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Convexity Properties of the Diestel-Leader Group Γ_3(2)

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Contents

Abstract

The Diestel-Leader groups, described in Section 1.4, are a family of groups first introduced in 2001 by Diestel and Leader in [7]. In this paper, we demonstrate that the Diestel-Leader group $\Gamma_3(2)$ is not almost convex with respect to a particular generating set S, defined in Section 2.5. Almost convexity, described in Section 1.3, is a geometric property that has been shown by Cannon [3] to guarantee a solvable word problem (that is, in any almost convex group there is a finite-step algorithm to determine if two strings of generators, or "words", represent the same group element). Our proof relies on the word length formula given by Stein and Taback in [10], and we construct a family of group elements X in Section 3.0.3 that contradicts the almost convexity condition. We then go on to show that $\Gamma_3(2)$ is minimally almost convex with respect to S, a geometric property also defined in Section 1.3.

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Section 1

Introduction

1.1 Overview

We first show that the Diestel-Leader group $\Gamma_3(2)$ is not almost convex with respect to a particular generating set S.

Prior to the proof of this theorem, we provide an introduction to the notions of almost convexity and minimal almost convexity. In order to do so, we first give basic definitions of Cayley graphs and word length. We then provide an introduction to the Diestel-Leader groups, paying special attention to the group $\Gamma_3(2)$, taken to be generated by a set we call S. Harkening back to our description of Cayley graphs, we go on to show that the Diestel-Leader graph $\mathrm{DL}_3(2)$ is the Cayley graph of the group $\Gamma_3(2)$ with respect to S. Having gained a thorough understanding of $\Gamma_3(2)$, the generating set S, and the Cayley graph with respect to this generating set $\mathrm{DL}_3(2)$, we reprove a word length formula on $\Gamma_3(2)$, first given in [10].

These tools are sufficient for us to introduce a family of elements X that violate the almost convexity condition, thereby disproving the almost convexity of $\Gamma_3(2)$ with respect to S.

Finally, we show that the Diestel-Leader group $\Gamma_3(2)$ is minimally almost convex with respect to S.

1.2 Introduction to word length and Cayley graphs

Given a group G generated by a set S, the *word length* of an element g is defined as the minimal length of any string of generators representing g.

Given a group G generated by a set S, we can define a graph $\Gamma(G, S)$ (notation not to be confused with that of the Diestel-Leader groups) as follows:

- Establish a bijective correspondence between group elements in G and vertices in $\mathrm{vert}(\Gamma(G, S))$.
- Draw an edge between the vertices corresponding to group elements that differ by a generator s. Label that edge s.
  - i.e. v_g and v_h are connected by an edge iff $g = hs$ for some $s \in S$.

Note that in the coming sections it becomes tedious to constantly distinguish between group elements and vertices, so we will on occasion simply refer to vertices by the name of the group element they correspond to. We will likewise refer to edges by the name of the generator they are labelled by.

Notice that, for any group element g, a word that realizes the word length of g will be represented in the Cayley graph a minimal-length path from the identity to g (where length is measured by number of edges traversed).

Cayley graphs are often studied as metric spaces, with a metric defined from the word length. For a group G with word length formula $l(g)$ relative to generating set S, the *word metric* is given by

$$d(g, h) = l(g^{-1}h)$$

This correspondence allows us to think about groups as metric spaces.

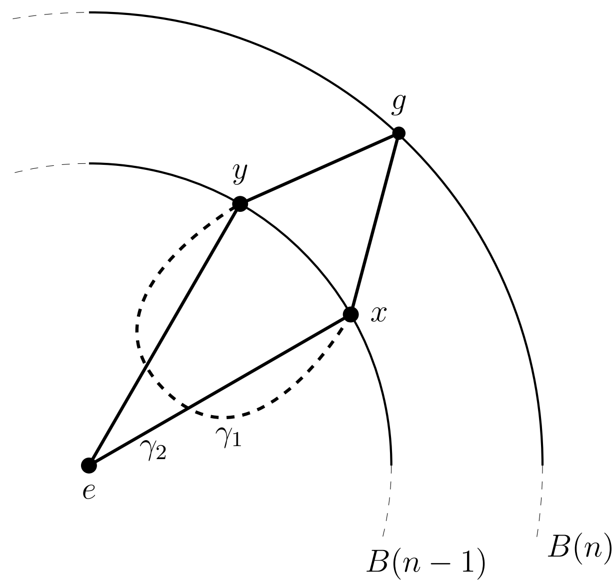


Figure 1.1: The almost convexity condition AC_2. If the points shown above are in the Cayley graph of a group G with respect to generating set S, then we say G is AC_2 with respect to S if for any such x, y there exists a path γ_1 of length less than or equal to some fixed number N that remains in $B(n-1)$. We say G is minimally almost convex if for any such x, y there exists a path γ_1 that is strictly shorter than the path described by γ_2. That is, the length of γ_1 is strictly less than $2n$.

1.3 Introduction to almost convexity

The notion of almost convexity was introduced by James Cannon [3] in 1987. It is defined for a group G, finite generating set S, and Cayley graph $\Gamma(G, S)$ with word length $l(g)$ and word metric $d(x, y)$. The almost convexity condition AC_k is satisfied when there is some number N such that for sufficiently large n, for any two points x and y inside the ball $B(n)$ that are joined by a path of length k, there exists a path from x to y of length at most N that lies entirely inside the ball $B(n)$. A diagram of the property AC_2 is shown in Figure 1.1.

We say that a group is almost convex, or AC, with respect to a particular generating set if it is AC_k for every k. Cannon [3] also showed that AC_2 implies AC, meaning that we only have to concern ourselves with points that are joined by a path of length 2. Finally, Cannon [3] showed that if a group is almost convex with respect to a given generating set, then it is recursive with respect to that generating set, where recursive means there is an algorithm to construct $B(k)$ for any positive integer k. Such an algorithm can be used to determine if two words represent the same group element, and thus solve the word problem–the question of when two different words describe the same group element. For this reason, almost convexity is a very useful property for a group to have.

The weaker notion of minimal almost convexity is a relaxation of the almost convexity condition. We say a group is minimally almost convex (MAC) with respect to a particular generating set if for any k and for any sufficiently large n, for any two points x and y inside the ball $B(n)$ that are joined by a path of length k, there exists a path from x to y of length at most $2n-1$ that lies entirely inside the ball $B(n)$. The notion encapsulated by minimal almost convexity is that if we have two points x and y in the ball $B(n)$, there is some path between them that remains in $B(n)$ that is shorter than the path described by simply going from x back to the identity and then out to y. It is clear that if G is almost convex with respect to some generating set S, then it is also minimally almost convex with respect to S. The MAC property is also illustrated graphically in Figure 1.1.

Cleary and Taback [6] have shown that the lamplighter group, described in the next section, is not minimally almost convex. In addition, Belk and Bux [2] have shown that Thompson's group F is not minimally almost convex, and Miller and Shapiro have shown that the solvable Baumslag-Solitar groups are not almost convex [9].

1.4 Introduction to the Diestel-Leader groups

In 2001 Diestel and Leader [7] introduced a family of graphs, which came to be called the Diestel-Leader graphs. The Diestel-Leader graphs are notated $\mathrm{DL}_d(m_1, m_2, \ldots, m_d)$ for some natural numbers $d, m_1, \ldots, m_n$. The Diestel-Leader graph $\mathrm{DL}_d(m_1, m_2, \ldots, m_d)$ is a subset of the Cartesian product of d trees, the first of valence m_1+1, the second of valence m_2+1, et cetera. The precise definition will be given for some concrete examples of Diestel-Leader graphs in Sections 2.2 and 2.4.

These groups were first introduced as an attempt to produce a connected, locally finite, vertex transitive graph that is not quasi-isometric with a Cayley graph of any finitely generated group. Five years later, in 2006, Eskin, Fisher, and Whyte [8] showed that if $m_1 \neq m_2$ then the Diestel-Leader graph $\mathrm{DL}_2(m_1, m_2)$ is not quasi-isometric with the Cayley graph of any finitely generated group. But when $m_1 = m_2$, the Diestel-Leader graph $\mathrm{DL}_2(m_1, m_2)$ is the Cayley graph of the lamplighter group with respect to a particular generating set.

It was later shown by Bartholdi, Neuhauser, and Woess [1] that $\mathrm{DL}_d(m_1, m_2, \dots, m_d)$ is not the Cayley graph of any finitely generated group if we do not have $m_1 = m_2 = \cdots = m_d$. However, it is known that if $m_1 = m_2 = \cdots = m_d$ and $d \leq p+1$ for all primes p dividing the m_i, then $\mathrm{DL}_d(m_1, m_2, \dots, m_d)$ is a Cayley graph of some finitely generated group. As these are the only Diestel-Leader graphs we will concern ourselves with in this paper, we simply write $\mathrm{DL}_d(q)$ for $\mathrm{DL}_d(q, q, \dots, q)$, since each tree is assumed to be of the same valence.

We will write the group for which $\mathrm{DL}_d(q)$ is a Cayley graph as $\Gamma_d(q)$.

Interestingly, if $m_1 = m_2 = \cdots = m_d$ and $d > p + 1$ for some prime p dividing the m_i, it is still not known if $\mathrm{DL}_d(m_1, m_2, \dots, m_d)$ is a Cayley graph of any finitely generated group.

Section 2

Background on Diestel-Leader Groups

In this paper we study convexity properties of one particular group in this family, $\Gamma_3(2)$, although our results will generalize to any Diestel-Leader group $\Gamma_3(q)$.

2.1 The lamplighter Group $\Gamma_2(2)$

The Diestel-Leader group $\Gamma_2(2)$ is more commonly referred to as the lamplighter group, and is usually denoted by L_2. We will simply call it L. It is a simple example of a wreath product, that is,

$$L = \mathbb{Z}_2 \wr \mathbb{Z}.$$

The family of lamplighter groups is given by $L_n = \mathbb{Z}_n \wr \mathbb{Z}$. It is particularly nice to study, as there is a convenient visualization of elements of this group, using a diagram like the one in Figure 2.1, called the lamplighter picture corresponding to a group element. The diagram is meant to encapsulate a bi-infinite sequence of lamps with one placed at each integer point on the number line, some finite number of which are illuminated, with a lamplighter positioned at one of them. This group is denoted L_2 because the lamps are modeled on $\mathbb{Z}_2$–that is, they can either be on or off. The lamplighter group L_n has lamps with n distinct states.

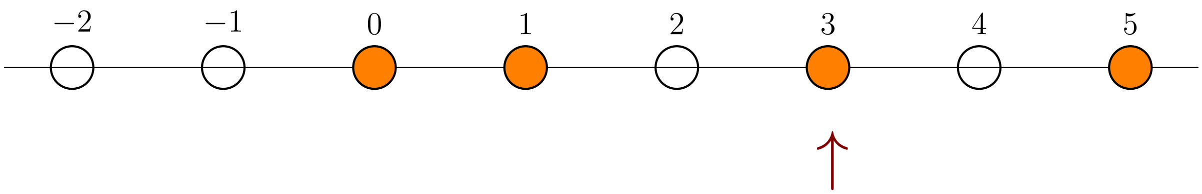


Figure 2.1: An element of L. The integer-indexed circles are meant to represent lamps, which can be either on (solid) or off (empty), and the arrow is meant to represent the position of the lamplighter. The infinite number of lamps not pictured are assumed to be off.

The identity is simply the element in which no bulbs are illuminated and the lamplighter is positioned at 0. With this convenient visualization in mind, we unpack the definition of a wreath product and express L as follows:

$$L = \bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_2)_i \rtimes_{\varphi} \mathbb{Z}$$

where $\varphi(n) \in Aut(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_2)_i)$ shifts the indices up by n.
The semidirect product of groups is by definition a group, so we know that L is a group. An element of L looks like $((\ldots 0, 0, 1, 0, 1, 1, 1, 0, 0, \ldots), n)$. The infinite tuple of 0's and 1's is meant to contain the information of which bulbs in our picture are illuminated, and the integer n represents the position of the lamplighter. In this notation, the element represented by Figure 2.1 is $((\ldots 0, 1, 1, 0, 1, 0, 1, 0 \ldots), 3)$, where the first 1 is in the copy of $\mathbb{Z}_2$ indexed by $0 \in \mathbb{Z}$.

The information captured in an element of the semidirect product can also be stored in a matrix. For instance, L is equivalently defined by

$$L = \{ \left(\begin{smallmatrix} t^x & P \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{Z}, P \in \mathbb{Z}_2[t, t^{-1}] \}$$

The definition of L as a matrix group is the most useful one, as we will be using the matrix presentation when we discuss its generalization, $\Gamma_3(2)$. P is a polynomial in t and t^{-1} with coefficients from $\mathbb{Z}_2$. Beginning with an infinite tuple as above, a_i in the i^{th} position becomes the term $a_i t^i$ in the corresponding polynomial. So, P encodes which lamps are on. The integer x is meant to represent the position of the lamplighter. In this notation, the element represented in Figure 2.1 is $\left(\begin{smallmatrix} t^3 & 1+t+t^3+t^5 \\ 0 & 1 \end{smallmatrix}\right)$. The binary operation is standard matrix multiplication:

$$\left(\begin{smallmatrix} t^x & P \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t^y & Q \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^{x+y} & t^xQ+P \\ 0 & 1 \end{smallmatrix}\right)$$

To see that the binary operation under this definition agrees with the binary operation under the preceding definition, we include a sample calculation:

Example 2.1.1. *Consider the group elements* $((\dots, 0, 1, 1, \bar{0}, 1, 0, 1, 0, \dots), 3)$ *and* $((\dots, 0, 1, \bar{1}, 0, \dots), -2)$*, where the bar represents the copy of* $\mathbb{Z}_2$ *indexed by* $0 \in \mathbb{Z}$*.*

Using the correspondence noted above, we see that these elements correspond to the matrices $\left(\begin{smallmatrix} t^3 & t^{-2}+t^{-1}+t+t^3 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} t^{-2} & t^{-1}+1 \\ 0 & 1 \end{smallmatrix}\right)$, respectively. First the product in the semidirect product notation:

$$((\dots, 0, 1, 1, \bar{0}, 1, 0, 1, 0, \dots), 3)((\dots, 0, 1, \bar{1}, 0, \dots), -2) = ((\dots, 0, 1, 1, \bar{1}, 0, 0, 1, 0, \dots), 1)$$

This element corresponds to the matrix $\left(\begin{smallmatrix} t & t^{-2}+t^{-1}+1+t^3 \\ 0 & 1 \end{smallmatrix}\right)$. We now check that the matrix product agrees:

$$\left(\begin{smallmatrix} t^3 & t^{-2}+t^{-1}+t+t^3 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t^{-2} & t^{-1}+1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} t & t(t^{-1}+1)+t^{-2}+t^{-1}+t+t^3 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} t & t^{-2}+t^{-1}+1+t^3 \\ 0 & 1 \end{smallmatrix}\right)$$

So multiplying matrices of the above form corresponds to semidirect product multiplication.

