrak{h}$, we determine the ordered pairs (a_1, a_2) which correspond to each root. Thus by the above computation, we have

$$\alpha_{12} = (2, -1) \qquad \alpha_{23} = (-1, 2)$$

Having determined this for the simple roots, we can easily calculate $-\alpha_{12} = (-2, 1)$, $-\alpha_{23} = (1, -2)$, $\alpha_{12} + \alpha_{23} = (1, 1)$, and $-\alpha_{12} - \alpha_{23} = (-1, -1)$. It is clear that there is a geometric correspondence between the dominant integral weights and the roots, which is depicted in the picture below, using the straightforward choice of basis for the space of weights where the x-coordinate is the eigenvalue k_1 of $\pi(H_1)$ and the y-coordinate is the eigenvalue k_2 of $\pi(H_2)$.



While this picture shows a basic relationship between the roots and the weights, we will introduce a more useful geometric picture relating the two after introducing the Weyl group in the next section.

Having this basic correspondence with our earlier notion of roots established, we proceed with classifying the representations of $\mathfrak{sl}(3,\mathbb{C})$ by defining an ordering on weights of a representation.

Definition 5.6. Let μ_1, μ_2 be weights. We say μ_1 is **higher** than μ_2 (or equivalently, μ_2 is **lower** than μ_1) if $\mu_1 - \mu_2$ can be written in the form

$$\mu_1 - \mu_2 = a\alpha_{12} + b\alpha_{23}$$

where $a, b \in \mathbb{R}$ such that $a \geqslant 0$ and $b \geqslant 0$.

If π is a representation of $\mathfrak{sl}(3,\mathbb{C})$, then a weight μ_0 for π is said to be a **highest weight** if for all weights μ of π, μ_0 is higher than μ. We note that the relation of a weight being "higher" is only a partial ordering since not all pairs are comparable.

Theorem 5.7 (Theorem of the Highest Weight)**.**

(1) Every irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ has a highest weight.
(2) Two irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ with the same highest weight are equivalent.
(3) The highest weight of every irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ is a dominant integral element.
(4) Every dominant integral element occurs as the highest weight of an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$.

The full proof of this theorem can be found following Theorem 5.9 in [**2**]. What we take away from this statement is that all irreducible finite-dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$ can be indexed by pairs of non-negative integers, i.e., dominant integral elements. While the Theorem of the Highest Weight does not provide much detail about constructing the representations corresponding to each highest weight, it provides a useful means of indexing all such representations.

By Weyl's unitary trick, we can use this information to classify the finite-dimensional representations of $SU(2,1)$ by a similar correspondence. We will see later in our parameterization of representations of $SU(2,1)$ that knowing the parameters associated to finite-dimensional representations will be provide useful insight to determining which infinite-dimensional representations are reducible. The weights corresponding to dominant integral elements will be explored in more detail in the next section.

6. The Weyl Group and its Action on the Fundamental Weights Λ_i

Definition 6.1 (Definition 8.1 in [**2**])**.** A **root system** is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$, together with a finite collection R of nonzero vectors in E satisfying the following:

(1) The vectors in R span E.
(2) If $\alpha \in R$, then $-\alpha \in R$.
(3) If $\alpha \in R$, then the only multiples of α in R are α and $-\alpha$.

(4) If α, β in R, then $S_\alpha(\beta) \in R$ where S_α is the linear transformation of E defined by

$$S_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \quad \text{for all } \beta \in E$$

(5) If $\alpha, \beta \in R$, then the quantity

$$2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer

The dimension of E is called the **rank** of the root system and the elements of R are called the **roots**. If (E, R) is a root system, then the **Weyl group** W of R is the subgroup of orthogonal group of E generated by the reflections S_α for $\alpha \in R$.

In the context of the root space decomposition of $\mathfrak{sl}(3, \mathbb{C})$, our root system will be $(\mathfrak{h}_\mathbb{R}^*, \Phi)$ and the Weyl group W will be the group consisting of S_α for all $\alpha \in \Phi$. The space $\mathfrak{h}_\mathbb{R}^*$ will equal $\text{Span}_\mathbb{R}\{\alpha_{12}, \alpha_{23}\}$,i.e., the linear span of α_{12} and α_{23} with real coefficients. We now define the necessary inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}_\mathbb{R}^*$.

Let $\text{diag}(3, \mathbb{C})$ denote the space of 3×3 diagonal matrices with complex entries. We see that

$$\mathfrak{h} \subseteq \text{diag}(3, \mathbb{C}) \cong \mathbb{C}^3$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^3$:

$$\langle w, z \rangle = w_1\overline{z_1} + w_2\overline{z_2} + w_3\overline{z_3} \text{ for all } w, z \in \mathbb{C}^3$$

We now restrict $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}$. We can now use $\langle \cdot, \cdot \rangle$ to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. For any $X \in \mathfrak{h}$, define $\alpha_X \in \mathfrak{h}^*$ via

$$\alpha_X(H) = \langle H, X \rangle \text{ for all } H \in \mathfrak{H}$$

Then the mapping $X \mapsto \alpha_X$ yields a conjugate-linear isomorphism of $\mathfrak{h}$ onto $\mathfrak{h}^*$. In turn, this yields an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ defined by

$$\langle \alpha_X, \alpha_Y \rangle = \langle Y, X \rangle \text{ for all } X, Y \in \mathfrak{h}$$

The desired $\mathbb{R}$ inner product on $\mathfrak{h}_\mathbb{R}^* \subseteq \mathfrak{h}^*$ is the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}_\mathbb{R}^*$.