We wish to examine a Cayley graph for L, so we must determine a generating set and presentation for the group.

A presentation that corresponds nicely with the lamplighter picture in Figure 2.1, and is therefore somewhat intuitive, is given by

$$L = \langle a, t \mid a^2 = e, \forall i, j \in \mathbb{Z} : (t^i a t^{-i})(t^j a t^{-j}) = (t^j a t^{-j})(t^i a t^{-i}) \rangle$$

The generator a is meant to correspond with the action of switching a bulb on or off, and the generator t is meant to represent the action of moving the lamplighter one bulb to the right. The relation $a^2 = e$ can be understood to mean that toggling a bulb twice (i.e. turning it on and then off again) has no net effect. The family of relations defined by

$$\forall i, j \in \mathbb{Z} : (t^i a t^{-i})(t^j a t^{-j}) = (t^j a t^{-j})(t^i a t^{-i})$$

encapsulate the notion that moving to a bulb at index i, toggling it, returning to the origin, moving to a bulb at index j, toggling it, and returning to the origin produces the same element as instead toggling j first and then toggling i. That is, these "operations" commute. In this notation, the element represented by Figure 2.1 is $atat^2at^2at^{-2}$.

We can give an equivalent presentation of L in terms of matrices with entries in $\mathbb{Z}_2[t, t^{-1}]$, using the generating set $\left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$. We can do some simple multiplication to figure out which matrix in this presentation is meant to correspond to which generator in the previous presentation. Take a generic element $\left(\begin{smallmatrix} t^x & P \\ 0 & 1 \end{smallmatrix}\right)$ of L and compute

$$\left(\begin{smallmatrix} t^x & P \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^x & t^x + P \\ 0 & 1 \end{smallmatrix}\right)$$

So, in the polynomial of the resultant matrix, the $\mathbb{Z}_2$-coefficient of t^x has changed, and the exponent of t is unchanged in the first entry. Recall that x is meant to encode the position of the lamplighter, and the coefficient of t^x in P is meant to encode the state of the bulb indexed by x. So multiplying by this generator toggles the bulb at the position of the lamplighter–it is equivalent to a in the above presentation. Multiplication by the other generator yields

$$\begin{pmatrix} t^x & P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{x+1} & P \\ 0 & 1 \end{pmatrix}$$

So, in the first entry of the resultant matrix, the exponent of t has increased by 1, and the polynomial is unchanged. Recall that the exponent of t encodes the position of the lamplighter. So multiplying by this generator moves the position of the lamplighter to the right–it is the equivalent of t in the above presentation.

Of course, the graph that we would like to investigate as a Cayley graph of this group is $\mathrm{DL}_2(2)$. In order to get that $\mathrm{DL}_2(2)$ is a Cayley graph of L, we must choose an alternate generating set to work with. The generating set we wish to use is

$$T = \{\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\}$$

which corresponds to $\{t, at\}$ in our previous notation. Observe that this generating set generates the same set of group elements as the above generators, because

$$\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We claim that $\mathrm{DL}_2(2)$ is the Cayley graph of L with respect to T. Before this claim can be substantiated, we must first precisely define $\mathrm{DL}_2(2)$.

2.2 The Diestel-Leader Graph $\mathrm{DL}_2(2)$

The Diestel-Leader graph $\mathrm{DL}_2(2)$ is a subset of the Cartesian product of two binary trees. Figure 2.2 shows the two trees with one sample point from $\mathrm{DL}_2(2)$ marked. We must define two height functions, which are maps $h : T \to \mathbb{R}$ with $\mathrm{Vert}(T)$ mapping onto $\mathbb{Z}$ such that each level is assigned a height value and adjacent levels of vertices are mapped to consecutive

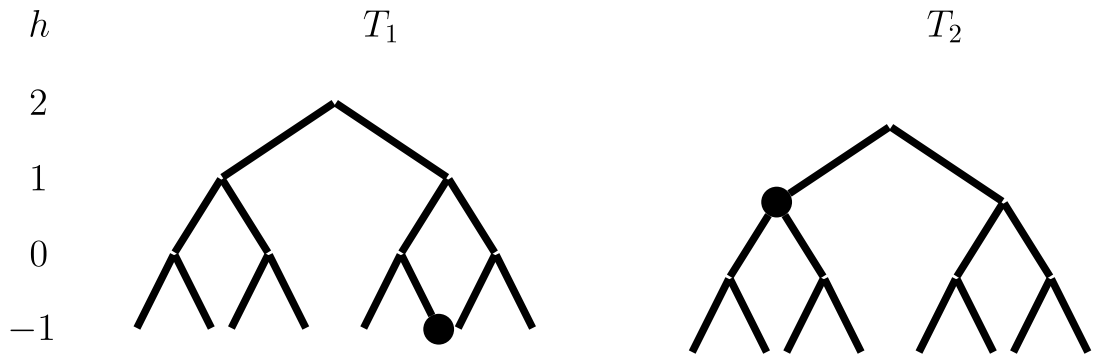


Figure 2.2: A picture of part of the Cartesian product of two binary trees. Note that we have defined a height function $h : T \to \mathbb{Z}$ on each tree, and then lined the levels up by height. The vertices in this Cartesian product are ordered pairs of vertices, one from each tree. The graph $\mathrm{DL}_2(2)$ is a subset of this graph. The indicated point is one example of a vertex of $\mathrm{DL}_2(2)$.

integers. So call the height function on T_1 "h_1" and the height function on T_2 "h_2." The vertices of $\mathrm{DL}_2(2)$ are defined by

$$\mathrm{Vert}(DL_2(2)) = \{(v_1, v_2) \in \mathrm{Vert}(T_1) \times \mathrm{Vert}(T_2) \mid h_1(v_1) + h_2(v_2) = 0\}$$

Using this fact we can readily see that the point indicated in Figure 2.2 is indeed in $\mathrm{DL}_2(2)$. We define the edges of $\mathrm{DL}_2(2)$ as follows:

$$\begin{aligned}\mathrm{Edges}(DL_2(2)) = \big\{\big((v_1, v_2), (w_1, w_2)\big) \in &\big(\mathrm{Vert}(\mathrm{DL}_2(2)) \times \mathrm{Vert}(\mathrm{DL}_2(2))\big)^2 \\ &\mid (v_1, w_1) \in \mathrm{Edges}(T_1), (v_2, w_2) \in \mathrm{Edges}(T_2)\big\}\end{aligned}$$

Figure 2.3 demonstrates that the valence of each vertex in $\mathrm{DL}_2(2)$ is 4. This fact will become important when we argue that $\mathrm{DL}_2(2)$ is the Cayley graph of $\Gamma_2(2)$ with respect to T.

2.3 Identification between L generated by T and $\mathrm{DL}_2(2)$

Recall from Section 1.2 that, to establish that $\mathrm{DL}_2(2)$ is the Cayley graph of L with respect to T, we must show that each element of L corresponds with a unique vertex in $\mathrm{DL}_2(2)$, and that for any $g \in L$ and $s \in T$, the vertex corresponding to g is connected by an edge to the vertex corresponding to gs.

We identify group elements with vertices as follows:
Consider a binary tree with edges labelled as in T_1 in Figure 2.4. Then, for any vertex v at height k, there is a unique sequence of zeros and ones corresponding to the labels on the edges from v to the parent of v, then to the parent above that, and so on. Notice that this string will eventually be only zeros. In this way we can see that vertices of a tree can be uniquely identified with an integer and a series of zeros and ones.

We now wish to give an identification procedure between elements of L and vertices of $\mathrm{DL}_2(2)$. It will use the same strategy, but we will need two sequences to pick out vertices in two trees. So, for an element of L of the form $g = \left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right)$ we first use the integer k to obtain two sequences of zeros and ones of the form $(a_k, a_{k+1}, \dots)$ and $(a_{k-1}, a_{k-2}, \dots)$ where a_i is the coefficient of x^i in P. Note that there can only be a finite number of nonzero coefficients in P, so both of these sequences become all zeroes after a finite length. The vertex associated with this element will have height k in tree 1 and height $-k$ in tree 2. So, we use the integer k and the former sequence to identify a vertex in tree one as described above, and the integer $-k$ and the latter sequence to identify a vertex in tree two as described above. Since the the height of the vertex in tree 1 is k and the height of the vertex in tree 2 is $-k$, this vertex is

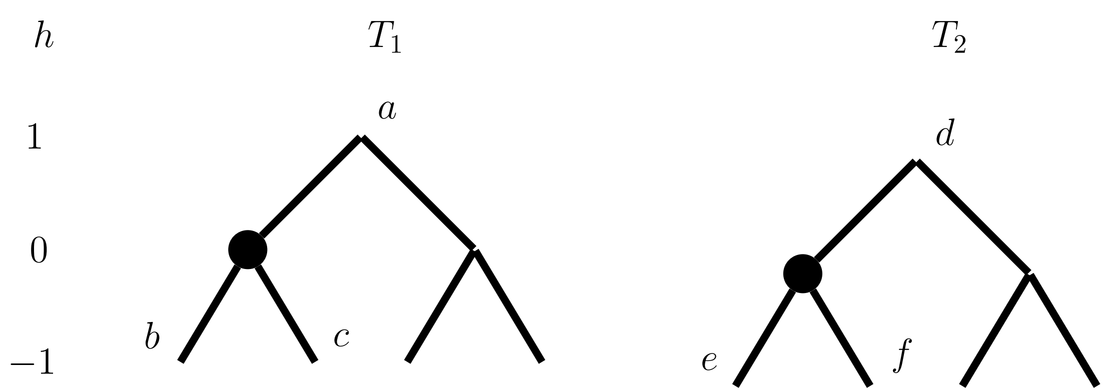


Figure 2.3: The marked vertex in $\mathrm{DL}_2(2)$ is connected by an edge in $\mathrm{DL}_2(2)$ to the vertices (a, e), (a, f), (d, b), and (d, c).

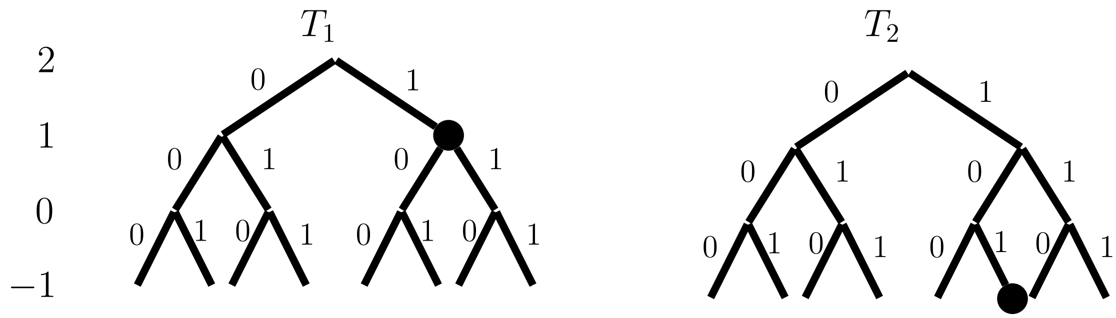


Figure 2.4: The vertex in $\mathrm{DL}_2(2)$ corresponding to $\left(\begin{smallmatrix} t & t+1+t^{-2} \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_2(2)$. The edges are labelled with zeros and ones to facilitate cross-referencing between the trees and the sequences associated with the group element. It is important to note that we assume that the left side of both trees is pictured–that is, the top node is a left child, as is its parent, and so on.

indeed in $\mathrm{DL}_2(2)$.

We have noted in the caption of Figure 2.4 that the vertex indicated corresponds to the group element $\left(\begin{smallmatrix} t & t+1+t^{-2} \\ 0 & 1 \end{smallmatrix}\right)$. We can see that the associated strings for this group element are $(1, 0, 0, \dots)$ and $(1, 0, 1, 0, 0, \dots)$. If we refer to Figure 2.4, it is plain to see that these strings correspond with the indicated vertices as described above.

Now that we have an identification procedure between elements of L and vertices of $\mathrm{DL}_2(2)$, we must show that vertices of $\mathrm{DL}_2(2)$ joined by an edge correspond to elements of L that differ by a member of the generating set T. We do so on the following page in Table 2.1. In this table we can see that for any $g \in L$ and any $s \in T$, the vertices corresponding to g and gs are connected by an edge in $\mathrm{DL}_2(2)$. So the four generators in T correspond to the four edges from any given point in $\mathrm{DL}_2(2)$.

We will now explore the generalization to the Diestel-Leader graph $\mathrm{DL}_3(2)$, which is a subset of the Cartesian product of three binary trees. Once we have a handle on this graph we will investigate the group for which it is a Cayley graph. That group is $\Gamma_3(2)$ with respect to the generating set S defined in Section 2.5.

Generator s	$\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} s$	How to identify the vertex corresponding to $\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} s$ in $\mathrm{DL}_2(2)$
$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{k+1} & P \\ 0 & 1 \end{pmatrix}$	Move up one edge in height in T_1, move down along edge marked a_k (i.e. the coefficient of t^k in P) in T_2
$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1}$	$\begin{pmatrix} t^{k-1} & P \\ 0 & 1 \end{pmatrix}$	Move down along edge marked a_k (i.e. the coefficient of t^k in P) in T_1, move up one edge in height in T_2
$\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{k+1} & P + t^k \\ 0 & 1 \end{pmatrix}$	Move up one edge in height in T_1, move down along edge marked $a_k + 1$ (i.e. the opposite of the coefficient of t^k in P) in T_2
$\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}^{-1}$	$\begin{pmatrix} t^{k-1} & P + t^{k-1} \\ 0 & 1 \end{pmatrix}$	Move down along edge marked $a_k + 1$ (i.e. the opposite of the coefficient of t^k in P) in T_1, move up one edge in height in T_2

Table 2.1: Every generator in $s \in T$, along with, for some arbitrary group element g, instructions on how to find the vertex corresponding to gs from the vertex corresponding with g.

2.4 The Diestel-Leader Graph $\mathbf{DL}_3(2)$

$\mathrm{DL}_3(2)$ is defined analogously to $\mathrm{DL}_2(2)$, as a subset of the Cartesian product of three trees of valence 3: $\mathrm{DL}_3(2) \subset T_1 \times T_2 \times T_3$, where T_1, T_2, and T_3 are binary trees as in the construction of $\mathrm{DL}_2(2)$. We associate a height function with each tree, $h_i : T_i \to \mathbb{R}$ as in Section 2.2, with $h_i(\mathrm{Vert}(T_i)) = \mathbb{Z}$. The set of vertices of $\mathrm{DL}_3(2)$ is given by

$$\mathrm{Vert}(\mathrm{DL}_3(2)) = \{(v_1, v_2, v_3) \in \mathrm{Vert}(T_1) \times \mathrm{Vert}(T_2) \times \mathrm{Vert}(T_3) \mid h_1(v_1) + h_2(v_2) + h_3(v_3) = -1\}.$$

There is a conspicuous -1 in this definition. Recall that in Section 2.2, we set the sum of the heights of the vertices equal to zero. This was merely out of convenience; we could have set this sum equal to any integer n and obtaining an isomorphic graph. In this setting we need the sum to be equal to -1 to ensure that the group $\Gamma_3(2)$ acts faithfully on the graph $\mathrm{DL}_3(2)$.

An edge connects two vertices of $\mathrm{DL}_3(2)$ if and only if the two vertices differ by a single edge in two of the trees and do not differ in the other tree.

We can see that moving along an edge in $\mathrm{DL}_3(2)$ from point (v_1, v_2, v_3) to point (w_1, w_2, w_3) corresponds with moving up in height in one tree, moving down in height in another, and staying fixed in the third as follows: let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. We know that if an edge connects v and w, they have one coordinate in common. Without loss of generality, assume that $v_1 = w_1$. Then v_2 and w_2 differ by an edge in T_2. Again preserving generality, assume that $h_2(v_2) + 1 = h_2(w_2)$. Because both of these points are in $\mathrm{DL}_3(2)$,

$$h_1(v_1) + h_2(v_2) + h_3(v_3) = h_1(w_1) + h_2(w_2) + h_3(w_3).$$

Substituting, we get

$$h_1(v_1) + h_2(v_2) + h_3(v_3) = h_1(v_1) + h_2(v_2) + 1 + h_3(w_3)$$

so we can conclude that $h_3(v_3) - 1 = h_3(w_3)$. One can now see that to get from point v to point w, we must move up in height in tree 2 and down in height in tree 3, while not moving in tree 1. Note the similarities between Figure 2.2 and Figure 2.5. Now we investigate the group for which $\mathrm{DL}_3(2)$ is the Cayley graph with respect to a particular generating set.

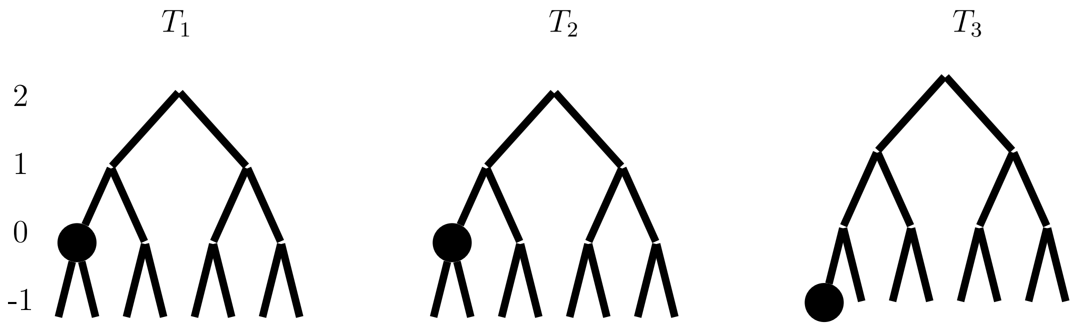


Figure 2.5: The Cartesian product of three binary trees. Note that we have defined a height function $h : T \to \mathbb{Z}$ on each tree, and then lined the levels up by height. The graph $\mathrm{DL}_3(2)$ is a subset of this graph. We will soon see that the marked point will be associated with the identity in $\Gamma_3(2)$.

2.5 The Diestel-Leader Group $\Gamma_3(2)$

Generalizing the matrix representation of the lamplighter group L, we follow Bartholdi, Neuhauser, and Woess in [1] and define and analogous group of matrices $\Gamma_3(2)$ with two formal variables, t and $1+t$. In general, elements of $\Gamma_3(2)$ are of the form $\begin{pmatrix} t^a(1+t)^b & P \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{Z}$ and $P \in \mathbb{Z}_2[t, t^{-1}, (1+t)^{-1}]$. This group is generated by the following set S of matrices:

$$S = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1+t & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} \frac{t}{1+t} & 0 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} \frac{t}{1+t} & \frac{1}{1+t} \\ 0 & 1 \end{pmatrix}^{\pm 1} \right\}$$

It is with respect to this generating set that we will investigate convexity properties of $\Gamma_3(2)$.

2.6 Identification between $\Gamma_3(2)$ and $\mathrm{DL}_3(2)$

We now show that $\mathrm{DL}_3(2)$ is the Cayley graph for $\Gamma_3(2)$ with respect to the generating set S presented in Section 2.5.

Let $g \in \Gamma_3(2)$ be given by $\begin{pmatrix} t^a(1+t)^b & P \\ 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{Z}$ and $P \in \mathbb{Z}_2[t, t^{-1}, (1+t)^{-1}]$.

2.6.1 Associating group elements to vertices

Analogous to what we did in Section 2.3, we need to get from this matrix a height value for each of the three trees, and we need to find a way to convert the information in the polynomial P into three separate sets of instructions that uniquely determine a vertex in each tree. To obtain height values, we use the top left entry $t^a(1+t)^b$. The height in the first tree will be a, the height in the second tree will be b, and, since the vertex must be in $\mathrm{DL}_3(2)$, we must have the height in the third tree equal to $-(a+b+1)$. To identify particular vertices at these heights, we will first associate each tree with a formal variable. Then, we

1. divide P into three separate polynomials, each in terms of one formal variable,
2. define an equivalence relation on polynomials, and use label the vertices of each tree with an equivalence class, and
3. associate each polynomial from step 1 with an equivalence class, then locate the vertex in the appropriate tree labelled by that equivalence class.

Each of these steps is described in a subsection, and the details of the identification are justified in [1].

Dividing P into three polynomials

We first rewrite P in three different Laurent polynomials in one variable, one polynomial for each of the formal variables $t, 1+t$, and t^{-1}. Note that each of these polynomials is infinite in at most one direction–that is to say, each has a minimal exponent. In order to ensure the identification is consistent, we find the Laurent polynomial not of the polynomial P, but instead of the polynomial $t^{-a}(1+t)^{-b}P$, which we will call Q. We denote these three Laurent polynomials by $L_t(Q), L_{1+t}(Q)$, and $L_{t^{-1}}(Q)$.

Consider a sample term $t^i(1+t)^j$ in Q. We show how to rewrite this term in each of our three formal variables. It is crucial to remember in these calculations that the coefficients of these polynomials come from $\mathbb{Z}_2$.

In terms of the formal variable t, rewriting will depend on the sign of j. If j is positive, it is clear that $t^i(1+t)^j$ can be calculated using the binomial theorem, and is equal to some finite polynomial in t.

If j is negative, then we observe that

$$\begin{aligned}
(1+t)^j &= ((1+t)^{-1})^{-j} \\
&= \left(\frac{1}{1+t}\right)^{-j} \\
&= \left(\frac{1}{1-t}\right)^{-j} \\
&= \left(\sum_{i\geq 0} t^i\right)^{-j}
\end{aligned}$$

which, since j is negative, can again be rewritten as a polynomial in t that is infinite in one direction. That is, there is a minimal degree n for which the coefficient of t^n is nonzero.

In terms of the formal variable $1+t$, rewriting will depend on the sign of i. If i is positive, it is clear that $t^i(1+t)^j = (1+t+(1+t)^0)^i(1+t)^j$ can be calculated using the binomial theorem, and is equal to some finite polynomial in $1+t$.

If i is negative, then we observe that $t^i = (t^{-i})^{-1}$. Since i is negative, t^{-i} is a positive power of t and can thus be rewritten as above as a finite polynomial in $(1+t)$. So $t^i = (t^{-i})^{-1}$ is merely the inverse of some finite polynomial in $1+t$. So it is equal to some finite polynomial in $1+t$.

In terms of the formal variable t^{-1}, rewriting will again depend on the sign of j. If j is positive, it is clear that $t^i(1+t)^j = (t^{-1})^{-i}(1+(t^{-1})^{-1})^j$ can be calculated using the binomial theorem, and is equal to some finite polynomial in t^{-1}.

If j is negative, then we observe that

$$\begin{aligned}
(1+t)^j &= ((1+t)^{-1})^{-j} \\
&= \left(\frac{1}{1+t}\right)^{-j} \\
&= \left(\frac{1}{1+\frac{1}{t^{-1}}}\right)^{-j} \\
&= \left(\frac{1}{1+\frac{1}{t^{-1}}} \cdot \frac{t^{-1}}{t^{-1}}\right)^{-j} \\
&= \left(t^{-1} \cdot \frac{1}{1+t^{-1}}\right)^{-j} \\
&= \left(t^{-1} \cdot \frac{1}{1-t^{-1}}\right)^{-j} \\
&= \left(t^{-1} \cdot \sum_{i\geq 0}(t^{-1})^i\right)^{-j} \\
&= \left(\sum_{i\geq 1}(t^{-1})^i\right)^{-j}
\end{aligned}$$

which, since j is negative, can again be rewritten as a polynomial in t^{-1} that is infinite in one direction. That is, there is a minimal degree n for which the coefficient of $(t^{-1})^n$ is nonzero.

So for any term of Q, we can rewrite in terms of any of the three formal variables. For a sample calculation, skip ahead to the initial steps of Example 2.6.1. Therefore we can rewrite the polynomial Q in terms of any of the three formal variables. We call these polynomials in one formal variable $L_t(Q), L_{1+t}(Q)$, and $L_{t^{-1}}(Q)$. We must now multiply the term $t^a(1+t)^b$ back into the polynomials, to get $(t^a(1+t)^b)L_t(Q)$, $(t^a(1+t)^b)L_{1+t}(Q)$, and $(t^a(1+t)^b)L_{t^{-1}}(Q)$. We have seen above that we can still rewrite the resulting polynomial in terms of the desired formal variable. So we will use the following notation for the three polynomials in one formal variable produced in this step:

$$\mathscr{L}_t, \mathscr{L}_{1+t}, \text{ and } \mathscr{L}_{t^{-1}}$$

where the subscript indicates which formal variable we have rewritten P in.

Labeling the vertices of binary trees with polynomials

We now define an equivalence relation on polynomials, and we label equivalence classes as "balls". Define $B(P, 2^n)$ to be the set of polynomials that agree with P on terms of degree less than or equal to $-(n+1)$. So if we fix n, we can have the same ball defined by many different polynomials. For instance,

$$B(t, 2^{-2}) = B(t + t^2, 2^{-2}) = B(t + \sum_{i \geq 2} t^i, 2^{-2}),$$

but

$$B(t, 2^{-2}) \neq B(t + 1, 2^{-2}).$$

This definition is meant to mimic the tree structure, with the exponent of 2 associated with height–consider how $B(0, 2^{-1})$ and $B(1, 2^{-1})$ are distinct, but $B(0, 2^0)$ and $B(1, 2^0)$ are the same. The relationship to binary trees is made explicit below.

We associate these balls with the vertices of the trees in $\mathrm{DL}_3(2)$. Consider as an example the tree associated with t; the procedure in the other trees will be analogous. For any height value n, we can label the points at height n using polynomials in t. We label the leftmost node at height n with the ball $B(0, 2^n)$. The node to its right is labelled $B(t^n, 2^n)$, and the labeling is continued as in Figure 2.6.

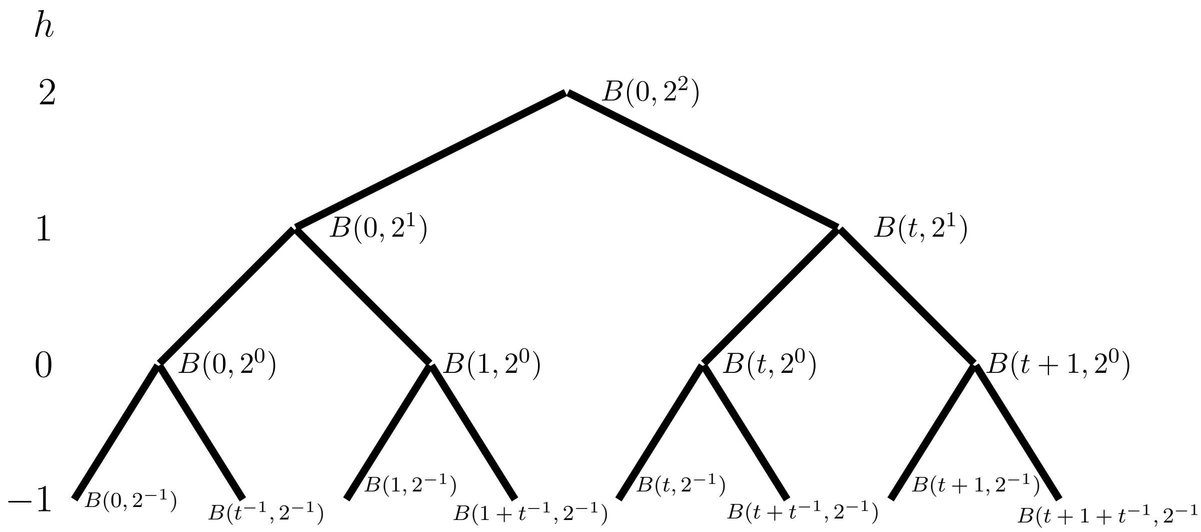


Figure 2.6: One of the three trees of $\mathrm{DL}_3(2)$, which we have associated with the formal variable t. We have also labelled the vertices with balls, or equivalence classes of polynomials.