With this inner product, our definition of S_α for each $\alpha \in \Phi$ is thus a reflection in the line through the origin in $\mathfrak{h}_\mathbb{R}^*$ perpendicular to α. This is an isometry of $\mathfrak{h}_\mathbb{R}^*$ relative to $\langle \cdot, \cdot \rangle$ and hence

$$\langle S_\alpha(\beta_1), S_\alpha(\beta_2) \rangle = \langle \beta_1, \beta_2 \rangle \text{ for all } \beta_1, \beta_2 \in \mathfrak{h}_\mathbb{R}^*$$

We now regard a weight α for (π, V) as a nonzero element of $\mathfrak{h}$ with the property that there exists a nonzero v in V such that

$$\pi(H)v = \langle H, \alpha \rangle v \quad \text{for all } H \in \mathfrak{h}$$

For example, the elements α_{12}, α_{23}, and α_{13} of $\mathfrak{h}^*$ are identified with the following elements of $\mathfrak{h}$:

$$\alpha_{12} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \alpha_{23} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \alpha_{12} + \alpha_{23} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

We now have the weights living in $\mathfrak{h}$ instead of $\mathfrak{h}^*$, and since the roots are just weights for the adjoint representation, we also have identified the roots with elements of $\mathfrak{h}$. Now with the inner product defined as above, we can determine the geometric relationship between the roots in Φ and the rest of the weights discussed in Section 5 by calculating

$$S_\alpha(\lambda) = \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha \quad \text{for all } \lambda \in \mathfrak{h}_{\mathbb{R}}^*$$

for $S_\alpha \in W$.

The Weyl group W for $\mathfrak{sl}(3, \mathbb{C})$ is generated by reflections S_1, S_2 over the simple roots, $\{\alpha_{12}, \alpha_{23}\}$ where $S_1 = S_{\alpha_{12}}$ and $S_2 = S_{\alpha_{23}}$. From here on, we will refer to α_{12} as α_1 and α_{23} as α_2. We can calculate the reflections S_1 and S_2 for each of the roots $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -\alpha_1-\alpha_2\} = \Phi$. We note that $S_i \in W$ are linear maps, so we need only calculate $S_i(\alpha_j)$ for $1 \leqslant i, j \leqslant 2$:

$$\begin{aligned} S_1(\alpha_1) &= -\alpha_1 & \qquad S_2(\alpha_1) &= \alpha_1 + \alpha_2 \\ S_1(\alpha_2) &= \alpha_1 + \alpha_2 & \qquad S_2(\alpha_2) &= -\alpha_2 \end{aligned}$$

Given that α_1 and α_2 are the simple roots of Φ^+, we define the **fundamental weights** $\Lambda_1, \Lambda_2 \in \mathfrak{h}^*$ for this ordering by

$$\frac{2\langle \Lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \quad 1 \leqslant i, j \leqslant 2$$

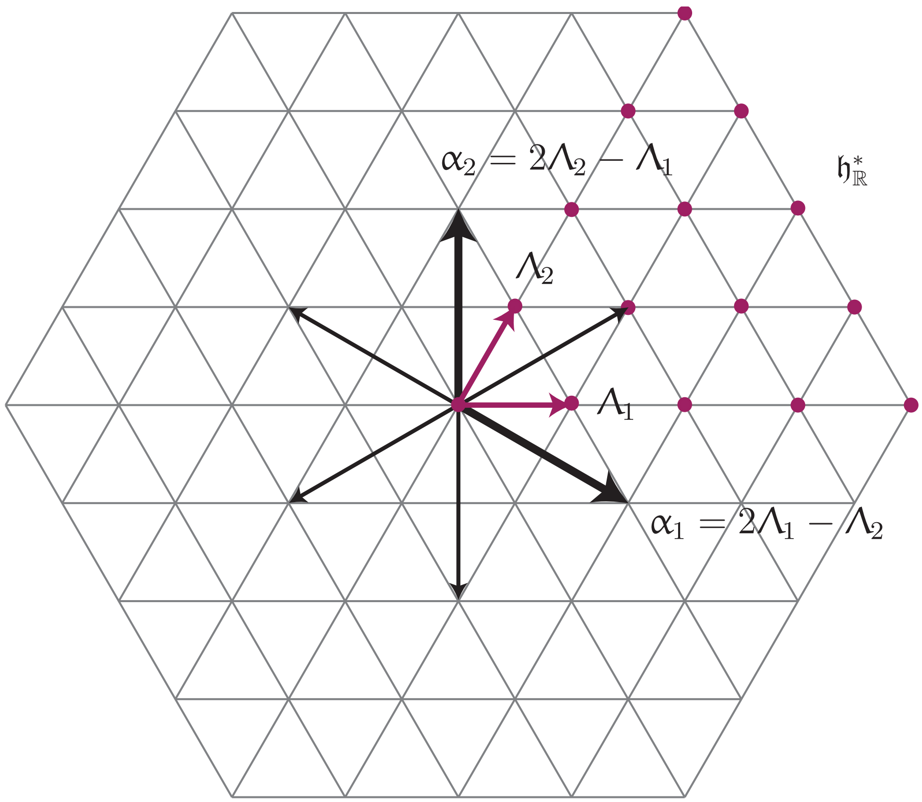
where δ_{ij} is the Kronecker delta function. Note that the fundamental weights are thus equivalent to the weights $(0, 1)$ and $(1, 0)$ from the previous section, yielding a particularly nice basis for $\mathfrak{h}_{\mathbb{R}}^*$. With this in mind, we note we have the following expressions for α_1 and α_2 in terms of Λ_1 and Λ_2:

$$\alpha_1 = 2\Lambda_1 - \Lambda_2 \quad \text{and} \quad \alpha_2 = 2\Lambda_2 - \Lambda_1 \tag{5}$$

These identities allow us to translate easily from the roots α_1, α_2 to the fundamental weights Λ_1, Λ_2 which can both be used as a basis for the 2-dimensional vector space $\mathfrak{h}_{\mathbb{R}}^*$. From the definition of fundamental weights, we can also easily calculate $S_i(\Lambda_j)$ for $1 \leqslant i, j \leqslant 2$.

$$\begin{aligned} S_1(\Lambda_1) &= \Lambda_1 - \delta_{11}\alpha_1 = \Lambda_1 - \alpha_1 & \qquad S_1(\Lambda_2) &= \Lambda_2 - \delta_{21}\alpha_1 = \Lambda_2 \\ S_2(\Lambda_1) &= \Lambda_1 - \delta_{12}\alpha_2 = \Lambda_1 & \qquad S_2(\Lambda_2) &= \Lambda_2 - \delta_{22}\alpha_2 = \Lambda_2 - \alpha_2 \end{aligned}$$

These computations involving Weyl reflections of roots and weights will prove useful later in our discussion of the reducibility of the principal series representations. Also we have developed a better understanding of the geometric structure of $\mathfrak{h}_{\mathbb{R}}^*$ when considered with a Weyl-invariant inner product. The following diagram depicts $\mathfrak{h}_{\mathbb{R}}^*$, showing the relative lengths and angles between the roots and the fundamental weights as determined by the inner product discussed above.



We note that the pairs of the discrete points in the diagram correspond to dominant integral weights, i.e., non-negative integer combinations of the fundamental weights. Thus they are in one-to-one correspondence with the finite dimensional representations of $\mathfrak{sl}(3, \mathbb{C})$ discussed in the last section.

7. Restricted Root Space Decomposition

In the case of a complex semisimple Lie algebra $\mathfrak{g}$, the root space decomposition allowed us to break up $\mathfrak{g}$ into 1-dimensional subalgebras $\mathfrak{g}_\lambda$ along with the Cartan subalgebra. In an effort to split our real semisimple Lie group $SU(2,1)$ into more manageable subgroups, we seek a decomposition of its real semisimple Lie algebra $\mathfrak{su}(2,1)$ similar to the root space decomposition in the complex case. Before we can proceed, we must define a few terms.

Definition 7.1. Let $\mathfrak{g}$ be a real of complex Lie algebra $\mathfrak{g}$. If x and y are in $\mathfrak{g}$, then it is meaningful to define

$$B(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$$

Thus B is a symmetric bilinear form on $\mathfrak{g}$ known as the **Killing form** on $\mathfrak{g}$.

Definition 7.2. An **involution** θ is an automorphism of a Lie algebra such that $\theta^2 = 1$. An involution θ of a real semisimple Lie algebra $\mathfrak{g}$ such that the symmetric bilinear form

$$B_\theta(x, y) = -B(x, \theta(y))$$

is positive definite is called a **Cartan involution**. B_θ will then define an inner product on $\mathfrak{g}$.

The existence of a Cartan involution for any real semisimple Lie algebra is not trivial. This result is Corollary 6.18 in [**7**]. In addition, while there may exist more than one Cartan involution for a real semisimple Lie algebra $\mathfrak{g}$, it is also shown (Corollary 6.19) that the Cartan involution is unique up to inner automorphisms of $\mathfrak{g}$, i.e., conjugation by elements $g \in G$, the Lie group associated to $\mathfrak{g}$.

For any Cartan involution θ of $\mathfrak{g}$, the fact that $\theta^2 = 1$ and θ is linear gives an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$ into $+1$ and -1 eigenspaces. Since θ preserves the bracket, it follows that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}$$

The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is referred to as a **Cartan decomposition** of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$, i.e., $[\mathfrak{a}, \mathfrak{a}] = 0$ and if $\tilde{\mathfrak{a}} \subseteq \mathfrak{p}$ is a subalgebra such that $\mathfrak{a} \subsetneq \tilde{\mathfrak{a}}$, then $\tilde{\mathfrak{a}}$ is not abelian. The dimension of $\mathfrak{a}$ is the **real rank** of the Lie algebra $\mathfrak{g}$ and of its corresponding Lie group G.

By Lemma 6.27 in [**7**], we know that for a Cartan involution θ, $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ for all $X \in \mathfrak{g}$ where the adjoint $(\cdot)^*$ is relative to the inner product B_θ of Definition 7.2. It follows that the members of $\operatorname{ad}(\mathfrak{a})$ form a commuting family of self-adjoint transformations on $\mathfrak{g}$ because for all $H \in \mathfrak{a}$, $(\operatorname{ad} H)^* = -\operatorname{ad} \theta H = -\operatorname{ad}(-H) = \operatorname{ad} H$. Therefore $\operatorname{ad}(\mathfrak{a})$ can be simultaneously diagonalized with real eigenvalues, so we can write $\mathfrak{g}$ as the direct sum of simultaneous eigenspaces. For each linear function $\lambda \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [H, x] = \lambda(H)x, \text{ for all } H \in \mathfrak{a}\}$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, we say that λ is a **restricted root** of $\mathfrak{g}$ and the corresponding subspace $\mathfrak{g}_\lambda$ of $\mathfrak{g}$ is a **restricted root space**. Unlike in the case of roots for a complex semisimple Lie algebra, the $\mathfrak{g}_\lambda$ are not necessarily one-dimensional.

Proposition 7.3. We have the following restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi} \mathfrak{g}_\lambda$$

Moreover, the restricted roots and their root spaces satisfy the following

(i) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$
(ii) $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ and $\lambda \in \Phi$ implies $-\lambda \in \Phi$
(iii) $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to B_θ when $\lambda \neq \mu$
(iv) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \{k \in \mathfrak{k} : [a, k] = 0 \text{ for all } a \in \mathfrak{a}\}$, and the sum is an orthogonal sum

The proof of this proposition can be found in Chapter 10.2 of [**5**]. Now define an ordering on $\mathfrak{a}^*$ and hence a notion of positivity for functionals in $\mathfrak{a}^*$. Let Φ^+ be the set of positive restricted roots so defined. We set

$$\mathfrak{n} = \sum_{\lambda \in \Phi^+} \mathfrak{g}_\lambda$$

The three subalgebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ yield an important decomposition of $\mathfrak{g}$.

Theorem 7.4 (Iwasawa Decomposition)**.**

The Lie algebra $\mathfrak{g}$ decomposes as a vector space direct sum

$$\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{a} \bigoplus \mathfrak{n}$$

In particular, $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent and $\mathfrak{k}$ is the Lie algebra of a compact Lie group.

Nilpotent means that for some $n \in \mathbb{N}$, $\mathfrak{g}_n = 0$ where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_\mathfrak{i}]$. To prove the above decomposition, we must show that each pair of components is disjoint and then show that any element $Y \in \mathfrak{g}$ can be written in the form $Y = k + a + n$ for $k \in \mathfrak{k}$, $a \in \mathfrak{a}$, and $n \in \mathfrak{n}$. First we note that $\mathfrak{a} \subset \mathfrak{g}_0$, and $\mathfrak{g}_0 \cap \mathfrak{n} = 0$, so $\mathfrak{a} + \mathfrak{n}$ is a direct sum. By property (ii) of Proposition 7.3, it follows that $\mathfrak{k} \cap \mathfrak{n} = 0$. Since $\mathfrak{a}$ is abelian, we know $\mathfrak{k} \cap \mathfrak{a} \subset Z_\mathfrak{k}(a) = \mathfrak{m}$, however by (iv), we know $\mathfrak{a} \cap \mathfrak{m} = 0$, so we conclude $\mathfrak{k} \cap \mathfrak{a} = 0$. Thus we have shown that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is a direct sum.