Association between polynomials and vertices

Recall that the group element we are trying to associate with a vertex of $\mathrm{DL}_3(2)$ is

$$\begin{pmatrix} t^a(1+t)^b & P \\ 0 & 1 \end{pmatrix}$$

where $a, b \in \mathbb{Z}$ and $P \in \mathbb{Z}_2[t, t^{-1}, (1+t)^{-1}]$. We have seen that the height in the tree associated with tree t is a, the height in the tree associated with tree $1+t$ is b, and the height in the tree associated with tree t^{-1} is $-(a+b+1)$. Furthermore, we have seen in step 1 that we can get from P the polynomials

$$\mathscr{L}_t, \mathscr{L}_{1+t}, \text{ and } \mathscr{L}_{t^{-1}},$$

each of which is infinite in at most one direction. We associate these polynomials with balls as described in step 2 as follows:

$$B(\mathscr{L}_t, 2^{-a}), B(\mathscr{L}_{1+t}, 2^{-b}), \text{ and } B(\mathscr{L}_{t^{-1}}, 2^{a+b-1}).$$

Finally, using the labeling described in step 2, we can find a unique vertex in each tree at the specified height with a label equivalent to the ball in the appropriate formal variable.

Example 2.6.1. *Let $g \in \Gamma_3(2)$ be defined by*

$$g = \begin{pmatrix} t(1+t)^{-1} & t^3(1+t)^{-1} \\ 0 & 1 \end{pmatrix}$$

So $Q = (t^3(1+t)^{-1})(t^{-1}(1+t)) = t^2$.

The first step is to identify the three Laurent polynomials $\mathscr{L}_t(Q), \mathscr{L}_{1+t}(Q)$, and $\mathscr{L}_{t^{-1}}(Q)$:

$$\begin{aligned} \mathscr{L}_t(P) &= t^2, \\ \mathscr{L}_{1+t}(P) &= (1+t)^2 + 1, \text{ and} \\ \mathscr{L}_{t^{-1}}(P) &= (t^{-1})^{-2}. \end{aligned}$$

Now we can associate the polynomial with the balls

$$\begin{aligned} &B(t(1+t)^{-1}(t^2), 2^{-1}) \text{ in } T_1, \\ &B(t(1+t)^{-1}((1+t)^2+1), 2^1) \text{ in } T_2, \text{ and} \\ &B(t(1+t)^{-1}((t^{-1})^{-2}), 2^{-1+1-1}) \text{ in } T_3, \end{aligned}$$

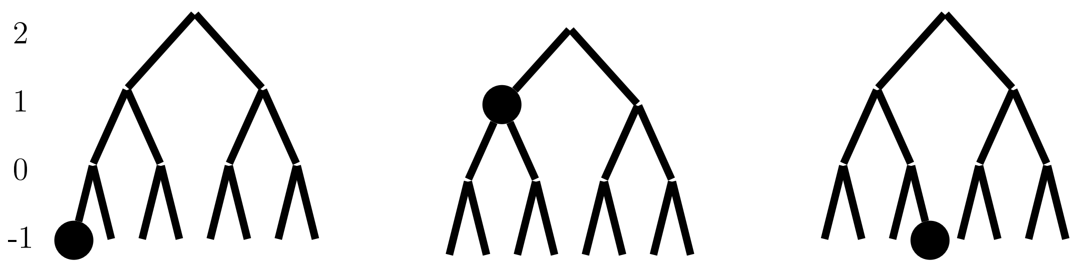
which, after some arithmetic and using the equivalence relation, can be simplified to

$B(0,2^{-1})$ in T_1, because $t(1+t)^{-1}(t^2) = \sum_{i\geq 3} t^i$, which has no terms of degree ≤ 0,

$B(0,2^1)$ in T_2, because $t(1+t)^{-1}((1+t)^2+1) = (1+t)^{-1} + 1 + (1+t) + (1+t)^2$, which has no terms of degree ≤ -2, and

$B((t^{-1})^{-2} + (t^{-1})^{-1} + 1, 2^{-1})$ in T_3, because $t(1+t)^{-1}((t^{-1})^{-2}) = \sum_{i\geq -2} (t^{-1})^i$.

Now we can locate these points on $\mathrm{DL}_3(2)$ using the labeling described in Section 2.6.1. So this group element corresponds to the vertex



We have now seen that each group element can be matched to a vertex. It remains to be shown that right multiplication by a member of the generating set corresponds with traversing an edge in $\mathrm{DL}_3(2)$.

2.6.2 Relationship between generators and edges

We show that for any $g \in \Gamma_3(2)$ and $s \in S$, the elements g and gs correspond to vertices which differ by an edge in $\mathrm{DL}_3(2)$. Consider some point $g \in \Gamma_3(2)$ given by $\begin{pmatrix} t^a(1+t)^b & P \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{Z}$ and $P \in \mathbb{Z}_2[t, t^{-1}, (1+t)^{-1}]$. Associate T_1 with the formal variable t, T_2 with the formal variable $1+t$, and T_3 with the formal variable t^{-1}. We show the result of multiplying g by each generator in S in Table 2.2.

Instructions on how to find the vertex corresponding to gs starting from the vertex corresponding to g can be obtained from Table 2.2 in two steps:

1. To find the heights, refer to the third column, which says in which tree the vertex associated with gs is one edge above the vertex associated with g, and in which tree the vertex associated with gs is one edge below the vertex associated with g. These changes in height are obtained by looking at the exponents in the top-left entry in the product matrix (the second column), which, as we have seen, encode the height of the vertices in each tree.

2. Notice that each pair of height changes in the third column appears twice. For each pair with identical height changes, note that in one case there is a new term introduced to the polynomial and in one case there is not. Recall that the polynomials encode the position of the vertex in each tree. Referring back to Figure 2.6, we can see that two nodes that share a common parent have labels that differ by exactly one term in the polynomial defining the equivalence class. So, these two pairs indicate two different ways to go down one edge (right or left) in the latter tree.

Generator s	gs	Trees in which height increases and decreases, respectively
$\begin{pmatrix} t(1+t)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a+1}(1+t)^{b-1} & P \\ 0 & 1 \end{pmatrix}$	T_1, T_2
$\begin{pmatrix} t(1+t)^{-1} & (1+t)^{-1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a+1}(1+t)^{b-1} & t^a(1+t)^{b-1}+P \\ 0 & 1 \end{pmatrix}$	T_1, T_2
$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a+1}(1+t)^{b} & P \\ 0 & 1 \end{pmatrix}$	T_1, T_3
$\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a+1}(1+t)^{b} & t^{a+1}(1+t)^b+P \\ 0 & 1 \end{pmatrix}$	T_1, T_3
$\begin{pmatrix} t^{-1}(1+t) & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a-1}(1+t)^{b+1} & P \\ 0 & 1 \end{pmatrix}$	T_2, T_1
$\begin{pmatrix} t^{-1}(1+t) & -t^{-1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a-1}(1+t)^{b+1} & t^{a-1}(1+t)^b+P \\ 0 & 1 \end{pmatrix}$	T_2, T_1
$\begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a}(1+t)^{b+1} & P \\ 0 & 1 \end{pmatrix}$	T_2, T_3
$\begin{pmatrix} 1+t & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a}(1+t)^{b+1} & t^a(1+t)^{b+1}+P \\ 0 & 1 \end{pmatrix}$	T_2, T_3
$\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a-1}(1+t)^{b} & P \\ 0 & 1 \end{pmatrix}$	T_3, T_1
$\begin{pmatrix} t^{-1} & -t^{-1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a-1}(1+t)^{b} & t^{a-1}(1+t)^b+P \\ 0 & 1 \end{pmatrix}$	T_3, T_1
$\begin{pmatrix} (1+t)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a}(1+t)^{b-1} & P \\ 0 & 1 \end{pmatrix}$	T_3, T_2
$\begin{pmatrix} (1+t)^{-1} & -(1+t)^{-1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^{a}(1+t)^{b-1} & t^a(1+t)^{b-1}+P \\ 0 & 1 \end{pmatrix}$	T_3, T_2

Table 2.2: Every generator in $s \in S$, along with, for some arbitrary group element g, the matrix corresponding to gs and the height differences between the vertices associated with g and gs.

We have seen in Section 2.4 that moving along an edge in $\mathrm{DL}_3(2)$ corresponds to moving up in one tree and down in another. As shown by Table 2.2, multiplying an arbitrary $g \in \Gamma_3(2)$ by the each generator in S produces twelve group elements that correspond to the twelve vertices connected by an edge to the vertex associated with g. So we have produced the desired correspondence between generators and edges, and conclude that $\mathrm{DL}_3(2)$ is the Cayley graph of $\Gamma_3(2)$ with respect to S.

We will introduce the following notation for our generators: fix a vertex g in $\mathrm{DL}_3(2)$ such that $g = \begin{pmatrix} t^a(1+t)^b & P \\ 0 & 1 \end{pmatrix}$. We let e_{ij} denote the generator such that ge_{ij} is located one edge higher in tree i and one edge lower in tree j, either preserving or introducing a term to the polynomial (depending on the group element g). So, this generator can be seen in $\mathrm{DL}_3(2)$ as the label of the edge which contains the edge which increases in height in T_i and decreases in height to the right in T_j. Similarly let $\overline{e}_{ij}$ denote the generator such that $g\overline{e}_{ij}$ is located one edge higher in tree i and one edge lower in tree j, either failing to introduce or removing a term from P. So, this generator can be seen in $\mathrm{DL}_3(2)$ as the label of the edge which contains the edge which increases in height in T_i and decreases in height to the left in T_j. Notice that the precise generator that has these properties will vary based on g, but for any group element g, there exists some generator with these properties. This notation will henceforth be the only notation we use to refer to the generators in S.

2.6.3 The projection function

We now define a projection function Π from $\mathrm{DL}_3(2)$ to $(\mathbb{Z}^2)^3$ which keeps track of some combinatorial information describing the location of the vertex in all 3 trees. This projection will be useful as it is much simpler to use than the matrix representation of group elements, and it is sufficient to determine the word length of a group element (as shown below in

Section 2.7, and was originally shown in [10]). It is defined by

$$\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$$

where (m_i, l_i) is defined as follows: find a path γ from the coordinate of the identity in tree i to the coordinate of g in tree i. Since we are in a tree, this path is unique. We can see that this path can be divided neatly into two subpaths: the path that strictly goes up in height (call this γ_1), and the path that strictly goes down in height (call this γ_2). So m_i is defined as the length of γ_1 (where length is measured by number of edges), and l_i is defined as the length of γ_2. That is, m_i is how far up in height the path goes, and l_i is how far down in height from the highest point the path goes.

Notice that $m_1 - l_1$ is the height in tree 1 and $m_2 - l_2$ is the height in tree 2, since the coordinate of the identity in trees 1 and 2 is at height 0. Similarly, $m_3 - l_3 - 1$ is the height in tree 3, since the coordinate of the identity in tree 3 is at height -1. We defined the vertices of $\Gamma_3(2)$ such that their heights sum to -1. So $(m_1 - l_1) + (m_2 - l_2) + (m_3 - l_3 - 1) = -1$, or

$$m_1 + m_2 + m_3 = l_1 + l_2 + l_3 \tag{2.1}$$

This equation will be key in many future calculations.

Notice that there is only one projection the corresponds to a unique group element:

$$\Pi(e) = \big((0,0), (0,0), (0,0)\big)$$

Any point with a nontrivial projection coordinate must have a nontrivial l_i value (by Equation 2.1), and there will be at least one other group element with the same projection but with a different coordinate (at the same height) in tree i. This is demonstrated in Figure 2.7.

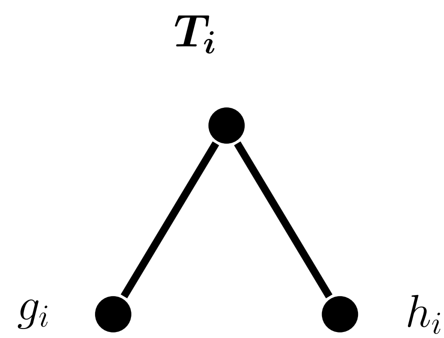


Figure 2.7: Notice that if g_i and h_i represent the coordinates of the vertices associated with group elements g and h in T_i, respectively, then g and h are distinct group elements that have the same projection coordinate in T_i.

2.7 Word Length in $\Gamma_3(2)$

A general formula for the Diestel-Leader groups $\Gamma_d(q)$ which computes word length with respect to the generating set S was proven by Stein and Taback in [10]. In this section we reprove their word length formula specifically for $\Gamma_3(2)$. As mentioned above, this word length formula relies solely on the combinatorial information contained in the projections of the group elements. We will prove below that the word metric on $\Gamma_3(2)$ with respect to S is given by:

$$\boxed{l(g) = \min_{\sigma \in S_3}\{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}\}.}$$

The proof relies on the following lemma:

Lemma 2.7.1 ([10],Lemma 1)**.** *Given a group G with generating set S, let $l : G \to \mathbb{N}$ be the word length with respect to S. If $f : G \to \mathbb{N}$ is another function satisfying:*

1. *$f(g) = 0$ iff g is the identity element of G.*
2. *For every $g \in G$, we have $l(g) \geq f(g)$.*
3. *For every nontrivial $g \in G$ there exists some $s \in S$ with $f(gs) = l(g) - 1$.*

then $l(g) = f(g)$ for every $g \in G$.