This sum is all of $\mathfrak{g}$ because, from Proposition 7.3, we can write any element $Y \in \mathfrak{g}$ in the form $Y = (H + Z) + \sum_{\lambda \in \Phi} X_\lambda$ with $H \in \mathfrak{a}$, $Z \in \mathfrak{m}$, and $X_\lambda \in \mathfrak{g}_\lambda$. Reorganizing gives

$$Y = \underbrace{Z + \sum_{\lambda \in \Phi^+} (X_{-\lambda} + \theta X_{-\lambda})}_{\in \mathfrak{k}} + \underbrace{H}_{\in \mathfrak{a}} + \underbrace{\sum_{\lambda \in \Phi^+} (X_\lambda - \theta X_{-\lambda})}_{\in \mathfrak{n}}$$

which establishes the Iwasawa decomposition for any $Y \in \mathfrak{g}$.

The Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ gives rise to a global decomposition of the Lie group G into subgroups K, A, and N via the exponential map as follows.

Theorem 7.5 (Global Iwasawa Decomposition - Theorem 5.12 in [**6**])**.** *If G is a connected, linear semi-simple Lie group, then there exist analytic subgroups K, A, N such that*

$$(k, a, n) \mapsto kan \text{ from } K \times A \times N \to G$$

is a diffeomorphism, where the Lie algebras of K, A, and N are $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Moreover, K is compact, A is abelian, and N is nilpotent.

A Lie group N is nilpotent if it is connected and its Lie algebra $\mathfrak{n}$ is nilpotent. Although it is true that $\exp(\mathfrak{k}) = K$, $\exp(\mathfrak{a}) = A$, and $\exp(\mathfrak{n}) = N$ in our particular situation (see Corollary VI.4.4 in [**3**] and Corollary 4.48 in [**7**]), the proof of the global Iwasawa decomposition is non-trivial.

Now we will discuss the above theory for the case of $\mathfrak{su}(2,1)$. Let θ be the Cartan involution of $\mathfrak{g} = \mathfrak{su}(2,1)$ given by $\theta(x) = -\overline{x}^T$ for $x \in \mathfrak{su}(2,1)$, i.e., the involution of an element x is its negative conjugate transpose. Consider the following basis $\mathcal{B}$ for $\mathfrak{su}(2,1)$:

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} \qquad K_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can therefore partition the basis vectors into their respective places in either $\mathfrak{k}$ or $\mathfrak{p}$ based on their eigenvalues under the mapping θ.

$$\begin{gathered} \theta(H) = -H \qquad \theta(P_1) = -P_1 \qquad \theta(P_2) = -P_2 \qquad \theta(P_3) = -P_3 \\ \theta(Z) = Z \qquad \theta(K_1) = K_1 \qquad \theta(K_2) = K_2 \qquad \theta(K_3) = K_3 \end{gathered}$$

Thus we have $\mathfrak{k} = \mathbb{R}Z \oplus \mathbb{R}K_1 \oplus \mathbb{R}K_2 \oplus \mathbb{R}K_3$ and $\mathfrak{p} = \mathbb{R}H \oplus \mathbb{R}P_1 \oplus \mathbb{R}P_2 \oplus \mathbb{R}P_3$.

We are given some choice involved with which maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ we consider. To follow the convention of [**9**], we choose

$$\mathfrak{a} = \mathrm{Span}_{\mathbb{R}}\{H\} = \mathrm{Span}_{\mathbb{R}}\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\}$$

By Proposition 7.3, we have that the zero eigenspace, $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \mathrm{Span}\, Z$. We can build the rest of the restricted root spaces using the above basis vectors. We define a linear form λ on $\mathfrak{a}$ by $\lambda(H) = 1$. Therefore

$$\begin{aligned} \mathfrak{g}_{2\lambda} &= \mathbb{R}(P_1 - K_1) \qquad & \mathfrak{g}_{-2\lambda} &= \mathbb{R}(P_1 + K_1) \\ \mathfrak{g}_{\lambda} &= \mathbb{R}(P_2 - K_2) \oplus \mathbb{R}(P_3 - K_3) \qquad & \mathfrak{g}_{-\lambda} &= \mathbb{R}(P_2 + K_2) \oplus \mathbb{R}(P_3 + K_3) \end{aligned}$$

Thus our restricted root space decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \underbrace{\mathfrak{a} \oplus \mathfrak{m}}_{\mathfrak{g}_0} \oplus \mathfrak{g}_{\lambda} \oplus \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{2\lambda} \oplus \mathfrak{g}_{-2\lambda}$$

The positive restricted roots for $\mathfrak{su}(2,1)$ are $\Phi^+ = \{\lambda, 2\lambda\}$, giving that

$$\mathfrak{n} = \mathbb{R}(P_1 - K_1) \oplus \mathbb{R}(P_2 - K_2) \oplus \mathbb{R}(P_3 - K_3)$$

Thus since clearly $\mathfrak{a} = \mathbb{R}H$, we have an explicit Iwasawa decomposition of $\mathfrak{su}(2,1)$, where $\mathfrak{k} = \mathbb{R}Z \oplus \mathbb{R}K_1 \oplus \mathbb{R}K_2 \oplus \mathbb{R}K_3$. This gives rise to a KAN decomposition of the group $SU(2,1)$ by Theorem 7.5. In addition, considering the subalgebra $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we describe an important subgroup $S = MAN$ of $SU(2,1)$ which will be instrumental in determining a continuous spectrum of representations of $SU(2,1)$. These representations will be referred to as the principal series and are to be discussed in detail in the next section.

8. Induced Representations and Principal Series

We will construct the principal series for $G = SU(2,1)$ following usual conventions of inducing representations from a parabolic subgroup of G.

Definition 8.1 (A.1.14 [**4**])**.** Suppose H is a closed subgroup of G. Let (π, V^π) be a continuous representation of H into a Hilbert space V^π and let $C_0(G, V^\pi)$ denote the space of continuous functions $f : G \to V^\pi$ such that the support of f is contained in a set CH where C is a compact set in G. Then the **induced representation** $\Phi = \mathrm{ind}_H^G\, \pi$ is a representation of G acting on the space

$$V^\Phi = \{f \in C_0(G, V^\pi) : f(gh) = \pi(h)^{-1} f(g) \text{ for all } h \in H\}$$

by

$$\Phi(g)(f(x)) = f(g^{-1}x)$$

The representation $\Phi = \mathrm{ind}_H^G\, \pi$ of G on V^Φ extends via continuity to a representation of G on the completion of V^Φ in its own natural topology. This extended representation is also denoted as $\mathrm{ind}_H^G\, \pi$ (A.1.14 [**4**]). This means that the group acts on the induced representation space by the left regular representation.

The definition given here is a general form of induction, and there are a number of ways of customizing the process based on the desired application. We will be using a version of **normalized** induction to ensure that if our representation π of the subgroup H is unitary, then the the induced representation $\Phi = \mathrm{Ind}_H^G(\pi)$ is unitarizable. For more on normalized induction, see pages 130-131 in [**1**], as well as the discussion below.

Definition 8.2 (Pg.132 [**6**])**.** A **parabolic subgroup** of G is a closed subgroup containing some conjugate of MAN, where A and N arise from a particular choice of $\mathfrak{a} \in \mathfrak{p}$ when constructing the Iwasawa decomposition, and $M = Z_K(A)$. The conjugates of MAN are called **minimal parabolic subgroups**.

$G = SU(2,1)$ is an example of a semisimple Lie group of **real rank one**, meaning any abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$ has dimension one. For such groups, we only need to consider one fixed minimal parabolic subgroup

$$S = MAN$$

for use in creating the necessary induced representations of G. To create the desired representations, start from an irreducible unitary representation (σ, V^σ) of M, necessarily finite-dimensional because M is compact, and a member ν of $(\mathfrak{a}^*)^{\mathbb{C}}$. We define a representation $(\sigma \otimes e^\nu \otimes 1)$ of S by

$$(\sigma \otimes e^\nu \otimes 1)(man) = e^{\nu \log a}\sigma(m)$$

where the map $\log : A \to \mathfrak{a}$ is the inverse of $\exp : \mathfrak{a} \to A$ described earlier so that $\log a \in \mathfrak{a}$ for any $a \in A$. Ordinary induction from $S = MAN$ would be given by

$$\mathrm{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1).$$

However, in our situation, the normalized induction from $\sigma \otimes e^\nu \otimes 1$ on MAN to G is given by

$$\pi_{\sigma,\nu} = \mathrm{Ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1) = \mathrm{ind}_{MAN}^G(\sigma \otimes e^{\nu+\rho} \otimes 1)$$

where $\rho = \frac{1}{2}\sum_{\lambda\in\Phi^+}(\dim \mathfrak{g}_\lambda)\lambda$, i.e., the half sum of the positive restricted roots (counting multiplicities). As noted earlier, this slight alteration of the definition of the induced representation will ensure that when $\sigma \otimes e^\nu \otimes 1$ is a unitary representation of MAN, then $\mathrm{Ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$ is a unitary representation of G. Thus the normalized induced representation space initially consists of all $f \in C_0(G, V^\sigma)$ such that

$$f(xman) = e^{-(\rho+\nu)\log a}\sigma(m)^{-1}f(x)$$

The action of $\Phi = \pi_{\sigma,\nu}$ is given by left-translation, i.e., $\pi_{\sigma,\nu}(g)f(x) = f(g^{-1}x)$. An inner product for $f, g \in V^\Phi$ is defined by

$$\langle f, g\rangle = \int_K \langle f(k), g(k)\rangle_{V^\sigma}\, dk \tag{6}$$

where $\langle\cdot,\cdot\rangle_{V^\sigma}$ denotes the inner product on V^σ. This pre-Hilbert space structure provides the "natural" topology for V^Φ. By taking the completion with respect to the norm arising from the inner product, we have the Hilbert space $H_{\sigma,\nu}$ which is our normalized induced representation space. These representations $(\pi_{\sigma,\nu}, H_{\sigma,\nu})$ are unitary when $\nu \in i\mathfrak{a}^*$ and we say that they comprise the **unitary principal series** (Pg. 38 [**8**]). We refer to this realization of the principal series as the **induced picture**. By restricting our induced picture to K, we have the initial representation representation space

$$\{f \in C(K, V^\sigma) : f(km) = \sigma(m)^{-1}f(k)\}$$

with inner product defined as in (6). Taking the completion of this space we have the **compact picture** realization of the principal series representation which we denote by H_σ. If $g \in G$ decomposes as $KMAN$ as

$$g = \kappa(g)\mu(g)e^{H(g)}n$$

then the action is

$$\pi_{\sigma,\nu}(g)f(k) = e^{-(\nu+\rho)H(g^{-1}k)}\sigma(\mu(g^{-1}k))^{-1}f(\kappa(g^{-1}k))$$

as given in equation (7.3a) and the subsequent formula in [**6**]. The action of $\pi_{\sigma,\nu}$ in the compact picture is significantly more complicated than in the induced picture, but since H_σ is independent of ν, it is easier to use the compact picture to study dependencies on ν.

The principal series described above, with $\nu \in i\mathfrak{a}^*$, were constructed to give a continuous series of unitary representations of G. We note that by relaxing the condition that ν is imaginary to a general $\nu \in (\mathfrak{a}^*)^{\mathbb{C}}$, we get the full **nonunitary principal series** of representations of G. While we are concerned with finding the unitary representations of $G = SU(2,1)$, we will need to work with the full principal series since there exist unitary representations beyond the unitary principal series which occur as subrepresentations and subquotients in the nonunitary principal series representations. These will include the critically important discrete series representations to be discussed in the next section.