Proof. Let $g \in G$, and suppose $f(g) = n$. Then by property (3) there exist $s_1, s_, \ldots, s_n \in S$ satisfying $f(gs_1s_2 \ldots s_n) = 0$. By property (1), $g = s_n^{-1} \ldots s_2^{-1} s_1^{-1}$, so $l(g) \leq f(g)$. Hence by property (2) we have $l(g) = f(g)$. $\square$

So we must show that the above function has the three properties listed in the lemma. We now use Lemma 2.7.1 to show that $l_{\Gamma_3(2)}$ is indeed the word metric on $\Gamma_3(2)$. To facilitate use of this lemma, assume that w represents the word metric on $\Gamma_3(2)$.

Theorem 2.7.2 ([10],Propositions 2,8)**.** *Let g be in $\Gamma_3(2)$. Then the word length of g with respect to the generating set S is given by*

$$l(g) = \min_{\sigma \in S_3} \{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}\}$$

Proof. We show each of the three properties of Lemma 2.7.1 as follows:

Property (1):

First suppose $l(g) = 0$, i.e.

$$\min_{\sigma \in S_3} \{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}\} = 0$$

Note that for any $i \in \{1, 2, 3\}$ we have $m_i \geq 0$ and $l_i \geq 0$, so

$$\min_{\sigma \in S_3} \{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}\} = 0 \text{ implies } m_1 + m_2 + m_3 = 0$$

regardless of σ. Furthermore,

$$m_1 = m_2 = m_3 = 0 \text{ implies } l_1 = l_2 = l_3 = 0$$

by Equation 2.1. So $l(g) = 0$ implies $\Pi(g) = \big((0, 0), (0, 0), (0, 0)\big) = \Pi(e)$. This projection determines a unique point in the graph, so it corresponds to a unique group element. That is, $\Pi(g) = \Pi(e)$ implies $g = e$. Therefore,

$$l(g) = 0 \text{ implies } g = e$$

Now suppose that $g = e$.

$$\begin{aligned} l(e) &= \min_{\sigma \in S_3}\{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\} \\ &= \min_{\sigma \in S_3}\{0 + 0 + \max\{0 + 0 + 0, 0 + 0\}\} \\ &= 0. \end{aligned}$$

So

$$g = e \text{ implies } l(g) = 0$$

We conclude that $l(g) = 0$ iff $g = e$.

Property (2):

Following [10], we introduce the following notation:

$$l_\sigma(g) = m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}$$

Using this notation, $l(g) = \min_{\sigma \in S_3}\{l_\sigma(g)\}$.

We aim to show for every $g \in G$ that there is some σ such that $w(g) \geq l_\sigma(g)$. It will then follow that for every $g \in G$ we have $w(g) \geq l(g)$, since for every $\sigma \in S_3$ we have $l_\sigma(g) \geq l(g)$.

Let g be such that $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$, and let γ describe a minimal-length path from e to g.

We define the three *summits* of γ as follows:

- s_1 is the first point $h_1 \in \gamma$ with $\Pi(h_1) = \big((m_1, 0), (a, b), (c, d)\big)$, where $a, b, c, d \in \mathbb{Z}$.
- s_2 is the first point $h_2 \in \gamma$ with $\Pi(h) = \big((e, f), (m_2, 0), (g, h)\big)$, where $e, f, g, h \in \mathbb{Z}$.
- s_3 is the first point $h_3 \in \gamma$ with $\Pi(h) = \big((i, j), (k, l), (m_3, 0)\big)$, where $i, j, k, l \in \mathbb{Z}$.

If s_1 does not come before s_2 in γ, then rename the summits such that s_1 is the first summit and s_2 is the second. Do the same with s_2 and s_3. Note that these three points are distinct. This can be demonstrated by contradiction as follows: assume $s_1 = s_2$. Then consider the point $x \in \gamma$ immediately preceding s_1. If we assume $x \neq s_1$ and $x \neq s_2$, then $\Pi(x)$ has one of two forms: either $\Pi(x)$ has tree 1 coordinates $(m_1+1, 0)$, in which case we must have already summited in s_1, or $\Pi(x)$ has tree 1 coordinates $(m_1 - 1, 0)$, in which case by Equation 2.1 $\Pi(x)$ either has tree 2 coordinates $(m_2 + 1, 0)$ or has tree 3 coordinates $(m_3 + 1, 0)$, i.e. we have already summited in s_2 or s_3. So if we assume $s_1 = s_2$ then we have mislabeled one or both of these points. We can apply the same argument to show that $s_2 \neq s_3$ and $s_1 \neq s_3$. Let $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4$, where:

- γ_1 is the path from e to s_1.
- γ_2 is the path from s_1 to s_2.
- γ_3 is the path from s_2 to s_3.
- γ_4 is the path from s_3 to g.

In the coming arguments we will be making inferences about the lengths of paths based on the differences of certain projection coordinates. Recall that each generator in S corresponds to an edge that goes up one edge in height in one tree and down one edge in height in another. So for two group elements with a projection coordinate that differs by n, any path between them must contain at least n generators, and therefore have at least length n. This type of argument will be tacitly employed each time we make an inference about the length of a path.

We have labelled the summits in such a way that the desired permutation will be ε, the identity permutation. Since we saw that if $l_\sigma(g) \leq w(g)$ for any $\sigma \in S_3$ then $l(g) \leq w(g)$, it will suffice to simply show this property for $l_\varepsilon(g)$.

$$l_\varepsilon(g) = m_1 + l_3 + \max\{m_1 + m_2 + m_3, m_2 + l_2\}$$

We consider two cases, based on which expression realizes the maximal term in the $l_\varepsilon(g)$ expression.

Case 1: Suppose that $\max\{m_1 + m_2 + m_3, m_2 + l_2\} = m_1 + m_2 + m_3$.

So

$$l_\sigma(g) = m_1 + l_3 + m_1 + m_2 + m_3$$

Consider $\gamma_1\gamma_2\gamma_3$. By the reasoning above, we have

$$\text{length}(\gamma_1) \geq m_1$$

Note that in order to preserve Equation 2.1, the sum of the height of the vertex in T_2 and the height of the vertex in T_3 must have decreased by a total of m_1 at this point. Say we decreased the height in T_2 by n and the height in T_3 by $m_1 - n$. Then

$$\text{length}(\gamma_2) \geq m_2 + n, \text{ and}$$

$$\text{length}(\gamma_3) \geq m_3 + m_1 - n.$$

Furthermore, there can be no overlap between γ_2 and γ_3, because there is no edge of $\mathrm{DL}_3(2)$ that goes up an edge in two trees simultaneously. So

$$\begin{aligned} \text{length}(\gamma_1\gamma_2\gamma_3) &\geq m_1 + m_2 + n + m_3 + m_1 - n \\ &= m_1 + m_1 + m_2 + m_3 \end{aligned}$$

Now consider γ_4. By the reasoning above,

$$\text{length}(\gamma_4) \geq l_3$$

So

$$\text{length}(\gamma) = \text{ length}(\gamma_1\gamma_2\gamma_3\gamma_4) \geq m_1 + l_3 + m_1 + m_2 + m_3 = l_\varepsilon(g)$$

By our choice of γ, we have $w(g) = \text{length}(\gamma)$, so

$$w(g) \geq l_\varepsilon(g)$$

as desired.

Case 2: Suppose that $\max\{m_1 + m_2 + m_3, m_2 + l_2\} = m_2 + l_2$.

So

$$l_\sigma(g) = m_1 + l_3 + m_2 + l_2$$

Consider $\gamma_1\gamma_2$. By the reasoning above, we have

$$\text{length}(\gamma_1) \geq m_1, \text{ and}$$

$$\text{length}(\gamma_2) \geq m_2.$$

Furthermore, there can be no overlap between these two paths, because there is no edge of $\mathrm{DL}_3(2)$ that goes up an edge in two trees simultaneously. So

$$\text{length}(\gamma_1\gamma_2) \geq m_1 + m_2$$

Now consider the rest of γ, namely $\gamma_3\gamma_4$:

$$\text{length}(\gamma_3\gamma_4) \geq l_2, \text{ because we must get to height } m_2 - l_2 \text{ in } T_2, \text{ and}$$

$$\text{length}(\gamma_3\gamma_4) \geq l_3, \text{ because we must get to height } m_3 - l_3 \text{ in } T_3.$$

Furthermore, there can be no overlap between these two parts of the path, because there is no edge of $\text{DL}_3(2)$ that goes down an edge in two trees simultaneously. So

$$\text{length}(\gamma_3\gamma_4) \geq l_2 + l_3$$

So then

$$\text{length}(\gamma) \geq m_1 + l_3 + m_2 + l_2 = l_\varepsilon(g).$$

By our choice of γ, we have $w(g) = \text{length}(\gamma)$, so

$$w(g) \geq l_\varepsilon(g)$$

as desired.

In either case, $w(g) \geq l_\varepsilon(g)$, and we seen that $l_\varepsilon(g) \geq l(g)$, so we conclude that $w(g) \geq l(g)$.

Property (3):

Consider $g \neq e \in \Gamma_3(2)$ with projection $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$.

General observation: If $m_1 = m_2 = m_3 = 0$ then $l_1 + l_2 + l_3 = 0$ by Equation 2.1, so then if $\Pi(g)$ is such that $m_1 = m_2 = m_3 = 0$, then $g = e$. So in general we can say that $g \neq e$ implies at least one of m_1, m_2, m_3 is nonzero and at least one of l_1, l_2, l_3 is nonzero.

Assume we have $\sigma \in S_3$ such that $l(g) = l_\sigma(g)$.

For ease of notation, assume $\sigma = \varepsilon$. It will be clear how to produce the necessary generators for other values of σ as well. So

$$l(g) = m_1 + l_3 + \max\{m_1 + m_2 + m_3, m_2 + l_2\}$$

We consider two cases, based on which expression realizes the maximal term in the $l_\varepsilon(g)$ expression.

Case 1: Suppose that $\max\{m_1 + m_2 + m_3, m_2 + l_2\} = m_2 + l_2$, so $l(g) = m_1 + l_3 + m_2 + l_2$.

- Subcase 1a: If $l_2 = 0$ and $l_3 > 0$, then $m_2 + 0 \geq m_1 + m_2 + m_3$, so $m_1 = m_3 = 0$. But then

  $$l_{(23)}(g) = m_2 < l_{\varepsilon(3)} + m_2 = l_\varepsilon(g)$$

  contradicting the assumption that $l_\varepsilon(g) = l(g)$. So this case does not arise.

- Subcase 1b: If $l_2 > 0$ then $l(g) = m_1 + 0 + m_2 + l_2 = m_1 + l_3 + m_2 + l_2$. Note that $\Pi(ge_{21}) = \big((m_1, l_1 + 1), (m_2, l_2 - 1), (m_3, l_3)\big)$. So

  $$l(ge_{21}) \leq l\varepsilon(ge_{21}) = m_1 + l_3 + m_2 + l_2 - 1 = l(g) - 1$$

- Subcase 1c: If $l_3 = 0$ then $l(g) = m_1 + m_2 + l_2$. Recall from subcase 1a that we must have $l_2 > 0$. So then $\Pi(ge_{21}) = \big((m_1, l_1 + 1), (m_2, l_2 - 1), (m_3, l_3)\big)$. So

  $$l(ge_{21}) \leq l\varepsilon(ge_{21}) = m_1 + m_2 + l_2 - 1 = l(g) - 1$$

Case 2: $\max\{m_1+m_2+m_3, m_2+l_2\} = m_1+m_2+m_3$, and thus $l(g) = m_1+l_3+m_1+m_2+m_3$.

- Subcase 2a: If $l_3 > 0$, then $\Pi(ge_{31}) = \big((m_1, l_1 + 1), (m_2, l_2), (m_3, l_3 - 1)\big)$. So

  $$l(ge_{31}) \leq l_\varepsilon(ge_{31}) = m_1+l_3-1+m_1+m_2+m_3 = m_1+l_3+m_1+m_2+m_3-1 = l(g)-1$$

- Subcase 2b: If $l_3 = 0$, then it follows from the general observation that $l_1 > 0$ or $l_2 > 0$. Assume without loss of generality that $l_1 > 0$. So

$$l(g) = m_1 + 0 + m_1 + m_2 + m_3 = m_1 + m_1 + m_2 + m_3$$

Note that $\Pi(g\bar{e}_{13}) = \big((m_1, l_1 - 1), (m_2, l_2), (m_3 - 1, 0)\big)$, so

$$l(g\bar{e}_{13}) \leq l_\varepsilon(g\bar{e}_{13}) = m_1 + 0 + m_1 + m_2 + m_3 - 1 = l(g) - 1$$

So, in either case, there is a generator s such that $l(gs) \leq l(g) - 1$. Since g and gs are separated by an edge in $\mathrm{DL}_3(2)$, we must have $l(gs) = l(g) - 1$. We can run the above argument assuming any $\sigma \in S_3$ is such that $l_\sigma(g) = l(g)$–the generators produced in each case will be the same as above but with subscripts permuted by σ.
So we conclude that for every $g \in G$ there exists some $s \in S$ such that $l(gs) = l(g) - 1$.

We have now shown that $l(g)$ satisfies the three properties of Lemma 2.7.1, so $l(g)$ is the word length function on $\Gamma_3(2)$. $\square$

We will just write $l(g)$ to refer to the word length of a group element g, and we will say that a permutation $\sigma \in S_3$ *realizes* the word length of g iff $l(g) = l_\sigma(g)$. Note that it is possible for multiple distinct permutations in S_3 to realize the word length of an element g.

Section 3

$\Gamma_3(2)$ is not AC with respect to S

To show that $\Gamma_3(2)$ is not AC with respect to S, we must define a family of elements that will give rise to pairs of points that violate the AC_2 condition. That is, these pairs of points lie in $B(n)$ for some n, are connected by a path of length two, and there is no fixed constant N such that all of these pairs can be connected by a path inside $B(n)$ of length less that N. We first introduce the following notation: for g with $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$, we let $g_i = (m_i, l_i)$ for each $i \in \{1, 2, 3\}$.

Let the family $X \subset \Gamma_3(2)$ be defined by

$$X = \{g \in \Gamma_3(2) \mid \Pi(g) = \big((x, x+1), (2x+1, 2x+1), (x+1, x)\big), x \geq 1\}$$

Theorem 3.0.3. *The Diestel-Leader group* $\Gamma_3(2)$ *is not almost convex with respect to* S*.*

Before proving the theorem, we show a lemma which will allow us to use the combinatorial information encoded in the projections to determine lower bounds on lengths of paths connecting vertices in $\mathrm{DL}_3(2)$.

Lemma 3.0.4. *If $h, h' \in \Gamma_3(2)$ are such that*

$$\Pi(h) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big) \text{ and}$$

$$\Pi(h') = \big((m_1', l_1'), (m_2', l_2'), (m_3', l_3')\big)$$

then

- *if $l_i = l_i' + m$ for some $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$, then $|m|$ is a lower bound on the length of any path from h to h', and*
- *if $m_i \neq m_i'$ for some $i \in \{1, 2, 3\}$ then l_i is a lower bound on the length of any path from h to h'.*

Proof. Assume we have h, h' as above, with $l_i = l_i' + m$ for some $m \in \mathbb{Z}$. It follows from the definition of $\mathrm{DL}_3(2)$ in Section 2.4 that following any single edge of $\mathrm{DL}_3(2)$ will change the coordinate in T_i by at most one edge. So, in order for a path in $\mathrm{DL}_3(2)$ to change the coordinate in T_i by m edges, it must contain at least m edges. That is, such a path must be of length at least $|m|$, as desired.

Now assume we have h, h' as above, and without loss of generality assume that $m_i > m_i'$. Furthermore, assume $l_i > 0$, as the statement is trivially true when $l_i = 0$. Consider the subtree T_h of T_i whose root is at the leftmost vertex at height m_i. We can see that h_i is in the right subtree of T_h, while h_i' is in the left subtree of T_h. Since there are no closed loops, any path γ between these two coordinates must pass through the root of T_h. So the length of γ is greater than or equal to the length of the path from the root of T_h to h_i, which by definition has length equal to l_i. So any path between h_i and h_i' in T_i is of length greater than or equal to l_i. As observed above, this implies that any path in $\mathrm{DL}_3(2)$ from h to h' must have have length greater than or equal to l_i, as desired. □

We now prove the theorem:

Proof. Assume $g \in X$ as defined above, i.e. $\Pi(g) = \big((x, x+1), (2x+1, 2x+1), (x+1, x)\big)$ for some x, and assume $x > 0$.

We will begin by showing that $l(ge_{31}) = l(g\bar{e}_{31}) = l(g) - 1$. We compute word lengths below:

- $l(g) = l_\varepsilon(g) = 6x + 2$, as shown in the following table:

σ	$l_\sigma(g)$	Simplified $l_\sigma(g)$
ε	$x + x + \max\{x + 2x + 1 + x + 1, 2x + 1 + 2x + 1\}$	$6x + 2$
(12)	$2x + 1 + x + \max\{x + 2x + 1 + x + 1, x + x + 1\}$	$7x + 3$
(13)	$x + 1 + x + 1 + \max\{x + 2x + 1 + x + 1, 2x + 1 + 2x + 1\}$	$6x + 4$
(23)	$x + 2x + 1 + \max\{x + 2x + 1 + x + 1, x + 1 + x\}$	$7x + 3$
(123)	$2x + 1 + x + 1 + \max\{x + 2x + 1 + x + 1, x + 1 + x\}$	$7x + 4$
(132)	$x + 1 + 2x + 1 + \max\{x + 2x + 1 + x + 1, x + x + 1\}$	$7x + 4$

- $l(ge_{31}) = l_\varepsilon(ge_{31}) = 6x + 1$, as shown in the following table.

  Note that $\Pi(ge_{31}) = \Pi(g\bar{e}_{31}) = \big((x, x+2), (2x+1, 2x+1), (x+1, x-1)\big)$.

σ	$l_\sigma(ge_{31})$	Simplified $l_\sigma(ge_{31})$
ε	$x + x - 1 + \max\{x + 2x + 1 + x + 1, 2x + 1 + 2x + 1\}$	$6x + 1$
(12)	$2x + 1 + x - 1 + \max\{x + 2x + 1 + x + 1, x + x + 2\}$	$7x + 2$
(13)	$x + 1 + x + 2 + \max\{x + 2x + 1 + x + 1, 2x + 1 + 2x + 1\}$	$6x + 5$
(23)	$x + 2x + 1 + \max\{x + 2x + 1 + x + 1, x + 1 + x - 1\}$	$7x + 3$
(123)	$2x + 1 + x + 2 + \max\{x + 2x + 1 + x + 1, x + 1 + x - 1\}$	$7x + 5$
(132)	$x + 1 + 2x + 1 + \max\{x + 2x + 1 + x + 1, x + x + 2\}$	$7x + 4$

Since ge_{31} and $g\bar{e}_{31}$ have the same projection, they must have the same word length. So, we conclude that $l(ge_{31}) = l(g\bar{e}_{31}) = 6x + 1 = l(g) - 1$.

Now assume γ is a path from ge_{31} to $g\bar{e}_{31}$ such that γ remains entirely inside the ball $B(l(g) - 1)$. We produce a point h which must be on γ by considering the three projections

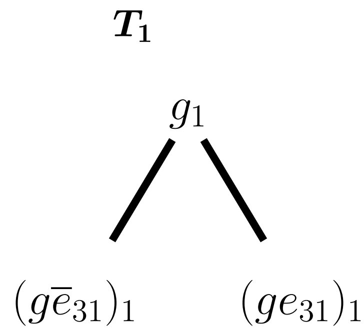


Figure 3.1: The relative positions in T_1 of ge_{31}, $g\bar{e}_{31}$, and g.

$\Pi(g)$, $\Pi(ge_{31})$, and $\Pi(g\bar{e}_{31})$.

First recall that

$$\Pi(g) = \big((x, x+1), (2x+1, 2x+1), (x+1, x)\big) \text{ and}$$

$$\Pi(ge_{31}) = \big((x, x+2), (2x+1, 2x+1), (x+1, x-1)\big) = \Pi(g\bar{e}_{31})$$

Since ge_{31} and $g\bar{e}_{31}$ differ only by their position in tree 1, that is, their coordinates in T_1 share a common parent, any path connecting them in $\mathrm{DL}_3(2)$ must pass through a point h such that $g_1 = h_1$. As proof consider a path in $\mathrm{DL}_3(2)$ between ge_{31} and $g\bar{e}_{31}$ which does not contain such an intermediate point. We can see by our definition of the edges of $\mathrm{DL}_3(2)$ that connected paths in $\mathrm{DL}_3(2)$ define connected paths in each tree. So such a path defines a connected path in T_1–we can readily see from Figure 3.1 that such a path in T_1 would create a closed loop, and therefore must not exist. Since $g_1 = h_1$, we have $h_1 = (x, x+1)$.

Case 1: Assume that we can write

$$\Pi(h) = \big((x, x+1), (2x+1, 2x+1+\delta), (x+1, x-\delta)\big) \text{ for some } -(2x+1) \leq \delta \leq x$$

Since γ stays completely inside the ball $B(l(g)-1)$, the point h must be inside this ball. We will use this fact to produce a lower bound on $|\delta|$. We have that h is inside the ball, so $l(h) < l(g)$, or $l(g) - l(h) > 0$. Recall that $l(g) = 6x+2$. We can no longer say which permutation realizes $l(h)$, so we must test each of them. We first test four permutations, produce a lower bound on $|\delta|$ based on these four, and then verify that this lower bound holds for the other two permutations as well.

	σ	Sign of δ	$l_\sigma(h)$	Simplified $l_\sigma(h)$	$l(g) - l_\sigma(h)$
1	ε	$+$	$x + x - \delta + 2x + 1 + 2x + 1 + \delta$	$6x + 2$	0
2	ε	$-$	$x + x - \delta + x + 2x + 1 + x + 1$	$6x + 2 - \delta$	δ
3	(12)	Irrelevant	$2x + 1 + x - \delta + x + 2x + 1 + x + 1$	$7x + 3 - \delta$	$-x - 1 + \delta$
4	(13)	$+$	$x + 1 + x + 1 + 2x + 1 + 2x + 1 + \delta$	$6x + 4 + \delta$	$-3 - \delta$
5	(13)	$-$	$x + 1 + x + 1 + x + 2x + 1 + x + 1$	$6x + 4$	-2
6	(132)	Irrelevant	$x + 1 + 2x + 1 + \delta + x + 2x + 1 + x + 1$	$7x + 4 + \delta$	$-x - 2 - \delta$

We can see that no σ in rows 1,2,4, or 5 can realize the word length of h, as $l(g) - l_\sigma(h) < 0$ and we have seen that $l(g) - l(h) > 0$. So if h is inside the ball, then the only permutations in the above table that can realize $l(h)$ are those in row 3 or row 6. That is, $l(h) = l_{(12)}(h)$ or $l(h) = l_{(132)}(h)$.

If $l(g) - l_{(12)}(h) > 0$, we must have $-x - 1 + \delta > 0$ and therefore $\delta > x + 1$.

If $l(g) - l_{(132)}(h) > 0$, we must have $-x - 2 - \delta > 0$ and therefore $\delta < -(x + 2)$.

So if any of the permutations in the above table realize the word length of h we must have $|\delta| > x + 1$.

We are seeking a lower bound of δ, so to test the remaining two permutations, we assume $|\delta| \leq x + 1$. Notice that the maximal expression in both the $l_{(23)}(h)$ and the $l_{(123)}(h)$ expressions is

$$\begin{aligned} &\max\{x + 2x + 1 + x + 1, x + 1 + x - \delta\} \\ =&\max\{4x + 2, 2x + 1 - \delta\} \end{aligned}$$

Because $|\delta| \leq x + 1$, we can say that $4x + 2 \geq 2x + 1 - \delta$. So the maximal term in both expressions will be $4x + 2$.

	σ	$l_\sigma(h)$	Simplified $l_\sigma(h)$	$l(g) - l_\sigma(h)$
1	(23)	$x + 2x + 1 + \delta + 4x + 2$	$7x + 3 + \delta$	$-x - 1 - \delta$
2	(123)	$2x + 1 + x + 1 + 4x + 2$	$7x + 4$	$-x - 2$

We can see clearly that $l(g) - l_{(123)}(h) < 0$, so this scenario does not arise. Furthermore, since we assumed $|\delta| \leq x + 1$, we can see that $-x - 1 - \delta < 0$. So $l(g) - l_{(23)}(h) < 0$ as well. So, assuming $|\delta| \leq x + 1$, we can see that neither of these permutations realizes $l(h)$.

Combining the above, we get that if h is in the ball $B(l(g) - 1)$, then $|\delta| > x + 1$. Since $h \in \gamma$, the path γ must contain a subpath from ge_{31} to h, which by Lemma 3.0.4, has a length of at least $x + 2$. So γ has length at least $x + 2$.

Case 2: Assume that we cannot write $\Pi(h)$ as above. That is, either the first coordinate of h_2 or the first coordinate of h_3 differs from the corresponding coordinate in $\Pi(g)$.

Case 2a: Assume the first coordinate of h_2 differs from the first coordinate of g_2. Since $g_2 = (ge_{31})_2$, the first coordinate of h_2 differs from the first coordinate of $(ge_{31})_2$. Then by Lemma 3.0.4, the length of any path between ge_{31} and h is bounded below by $2x + 1$.

Case 2b: Assume the first coordinate of h_3 differs from the first coordinate of g_3. Since the first coordinate of g_3 equals the first coordinate of $(ge_{31})_3$, the first coordinate of h_3 differs from the first coordinate of $(ge_{31})_3$. Then by Lemma 3.0.4, the length of any path between ge_{31} and h is bounded below by $x - 1$.

Again because γ contains a subpath from ge_{31} to h, we conclude that the length of γ is at least $x - 1$. In either case, the length of γ is bounded below by $x - 1$. Thus for any N, we can select $g \in X$ such that $\Pi(g) = \big((N+2, N+3), (2(N+2)+1, 2(N+2)+1), (N+3, N+2)\big)$. Then the elements ge_{31} and $g\bar{e}_{31}$ cannot be connected by a path of length less than $N + 1$ inside the ball of radius $l(ge_{31})$. So $\Gamma_3(2)$ is not AC_2, and therefore not almost convex. $\square$

Section 4

$\Gamma_3(2)$ is MAC with respect to S

4.1 Proof Overview

We wish to show that $\Gamma_3(2)$ is minimally almost convex with respect to S, i.e. that for every pair of points of word length n joined by a path of length two, there exists a path between them remaining inside the closed ball of radius n, whose length is at most $2n - 1$.

We will begin by identifying every pair of points of word length n joined by a path of length two such that the intermediate point on the path is of word length $n+1$. These are precisely the points described in Figure 1.1. Note that if the intermediate point on the path is of length less than $n + 1$, it represents a fixed-length path that stays inside the required ball. So in this way we will enumerate every point for which it is actually necessary to produce a path of length less than $2n - 1$.

Our method for finding all of these points is to identify for each point $g \in \Gamma_3(2)$ every generator $s \in S$ with the property that $l(gs) = l(g) - 1$. Each of these points gs, assuming there is more than one of them, will represent an endpoint of a path we must produce, which stays inside the closed ball $B(l(gs))$ and is of length less than $2l(gs) - 1$. This process is carried out in three main cases, called the two-zero, one-zero, and no zero cases. These names are based on the number of l_i's in $\Pi(g)$ that are equal to zero. Notice that the three-zero case would consist solely of the element g with $\Pi(g) = \big((0,0),(0,0),(0,0)\big)$ (due to equation 2.1), which is the identity and does not give rise to any of the pairs we are interested in.

The two-zero case deals with $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1,l_1),(m_2,0),(m_3,0)\big)$. In this case, we must handle the possibility of some of the m_i's being zero in separate claims. The one-zero case deals with $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1,l_1),(m_2,l_2),(m_3,0)\big)$. In this case it is not important whether m_1 or m_2 are 0, but we must investigate the case where $m_3 = 0$ separately. Finally, the no-zero case deals with $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1,l_1),(m_2,l_2),(m_3,l_3)\big)$. In this case it does not matter if any m_i is 0.