Let $G = SU(2,1)$. The subgroups M and A can be realized by the Iwasawa decomposition discussed in the previous section and are given by

$$M = \{\exp(\theta Z) : 0 \leqslant \theta \leqslant 2\pi\} = \Big\{ \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} : 0 \leqslant \theta \leqslant 2\pi \Big\}$$

$$A = \{\exp(tH) : t \in \mathbb{R}\} = \{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R}\}$$

We identify $(\mathfrak{a}^*)^{\mathbb{C}}$ with the set of complex numbers by identifying a functional ν with the complex number $\nu(H) = 1$. Let ρ be the half sum of the positive restricted roots counting multiplicity, therefore given by

$$\rho = \frac{1}{2}[(\dim \mathfrak{g}_\lambda)\lambda + (\dim \mathfrak{g}_{2\lambda})2\lambda] = \frac{1}{2}(2\lambda + 2\lambda) = 2\lambda$$

Thus since we have chosen $\lambda(H) = 1$, we have that $\rho(H) = 2$. Since M is one-dimensional, abelian, we know that its irreducible unitary representations are one-dimensional acting on $\mathbb{C}$ and parameterized by $n \in \mathbb{Z}$. The action of each $\sigma_n \in \widehat{M}$ is multiplication by scalars given by

$$\sigma_n(m_\theta) = \sigma_n(\begin{pmatrix} e^i\theta & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} = e^{in\theta}$$

Thus for each $\sigma_n \in \widehat{M}$ and complex number ν we have a one-dimensional representation $(\sigma_n \otimes e^\nu \otimes 1)$ of the minimal parabolic subgroup MAN which we induce on the group G. The induced representation space of $\pi_{\sigma_n,\nu} = \mathrm{Ind}_{MAN}^G(\sigma_n \otimes e^\nu \otimes 1)$ initially consists of

$$\{f \in C_0(G, \mathbb{C}) : f(xman) = e^{-(\rho+\nu)\log a}\sigma_n(m)^{-1}f(x)\}$$

or equivalently (where $\log a = tH$)

$$\{f \in C_0(G, \mathbb{C}) : f(xm_\theta a_t n) = e^{-(2+\nu)t}e^{-in\theta}f(x)\}$$

We form the full representation space by completion. Thus the nonunitary principal series representations for $SU(2,1)$ are $(\pi_{\sigma_n,\nu}, H_{\sigma,\nu})$ for $n \in \mathbb{Z}$ and $\nu \in \mathbb{C}$ where $\pi_{\sigma_n,\nu}(g)f(x) = f(g^{-1}x)$. As discussed in the general theory, when ν is imaginary, the representations $(\pi_{\sigma_n,\nu}, H_{\sigma,\nu})$ comprise the unitary principal series.

If $g \in G$, we write $g = \kappa(g)\mu(g)e^{H(g)}n(g)$ with $\kappa(g) \in K$, $\mu(g) \in M$, $H(g) \in A$, and $n(g) \in N$. Note this is ambiguous because $M \subseteq K = U(2) = MSU(2)$. Thus we take $\kappa(g) \in SU(2)$, which we realize as the matrices

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \text{ for } u = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \in SU(2)$$

Then

$$\kappa(g)\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}z_1 & -\overline{z_2} & 0\\ z_2 & \overline{z_1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}z_1\\z_2\\1\end{pmatrix} = \begin{pmatrix}z\\1\end{pmatrix} \tag{7}$$

with $z = (z_1, z_2) \in S^3 = \{z \in \mathbb{C}^2 : |z_1| + |z_2| = 1\}$

Now moving to the compact picture, for $f \in L^2(S^3)$, we define

$$f_{\sigma_n,\nu}(g) = \mu(g)^{-n}H(g)^{-2-\nu}f(\kappa(g))$$

Therefore the group action in the compact picture is

$$\pi_{\sigma_n,\nu}(g)f(k) = f_{\sigma_n,\nu}(g^{-1}k) = \mu(g^{-1}k)^{-n}H(g^{-1}k)^{-2-\nu}f(\kappa(g^{-1}k))$$

This construction is standard for determining the nonunitary principal series representations. We will now consider an alternative construction of the principal series due to Nolan Wallach in [**9**]. The advantage of this realization of principal series will become apparent in the next section when we consider the problem of determining reducibility of principal series representations and the embedding of discrete series representations in the nonunitary principal series.

Following Section 7 in [**9**], let $g \in SU(2,1)$ act on $S^3 = \{z \in \mathbb{C}^2 : |z| = 1\}$ as follows:

$$g \cdot z = (\langle z, c\rangle + d)^{-1}(Az + b), \quad g = \begin{pmatrix}A & b\\ c^* & d\end{pmatrix}$$

where A is a 2×2 matrix and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{C}^2$. Set $h(g, z) = \overline{d} - \langle z, b\rangle$ for $z \in S^3$, $g \in SU(2,1)$. If $k_1, k_2 \in \mathbb{C}$ and $k_1 - k_2 \in \mathbb{Z}$, define

$$\pi_{k_1,k_2}(g)f(z) = h(g,z)^{k_1}\overline{h(g,z)}^{k_2}f(g^{-1}\cdot z)$$

for $f \in C^\infty(S^3)$, $g \in SU(2,1)$. Then $\pi_{k_1,k_2}(g)$ extends to a bounded operator on $L^2(S^3) = \mathcal{H}$ and $(\pi_{k_1,k_2}, \mathcal{H})$ defines a continuous representation of G for all $(k_1, k_2) \in \mathbb{C}^2$ such that $k_1 - k_2 \in \mathbb{Z}$. Note that the π_{k_1,k_2} representations are analogous to the compact picture discussed above where we restrict from $f \in C^\infty(G)$ to $f \in C^\infty(S^3)$.

Now let Λ_1 and Λ_2 be the fundamental weights as defined in (6). We will say that $\Lambda \in \mathfrak{h}^*$ is ***G*-integral** if and only if $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ for $k_i \in \mathbb{C}$, and $k_1 - k_2 \in \mathbb{Z}$. We will now denote the representations π_{k_1,k_2} by π_Λ where Λ is G-integral. The full induction procedure working through the induced picture to the compact picture given above is developed for π_Λ in Section 7 of [**9**].

We turn now to finding an explicit correspondence between these two versions of the principal series. First, we parameterize the quasi-characters of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We use the polar decomposition $z = ma$ with $m \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $a > 0$, $a \in \mathbb{R}$. Then $m = z/|z|$ and $a = |z|$. Note that $\overline{m} = m^{-1}$. Thus if $k_1, k_2 \in \mathbb{C}$ and $k_1 - k_2 \in \mathbb{Z}$, then

$$\begin{aligned} z^{k_1}\overline{z}^{k_2} &= (\frac{z}{|z|})^{k_1}(\frac{z}{|z|})^{k_2}|z|^{k_1+k_2}\\ &= m^{k_1}m^{-k_2}a^{k_1+k_2}\end{aligned}$$