The results of these sections will be enough to describe any element of g up to permutation. That is, to find out which generators s have the property that for $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1,0),(m_2,l_2),(m_3,0)\big)$ we have $l(gs) = l(g) - 1$, we can still use our result from the two-zero case, we merely need to apply the permutation (12) to the subscripts of each generator. The idea of proofs and results being the same up to permutation will arise again in the coming sections.

Note that we will be using the following notation: $m_n(g)$ is the coordinate m_n in $\Pi(g)$. If we are comparing two group elements g and ge_{ij} such that we know $m_n(g) = m_n(ge_{ij})$, we will simply write m_n. Note that this notation can sometimes look similar to the word length formula of an element–so we must remember that $l_\sigma(g)$ for some $\sigma \in S_3$ is an expression dealing with word length of g, while $l_x(g)$ for some $x \in \{1,2,3\}$ is a coordinate from $\Pi(g)$.

Also make note of the following lemmas, the first of which simplifies the process of determining when multiplication by a generator decreases word length. It will be used in the one-zero and no-zero cases, as well as when we go on to build paths.

Lemma 4.1.1. *Let $g \in \Gamma_3(2)$ and $s \in S$. If there is a $\sigma \in S_3$ such that $l_\sigma(gs) = l(g) - 1$ then $l_\sigma(gs) = l(gs)$.*

Proof. Assume we have σ such that $l_\sigma(gs) = l(g) - 1$, and by way of contradiction assume that there is some $\tau \in S_3$ such that $l_\tau(gs) < l_\sigma(gs)$. So $l_\tau(gs) \leq l(g) - 2$, and since $l(gs) \leq l_\tau(gs)$ we can see that $l(gs) \leq l(g) - 2$. So we have found two elements of $\Gamma_3(2)$ that differ by a single generator, whose word length differs by more than one. So l must not be the word length function for $\Gamma_3(2)$ generated by S, contradicting Section 2.7. We conclude that there is no such τ. So then for any other $\tau \in S_3$ we have $l_\tau(gs) \geq l_\sigma(gs)$. Therefore σ is minimal, i.e. $l_\sigma(gs) = l(gs)$, as desired. $\square$

The next lemma is a property on the ordering of the maximal terms $m_1 + m_2 + m_3$ and $m_{\sigma(2)} + l_{\sigma(2)}$.

Lemma 4.1.2. *Let $g \in \Gamma_3(2)$ and let $\{i, j, k\} = \{1, 2, 3\}$. If $\Pi(g)$ is such that $m_1 + m_2 + m_3 < m_i + l_i$ then $m_1 + m_2 + m_3 \geq m_j + l_j$ and $m_1 + m_2 + m_3 \geq m_k + l_k$.*

Proof. Assume without loss of generality that $i = 1$, $j = 2$, and $k = 3$, and assume by way of contradiction that $m_1 + m_2 + m_3 < m_1 + l_1$ and $m_1 + m_2 + m_3 < m_2 + l_2$.
So $2(m_1 + m_2 + m_3) < m_1 + m_2 + l_1 + l_2$.
From Equation 2.1 we have $m_1 + m_2 + m_3 - l_3 = l_1 + l_2$.
So

$$\begin{aligned} 2(m_1 + m_2 + m_3) &< m_1 + m_2 + m_1 + m_2 + m_3 - l_3 \\ m_1 + m_2 + m_3 &< m_1 + m_2 - l_3 \\ m_3 &< -l_3 \end{aligned}$$

Since projection values are nonnegative, this is a contradiction. $\square$

4.1.1 Two-Zero Case

In this section, we will enumerate all generators $s \in S$ with the property $l(gs) = l(g) - 1$ under the assumption that $\Pi(g)$ is of the form $\big((m_1, l_1), (m_2, 0), (m_3, 0)\big)$.

So assume $\Pi(g) = \big((m_1, l_1), (m_2, 0), (m_3, 0)\big)$. We will first concern ourselves with the case where $l_1 = 1$ in the following two lemmas:

Lemma 4.1.3. *If* $\Pi(g) = \big((1, 1), (0, 0), (0, 0)\big)$ *then* $e_{12}, \bar{e}_{12}, e_{13},$ *and* $\bar{e}_{13}$ *are the only generators* s *such that* $l(gs) = l(g) - 1$.

Proof. Clearly $l(g) = 2$, so any point of word length $l(g) - 1$ must be a generator.
We can see that $\Pi(ge_{12}) = \Pi(g\bar{e}_{12}) = \big((1, 0), (0, 1), (0, 0)\big) = e_{12}$.
Similarly $\Pi(ge_{13}) = \Pi(g\bar{e}_{13}) = \big((1, 0), (0, 0), (0, 1)\big) = e_{13}$.
The product of g and any other generator will have a projection with nonzero m-coordinates in two trees, and is therefore not a generator. □

Lemma 4.1.4. *If* $\Pi(g) = \big((0, 1), (1, 0), (0, 0)\big)$ *or* $\Pi(g) = \big((0, 1), (0, 0), (1, 0)\big)$ *then the only generator that decreases word length is* g^{-1}.

Proof. If $\Pi(g) = \big((0, 1), (1, 0), (0, 0)\big)$ then $g = e_{21}$, and if $\Pi(g) = \big((0, 1), (0, 0), (1, 0)\big)$ then $g = e_{31}$.
In either case $l(g) = 1$, so the only point at word length $l(g) - 1$ is the identity.
Because inverses are unique, g^{-1} is the only generator that decreases word length. □

Finally note that the three projections in Lemmas 4.1.3 and 4.1.4 are the only possible projections in the two-zero case with $l_1 = 1$, as a result of Equation 2.1.
So we assume $l_1 > 1$. We also assume m_2 and m_3 are nonzero. We will handle the case in which some of these values are zero separately.
First we simplify the word length formula for this case by enumerating the possible values over all $\sigma \in S_3$. We will use the fact that, from Equation 2.1, $m_1 + m_2 + m_3 = l_1$:

σ	$l_\sigma(g)$	Simplified $l_\sigma(g)$
ε	$m_1 + \max\{m_1 + m_2 + m_3, m_2\}$	$m_1 + m_1 + m_2 + m_3$
(12)	$m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1\}$	$m_2 + m_1 + m_1 + m_2 + m_3$
(13)	$m_3 + l_1 + \max\{m_1 + m_2 + m_3, m_2\}$	$m_3 + m_1 + m_2 + m_3 + m_1 + m_2 + m_3$
(23)	$m_1 + \max\{m_1 + m_2 + m_3, m_3\}$	$m_1 + m_1 + m_2 + m_3$
(123)	$m_2 + l_1 + \max\{m_1 + m_2 + m_3, m_3\}$	$m_1 + m_1 + m_2 + m_3 + m_1 + m_2 + m_3$
(132)	$m_3 + \max\{m_1 + m_2 + m_3, m_1 + l_1\}$	$m_3 + m_1 + m_1 + m_2 + m_3$

So for any $\sigma \in S_3$, we can see by direct comparison that $l_\varepsilon \leq l_\sigma(g)$, and therefore $l_\varepsilon(g) = l(g)$. So in the two zero case,

$$l(g) = m_1 + m_1 + m_2 + m_3 \tag{4.1}$$

Proposition 4.1.5. *For g as above and any generator $s \in S$, we have that $l(gs) = l(g) - 1$ iff $s = \overline{e}_{12}$ or $s = \overline{e}_{13}$*

Proof. We show the biconditional by enumerating all generators and explicitly showing which ones reduce word length. First consider those generators e_{ij} such that $l_2(ge_{ij}) = 0$ and $l_3(ge_{ij}) = 0$. Because we have assumed that $l_1 > 1$, we know that the word length formula specified in equation 4.1 will hold for ge_{ij}. We enumerate those generators and compute corresponding word lengths in Table 4.1. Notice that only $\overline{e}_{12}$ and $\overline{e}_{13}$ subtract 1 from the word length of g, given in Equation 4.1. The remaining generators not in the above table are given in Table 4.2. Let s be one of the generators in Table 4.2. Since each of the generators has the property that either $l_2(gs) \neq 0$ or $l_3(gs) \neq 0$, we can no longer apply Equation 4.1. Instead we divide the problem into two subcases, based on which term is maximal in the $l(gs)$ expression. Assume we have $l(gs) = l_\sigma(gs)$ for some $\sigma \in S_3$. First note that

$$l(gs) = m_{\sigma(1)}(gs) + l_{\sigma(3)}(gs) + \max\{m_1 + m_2(gs) + m_3(gs), m_{\sigma(2)}(gs) + l_{\sigma(2)}(gs)\}$$

Generator s	Differences between $\Pi(g)$ and $\Pi(gs)$	$l(gs)$
$\bar{e}_{12}$	$l_1(g\bar{e}_{12}) = l_1(g) - 1$ $m_2(g\bar{e}_{12}) = m_2(g) - 1$	$m_1 + m_1 + m_2(g) - 1 + m_3$
$\bar{e}_{13}$	$l_1(g\bar{e}_{13}) = l_1(g) - 1$ $m_3(g\bar{e}_{13}) = m_3(g) - 1$	$m_1 + m_1 + m_2 + m_3(g) - 1$
$\bar{e}_{23}$	$m_2(g\bar{e}_{23}) = m_2(g) + 1$ $m_3(g\bar{e}_{23}) = m_3(g) - 1$	$m_1 + m_1 + m_2(g) + 1 + m_3(g) - 1$
$\bar{e}_{32}$	$m_3(g\bar{e}_{32}) = m_3(g) + 1$ $m_2(g\bar{e}_{32}) = m_2(g) - 1$	$m_1 + m_1 + m_2(g) - 1 + m_3(g) + 1$
$\bar{e}_{21}$	$l_1(g\bar{e}_{21}) = l_1(g) + 1$ $m_2(g\bar{e}_{21}) = m_2(g) + 1$	$m_1 + m_1 + m_2(g) + 1 + m_3$
e_{21}	$l_1(ge_{21}) = l_1(g) + 1$ $m_2(ge_{21}) = m_2(g) + 1$	$m_1 + m_1 + m_2(g) + 1 + m_3$
$\bar{e}_{31}$	$l_1(g\bar{e}_{31}) = l_1(g) + 1$ $m_3(g\bar{e}_{31}) = m_3(g) + 1$	$m_1 + m_1 + m_2 + m_3(g) + 1$
e_{31}	$l_1(ge_{31}) = l_1(g) + 1$ $m_3(ge_{31}) = m_3(g) + 1$	$m_1 + m_1 + m_2 + m_3(g) + 1$

Table 4.1: List of generators that do not change the value of l_2 or l_3.

Generator	Projection
e_{12}	$\Pi(ge_{12}) = \big((m_1, l_1 - 1), (m_2, 1), (m_3, 0)\big)$
e_{13}	$\Pi(ge_{13}) = \big((m_1, l_1 - 1), (m_2, 0), (m_3, 1)\big)$
e_{23}	$\Pi(ge_{23}) = \big((m_1, l_1), (m_2 + 1, 0), (m_3, 1)\big)$
e_{32}	$\Pi(ge_{32}) = \big((m_1, l_1), (m_2, 1), (m_3 + 1, 0)\big)$

Table 4.2: The generators not listed in Table 4.1.

Case 1: $\max\{m_1 + m_2(gs) + m_3(gs), m_{\sigma(2)}(gs) + l_{\sigma(2)}(gs)\} = m_{\sigma(2)}(gs) + l_{\sigma(2)}(gs)$.

In this case $m_{\sigma(2)}(gs) + l_{\sigma(2)}(gs) \geq m_1 + m_2(gs) + m_3(gs)$ implies $\sigma(2) = 1$, since l_2 and l_3 are at most 1, and $m_2(gs), m_3(gs) > 0$.

And $l_1(g) = m_1 + m_2(g) + m_3(g)$, as noted above.

So

$$l(gs) = \begin{cases} m_{\sigma(1)}(gs) + l_{\sigma(3)}(gs) + \max\{m_1 + m_2(g) + m_3(g), m_1 + l_1(g)\} & : s = e_{23} \\ m_{\sigma(1)}(gs) + l_{\sigma(3)}(gs) + \max\{m_1 + m_2(g) + m_3(g), m_1 + l_1(g)\} & : s = e_{32} \\ m_{\sigma(1)}(g) + l_{\sigma(3)}(gs) + \max\{m_1 + m_2(g) + m_3(g), m_1 + l_1(g) - 1\} & : s = e_{12}, e_{13} \end{cases}$$

Notice that $m_1 + l_1(g) = m_1 + m_1 + m_2(g) + m_3(g) = l(g)$, so we can see right away that $l(ge_{23}) \geq l(g)$ and $l(ge_{32}) \geq l(g)$. Furthermore, since $m_{\sigma(1)}(g) \geq 1$, we also have $l(ge_{12}) = l(ge_{13}) \geq l(g)$.

Case 2: $\max\{m_1 + m_2(gs) + m_3(gs), m_{\sigma(2)}(gs) + l_{\sigma(2)(gs)}\} = m_1 + m_2(gs) + m_3(gs)$.

In this case $m_{\sigma(2)}(gs) + l_{\sigma(2)}(gs) \leq m_1 + m_2(gs) + m_3(gs)$ implies $\sigma(2) \neq 1$, since $l_1 = m_1 + m_2 + m_3$ and $m_2(gs), m_3(gs) > 0$. So $\sigma(1) = 1$ or $\sigma(3) = 1$. We examine the cases

separately:

Case 2a: $\sigma(1) = 1$

Then $l(gs) = m_1(g) + l_{\sigma(3)}(gs) + m_1 + m_2(gs) + m_3(gs) \geq l(g)$.