If we write $-n = k_1 - k_2$ and $k_1 + k_2 = -2 - \nu$ with $\nu \in \mathbb{C}$, then

$$z^{k_1}\overline{z}^{k_2} = m^{-n}a^{-2-\nu}$$

Comparing the actions of $\pi_{\sigma_n,\nu}$ and $\pi_\Lambda = \pi_{k_1,k_2}$ when restricted to the compact picture, we have

$$\pi_{\sigma_n,\nu}(g)f(k) = \mu(g^{-1}k)^{-n}H(g^{-1}k)^{-2-\nu}f(\kappa(g^{-1}k))$$
$$\pi_{k_1,k_2}(g)f(z) = h(g,z)^{k_1}\overline{h(g,z)}^{k_2}f(g^{-1}\cdot z)$$

If $f \in C^\infty(S^3)$, then thinking of $\mu(g^{-1}k)^{-n}H(g^{-1}k)^{-\nu} = ma$ as an element of $\mathbb{C}^\times$ in polar coordinates and $f(z) = f(k)$ with the relationship between k and z given in (7), we have

$$h(g,z) = \mu(g^{-1}k)H(g^{-1}k)$$

Thus we have the natural correspondence between $\pi_{\sigma_n,\nu}$ and π_Λ given by

$$k_1 - k_2 = -n \text{ and } \nu = -k_1 - k_2 - 2. \tag{8}$$

With this result, we can translate results, especially those regarding reducibility, found through the π_Λ realization to the usual principal series representations $\pi_{\sigma_n,\nu}$

9. Discrete Series and Its Embedding

Theorem 9.1 (Godement, pg 69 in [**8**])**.** *For an irreducible unitary representation π of a unimodular Lie group G, the following three conditions are equivalent:*

(a) Some non-zero matrix coefficient is in $L^2(G)$.
(b) All matrix coefficients are in $L^2(G)$.
(c) π is equivalent with a direct summand of the right regular representation of G on $L^2(G)$.

A representation satisfying these three equivalent conditions is said to be in the ***discrete series*** *of G.*

Theorem 9.2 (Theorem 12.20 in [**6**])**.** *A linear connected semisimple group G has discrete series representations if and only if* $\operatorname{rank} G = \operatorname{rank} K$*.*

Note the condition $\operatorname{rank} G = \operatorname{rank} K$ is equivalent to G having a compact Cartan subgroup. For $G = SU(2,1)$, we have the Cartan subalgebra

$$\mathfrak{t} = \mathbb{R}Z \oplus \mathbb{R}K_1 = \{i\operatorname{diag}(t_1,t_2,t_3) : t_1,t_2,t_3 \in \mathbb{R} \text{ and } t_1+t_2+t_3 = 0$$

which corresponds to the Cartan subgroup

$$T = \{\operatorname{diag}(z_1,z_2,z_3) : |z_1| = |z_2| = |z_3| = 1 \text{ and } z_1z_2z_3 = 1\}$$

We have that $T \subseteq K = U(2)$ where K is the maximal compact subgroup of G, so we have that T is, in fact, a compact Cartan subgroup, guaranteeing the existence of discrete series for $SU(2,1)$.

The discrete series are named for the fact that these representations are precisely those irreducible unitary representations of G which have positive Plancherel measure. The discrete series were classified for any semisimple G by Harish-Chandra in the 1950-60s. Under the condition that they exist for linear, connected, semi-simple G (Theorem 9.2), the series of discrete representations are identified and indexed by particular linear functionals λ acting on the compact Cartan subalgebra. Each λ defines an "infinitesimal character" χ_λ of a discrete series representation π_λ. The index λ is called the Harish-Chandra parameter and corresponds to the discrete series representation π_λ. For these results and more background on existence of discrete series, see Theorem 9.20 in [**6**]. In fact, by Theorem 12.21 in [**6**], the discrete series specified by Theorem 9.20 exhaust all discrete series representations of G up to equivalence.

For our purposes, we would like to determine the embedding of such discrete series representations in the non-unitary principal series. In other words, we want to the determine where principal series reduce to discrete series and in particular, to which discrete series representation in terms of their Harish-Chandra parameters. This involves first determining the points at which the nonunitary principal series is reducible. We return to $G = SU(2,1)$.

Lemma 9.3 (Lemma 7.1' in [**9**]). *The representation π_Λ of G is reducible if and only if Λ is **integral**, meaning $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ where $k_1, k_2, \in \mathbb{Z}$ and $\Lambda \neq -\rho$ where $\rho = \alpha_1 + \alpha_2$ is the half sum of roots of $\mathfrak{sl}(3,\mathbb{C})$ as discussed in Section 4.*

There are three types of discrete series which embed in the nonunitary principal series, namely the **holomorphic**, the **antiholomorphic**, and the **nonholomorphic** discrete series. A full description of these three classes, including their respective representation spaces is given on pg. 183 of [**9**] and more explicitly discussed on pg. 481-482 of [**10**].

We will rewrite $(D_\Lambda^+, V_+^\Lambda)$ for the holomorphic discrete series, $(D_\Lambda^-, V_-^\Lambda)$ for the antiholomorphic, and (D_Λ, W^Λ) for the nonholomorphic. From the perspective of these discrete series representations being subspaces of π_Λ, we also could write $D_\Lambda^+(g) = \pi_\Lambda(g)\big|_{V_+^\Lambda}$ for holomorphic, $D_\Lambda^-(g) = \pi_\Lambda(g)\big|_{V_-^\Lambda}$ for antiholomorphic, and $D_\Lambda(g) = \pi_\Lambda(g)\big|_{W^\Lambda}$ for nonholomorphic. Further, if π is a representation of G, then we say that $\pi \subset \pi_\Lambda$ if π is infinitesimally equivalent with a subquotient of π_Λ. With this notion of embedding, we have the following result.

Theorem 9.4 (Lemma 7.10 in [**9**]). *If $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, for $k_i \geqslant 0$ and $k_i \in \mathbb{Z}$, then*

(1) $D^+_{S_1S_2(\Lambda+\rho)-\rho} \subset \pi_\mu$ if and only if $\mu = S_1(\Lambda+\rho)-\rho$, or $\mu = S_1S_2(\Lambda+\rho)-\rho$.
(2) $D^-_{S_2S_1(\Lambda+\rho)-\rho} \subset \pi_\mu$ if and only if $\mu = S_2(\Lambda+\rho)-\rho$, or $\mu = S_2S_1(\Lambda+\rho)-\rho$.
(3) $D_{S_1S_2S_1(\Lambda+\rho)-\rho} \subset \pi_\mu$ if and only if $\mu = S_1S_2S_1(\Lambda+\rho)-\rho$.

where S_1, S_2 are the Weyl reflections over the simple roots α_1 and α_2, respectively.

This lemma describes the full embedding of the discrete series

$$\hat{G}_d = \{D^+_{S_1S_2(\Lambda+\rho)-\rho}, D^-_{S_2S_1(\Lambda+\rho)-\rho}, D_{S_1S_2S_1(\Lambda+\rho)-\rho}\}$$

in the nonunitary principal series.

We have worked thus far to construct the unitary principal series as well as determined the points of the nonunitary principal series that reduce to unitary representations, the discrete series. We note that the discrete series are named as such, since they are precisely those points of the unitary dual $\widehat{G}$ that have positive Plancherel measure. Thus by exhausting the discrete series and the unitary principal series we have sufficiently many representations to discuss the operator-valued Fourier transform for $G = SU(2,1)$ and its respective Plancherel Inversion formula. The determination of these representations was necessary since all form the support of the Plancherel measure, which has a fundamental role in the Inversion formula.

10. Further Work

We would like to further investigate how irreducible representations are embedded in the reducible nonunitary principal series. By the remark before Lemma 7.10 in [**9**], if π_Λ is irreducible and if $\pi_\Lambda \subset \pi_\mu$ for some μ, then $\mu = S_0(\Lambda+\rho)-\rho$. We interpret this result to mean that if we know an irreducible component π_Λ occurs at some principal series π_μ, we can determine the principal series in question by applying the equation above.

Thus suppose $\mu = k_1\Lambda_1 + k_2\Lambda_2$ for $k_1, k_2 \in \mathbb{Z}$ such that $(k_1, k_2) \neq (-1. -1)$ is some reducible principal series (see Lemma 9.3). We calculate $S_0(\mu+\rho)-\rho$ and write our answer in terms of the fundamental weights Λ_1, Λ_2 using our established relationship between roots and weights by Weyl reflections, so that we may consider the parameters of the irreducible component π_Λ. Note that we use the fact that $\rho = \Lambda_1 + \Lambda_2$ below which is easily justified by $\rho = \alpha_1 + \alpha_2 = (2\Lambda_1 - \Lambda_2) - (2\Lambda_2 - \Lambda_1) = \Lambda_1 + \Lambda_2$ from (5).

$$\begin{aligned}
\Lambda = S_0(\mu+\rho)-\rho &= S_1S_2S_1(k_1\Lambda_1 + k_2\Lambda_2 + \Lambda_1 + \Lambda_2) - \rho \\
&= S_1S_2((k_1+1)S_1(\Lambda_1) + (k_2+1)S_1(\Lambda_2)) - \rho \\
&= S_1S_2((k_1+1)(\Lambda_1 - \alpha_1) + (k_2+1)\Lambda_2) - \rho \\
&= S_1((k_1+1)(S_2(\Lambda_1) - S_2(\alpha_1)) + (k_2+1)S_2(\Lambda_2)) - \rho \\
&= S_1((k_1+1)(\Lambda_1 - (\alpha_1+\alpha_2)) + (k_2+1)(\Lambda_2 - \alpha_2)) - \rho \\
&= ((k_1+1)(S_1(\Lambda_1) - S_1(\alpha_1+\alpha_2)) + (k_2+1)(S_1(\Lambda_2) - S_1(\alpha_2))) - \rho \\
&= ((k_1+1)((\Lambda_1 - \alpha_1) - \alpha_2) + (k_2+1)(\Lambda_2 - (\alpha_1+\alpha_2))) - \rho \\
&= ((k_1+1)(\Lambda_1 - (\alpha_1+\alpha_2)) + (k_2+1)(\Lambda_2 - (\alpha_1+\alpha_2))) - (\Lambda_1 - \Lambda_2) \\
&= k_1\Lambda_1 + k_2\Lambda_2 - (k_1+1+k_2+1)(\alpha_1+\alpha_2) \\
&= k_1\Lambda_1 + k_2\Lambda_2 - (k_1+k_2+2)(2\Lambda_1 - \Lambda_1 + 2\Lambda_1 - \Lambda_2) \\
&= k_1\Lambda_1 + k_2\Lambda_2 - (k_1+k_2+2)(\Lambda_1+\Lambda_2) \\
&= -(k_2+2)\Lambda_1 + -(k_1+2)\Lambda_2
\end{aligned}$$

Thus we have $\pi_{-(k_2+2),-(k_1+2)} \subset \pi_{k_1,k_2}$. Translating this result to our other realization of principal series via (8), we have that

$$\pi_{\sigma_{(k_2-k_1)},(k_1+k_2+2)} \subset \pi_{\sigma_{(k_2-k_1)},(-k_1-k_2-2)}.$$

This implies that $\pi_{\sigma_n,-\nu} \subset \pi_{\sigma_n,\nu}$ which for a number of reasons seems odd. To check the validity of this calculation and its implications, we have to look closer at the discussion of embedding and the notion of infinitesimal equivalence as discussed on pg 185 of [**9**]. Beyond clarifying the meaning of this embedding result, we would also like to determine how the Harish-Chandra parameter which indexes the discrete series relates to the subrepresentations and subquotients we have identified in the nonunitary principal series in Theorem 9.4.

We have cited a result identifying the reducible principal series (Lemma 9.3) without proof. This can supposedly be confirmed by considering the representations of $SU(2)$ which are indexed by their dimension and are given explicitly on pg 185 in [**9**]. The representations of $SU(2)$ also provide clearer understanding of the principal series representations by determining a reasonable basis for each H_{σ_n} representation space in the compact picture. This would be helpful in our analysis of how the principal series representations reduce as it would allow an explicit description of subrepresentation spaces corresponding to discrete series.

While the results we have discussed give a parameterization of the irreducible unitary representations that support the Plancherel measure for $SU(2,1)$, we have not explicitly put them in the context of the Plancherel Inversion formula. To do this requires a careful understanding of the operator-valued Fourier transform, which is built from the matrix coefficients of the irreducible unitary representations of the principal and discrete series. We also note that we have not given the formula for the Plancherel measure on $\widehat{G}$ which appears in the Plancherel theorem for G. This can be found in a general form for linear connected groups G of real rank 1 in Theorem 13.5 in [**6**].

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