Case 2b: $\sigma(3) = 1$

So

$$l(gs) = \begin{cases} m_{\sigma(1)}(gs) + l_1(g) + m_1 + m_2(g) + m_3(g) & : s = e_{23}, e_{32} \\ m_{\sigma(1)}(g) + l_1(g) - 1 + m_1 + m_2(g) + m_3(g) & : s = e_{12}, e_{13} \end{cases}$$

Since $l_1(g) = m_1 + m_2(g) + m_3(g)$, we can see $l(ge_{23}) \geq l(g)$ and $l(ge_{32}) \geq l(g)$, and furthermore since each $\sigma(1) = 2$ or 3 and $m_2(g), m_3(g) \geq 1$ we have $l(ge_{12}) \geq l(g)$ and $l(ge_{13}) \geq l(g)$ as well.

We have shown that, regardless of what form the word length formula takes, $l(gs) \geq l(g)$ where s stands for any of the above 4 generators. So we have enumerated all generators $s \in S$, and found that exactly two of them have the property that $l(gs) = l(g) - 1$: $\overline{e}_{12}$ and $\overline{e}_{13}$. So then for any g as above, we can say $l(gs) = l(g) - 1$ iff $s = \overline{e}_{12}$ or $s = \overline{e}_{13}$. $\square$

We now move on to enumerate exactly which generators s have the property that $l(gs) = l(g) - 1$ in the cases in which some m_i values are zero:

Lemma 4.1.6. *If g is such that $\Pi(g) = \big((m_1, l_1), (0, 0), (m_3, 0)\big)$ and s is any generator then $l(gs) = l(g) - 1$ iff $s = \overline{e}_{12}$, $s = e_{12}$, or $s = \overline{e}_{13}$.*

Proof. The proof is almost identical to the proof of Proposition 4.1.5, with 0 substituted for m_2, with the exception of the $s = e_{12}$ case. To see that this generator also decreases word length, simply observe that $\Pi(ge_{12}) = \big((m_1, l_1 - 1), (0, 1), (m_3, 0)\big) = \Pi(g\bar{e}_{12})$, so $l(ge_{12}) = l(g\bar{e}_{12}) = l(g) - 1$. $\square$

Lemma 4.1.7. *If g is such that $\Pi(g) = \big((m_1, l_1), (m_2, 0), (0, 0)\big)$ and s is any generator then $l(gs) = l(g) - 1$ iff $s = \bar{e}_{12}$, $s = e_{13}$, or $s = \bar{e}_{13}$.*

Proof. The proof is identical to the proof of Lemma 4.1.6, with the permutation (23) applied to all subscripts. $\square$

Lemma 4.1.8. *If g is such that $\Pi(g) = \big((0, l_1), (0, 0), (m_3, 0)\big)$ and s is any generator then $l(gs) = l(g) - 1$ iff $s = \bar{e}_{13}$.*

Proof. The proof is almost identical to the proof of Proposition 4.1.5, with 0 substituted for m_1 and m_2, with the exception of the $s = \bar{e}_{12}$ case. To see that this generator no longer decreases word length, simply observe that $\Pi(g\bar{e}_{12}) = \big((0, l_1 - 1), (0, 1), (m_3, 0)\big) = \Pi(ge_{12})$, so $l(ge_{12}) = l(ge_{12}) \neq l(g) - 1$ $\square$

Lemma 4.1.9. *If g is such that $\Pi(g) = \big((0, l_1), (m_2, 0), (0, 0)\big)$ and s is any generator then $l(gs) = l(g) - 1$ iff $s = \bar{e}_{12}$.*

Proof. The proof is identical to the proof of Lemma 4.1.8, with the permutation (23) applied to all subscripts. $\square$

Claim 4.1.10. *If g is such that $\Pi(g) = \big((m_1, l_1), (0, 0), (0, 0)\big)$ and s is any generator then $l(gs) = l(g) - 1$ iff $s = e_{12}, e_{13}, \bar{e}_{12}$, or $\bar{e}_{13}$.*

Proof. The proof is almost identical to the proof of Proposition 4.1.5, with 0 substituted for m_2 and m_3, with the exception of the $s = e_{12}$ case and the $s = e_{13}$ case. To see the word length does decrease in both of these cases as well, simply apply the arguments from the proofs of Lemmas 4.1.6 and 4.1.7. $\square$

4.1.2 One-Zero Case

In this case we will be considering $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, 0)\big)$, where l_1, l_2, and m_3 are assumed to be nonzero. We will deal with the case when $m_3 = 0$ in a series of corollaries. We will say of any g with a projection of this form that it satisfies the "1 zero condition."

Proposition 4.1.11. *For g satisfying the 1 zero condition, if $\sigma \in S_3$ is such that $l(g) = l_\sigma(g)$ then $\sigma(3) = 3$. That is, $\sigma \in \{\varepsilon, (12)\}$.*

Proof. Assume that $l(g) = l_\sigma(g)$ for some $\sigma \in S_3$.

First note that $m_3 + l_3 \leq m_1 + m_2 + m_3$, because $l_3 = 0$.

We proceed by trying all permutations $\tau \in S_3$ and both maximal terms within the $l_\tau(g)$ expression. In this way we enumerate every possible form that the $l(g)$ expression can take, in Table 4.3

τ	$l_\tau(g)$
ε	$m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2\}$
(12)	$m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1\}$
(23)	$m_1 + l_2 + \max\{m_1 + m_2 + m_3, m_3\}$
(13)	$m_3 + l_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2\}$
(123)	$m_2 + l_1 + \max\{m_1 + m_2 + m_3, m_3\}$
(132)	$m_3 + l_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1\}$

Table 4.3: Every possible form of $l(g)$ for g satisfying the one-zero condition.

We have now enumerated all possible values for $l(g)$. Notice that $l_\varepsilon(g) < l_{(23)}(g)$ and $l_{(12)} < l_{(123)}$, so $l(g) \neq l_{(23)}(g)$ and $l(g) \neq l_{(123)}(g)$. Unfortunately, we cannot simply compare the other expressions directly. Instead, we can consider the problem in cases based on which term is maximal. Recall that $m_1 + m_2 + m_3 > m_3 + l_3$. The other potential maximal expressions can compare to one another in the following four ways:

Case 1: $m_1 + m_2 + m_3 \leq m_1 + l_1$ and $m_1 + m_2 + m_3 \leq m_2 + l_2$.

By Lemma 4.1.2 we have $m_1 + m_2 + m_3 = m_1 + l_1$ or $m_1 + m_2 + m_3 = m_2 + l_2$. So this scenario is covered by case 2 or case 3.

Case 2: $m_1 + l_1 \leq m_1 + m_2 + m_3 \leq m_2 + l_2$.

Then, given σ, we know what the maximal term will be, and we can enumerate the possible values of $l(g)$ as follows:

σ	$l_\sigma(g)$
ε	$m_1 + m_2 + l_2$
(12)	$m_2 + m_1 + m_2 + m_3$
(13)	$m_3 + l_1 + m_2 + l_2$
(132)	$m_3 + l_2 + m_1 + m_2 + m_3$

Direct comparison to $l_\varepsilon(g)$ rules out $l_{(132)}(g)$.

So we consider $l_{(13)}(g)$:

$l_{(13)}(g) = m_3 + m_2 + (l_1 + l_2) = m_3 + m_2 + m_1 + m_2 + m_3 > m_2 + m_1 + m_2 + m_3 = l_{(12)}(g)$.

So $l_{(13)}(g) > l_{(12)}(g)$.

So $l(g) = l_\varepsilon(g)$ or $l(g) = l_{(12)}(g)$.

In either case, $\sigma(3) = 3$.

Case 3: $m_2 + l_2 \leq m_1 + m_2 + m_3 \leq m_1 + l_1$.

Then, given σ, we know what the maximal term will be, and we can enumerate the possible values of $l(g)$ as follows:

σ	$l_\sigma(g)$
ε	$m_1 + m_1 + m_2 + m_3$
(12)	$m_2 + m_1 + l_1$
(13)	$m_3 + l_1 + m_1 + m_2 + m_3$
(132)	$m_3 + l_2 + m_1 + l_1$

Direct comparison to $l_{(12)}(g)$ rules out $l_{(13)}(g)$.

So we consider $l_{(132)}(g)$:

$l_{(132)}(g) = m_3 + m_1 + (l_1 + l_2) = m_3 + m_1 + m_1 + m_2 + m_3 > m_1 + m_1 + m_2 + m_3 = l_\varepsilon(g)$.

So $l_{(132)}(g) > l_\varepsilon(g)$.

So $l(g) = l_\varepsilon(g)$ or $l(g) = l_{(12)}(g)$.

In either case, $\sigma(3) = 3$.

Case 4: $m_1 + l_1 \leq m_1 + m_2 + m_3$ and $m_2 + l_2 \leq m_1 + m_2 + m_3$.

Then, given σ, we know what the maximal term will be, and we can enumerate the possible values of $l(g)$ as follows:

σ	$l_\sigma(g)$
ε	$m_1 + m_1 + m_2 + m_3$
(12)	$m_2 + m_1 + m_2 + m_3$
(13)	$m_3 + l_1 + m_1 + m_2 + m_3$
(132)	$m_3 + l_2 + m_1 + m_2 + m_3$

First consider $l_{(132)}(g)$:

$$l_{(132)}(g) = m_3 + l_2 + (m_1 + m_2 + m_3) \geq m_3 + (l_2 + l_1) + m_1 \geq m_3 + m_1 + m_2 + m_3 + m_1$$
$$> m_1 + m_1 + m_2 + m_3 = l_\varepsilon(g).$$

So $l_{(132)}(g) > l_\varepsilon(g)$.

Now consider $l_{(13)}(g)$:

$$l_{(13)}(g) = m_3 + l_1 + (m_1 + m_2 + m_3) \geq m_3 + (l_1 + l_2) + m_2 \geq m_3 + m_1 + m_2 + m_3 + m_2$$
$$> m_2 + m_1 + m_2 + m_3 = l_{(12)}(g).$$

So $l_{(13)}(g) > l_{(12)}(g)$.

So $l(g) = l_\varepsilon(g)$ or $l(g) = l_{(12)}(g)$.

In either case, $\sigma(3) = 3$.

So $\sigma(3) = 3$ for the minimal σ. □

Corollary 4.1.12. *For* $g \in \Gamma_3(2)$ *with* $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$,

- $l(g) = l_\varepsilon(g) = l_{(132)}(g)$, *or*
- $l(g) = l_{(12)}(g) = l_{(13)}(g)$

and $l(g) \neq l_{(23)}(g), l_{(123)}(g)$.

Proof. Assume $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$, where $m_1, l_1, m_2, l_2 > 0$.

First note that $l(g) \neq l_{(23)}(g)$ and $l(g) \neq l_{(123)}(g)$ by the same logic as in the proof of Proposition 4.1.11.

First note that $m_1 + m_2 = l_1 + l_2$, which will be used in the following calculations.

First we show $l_\varepsilon(g) = l_{(132)}(g)$:

So

$$\begin{aligned} l_\varepsilon(g) &= m_1 + \max\{m_1 + m_2, m_2 + l_2\} \\ &= m_1 + \max\{l_1 + l_2, m_2 + l_2\} \\ &= m_1 + l_2 + \max\{l_1, m_2\} \\ &= l_2 + m_1 + \max\{m_2, l_1\} \\ &= l_2 + \max\{m_1 + m_2, m_1 + l_1\} \\ &= l_{(132)}(g) \end{aligned}$$

So $l_\varepsilon(g) = l_{(132)}(g)$.

Next we show $l_{(12)}(g) = l_{(13)}(g)$:

So

$$
\begin{aligned}
l_{(12)}(g) &= m_2 + \max\{m_1 + m_2, m_1 + l_1\} \\
&= m_2 + \max\{l_1 + l_2, m_1 + l_1\} \\
&= m_2 + l_1 + \max\{l_2, m_1\} \\
&= l_1 + m_2 + \max\{m_1, l_2\} \\
&= l_1 + \max\{m_1 + m_2, m_2 + l_2\} \\
&= l_{(13)}(g)
\end{aligned}
$$

So $l_{(12)}(g) = l_{(13)}(g)$. □

Note that Corollary 4.1.12 allows us to conclude that for any g with

$$\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big),$$

the word length $l(g)$ is realized by ε or (12).

We now wish to find for g satisfying the one-zero condition, which values of $i, j \in \{1, 2, 3\}$ the generators e_{ij} and $\overline{e}_{ij}$ will have the property $l(ge_{ij}) = l(g) - 1$ or $l(g\overline{e}_{ij}) = l(g) - 1$, respectively. We begin with a series of propositions that identify circumstances in which particular values of i, j will never have this property. The first two propositions together say that no generator of the form e_{3j} or $\overline{e}_{3j}$ will have the property $l(gs) = l(g) - 1$.

Proposition 4.1.13. *For g satisfying the 1 zero condition and $j \in \{1, 2\}$, we have $l(ge_{3j}) \geq l(g)$ and $l(g\overline{e}_{3j}) \geq l(g)$.*

Proof. The following proof is for the generator e_{3j}. However, in this case $\Pi(g\overline{e}_{3j}) = \Pi(ge_{3j})$, so the same argument will work for $\overline{e}_{3j}$.

We know $m_3(ge_{3j}) = m_3(g) + 1$. Furthermore, since $j \neq 3$, we know $l_3(ge_{3j}) = l_3(g) = 0$.

So Proposition 4.1.11 applies to both group elements. We divide the proof into four cases according to which permutations in $\{\varepsilon, (12)\}$ minimize the $l(g)$ and $l(ge_{3j})$ expressions:

Case 1: Assume $l(g) = l_\varepsilon(g)$ and $l(ge_{3j}) = l_\varepsilon(ge_{3j})$.

So

$$l(g) = m_1 + l_3(g) + \max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\}$$

and

$$l(ge_{3j}) = m_1 + l_3(ge_{3j}) + \max\{m_1 + m_2 + m_3(ge_{3j}), m_2 + l_2(ge_{3j})\}$$

Since $m_3(ge_{3j}) = m_3(g) + 1$ and $l_3(ge_{3j}) = l_3(g) = 0$, we can say

$$l(g) = m_1 + \max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\}$$

and

$$l(ge_{3j}) = m_1 + \max\{m_1 + m_2 + m_3(g) + 1, m_2 + l_2(ge_{3j})\}$$

Since $j \neq 3$, we know that $l_2(ge_{3j}) \geq l_2(g)$. So, regardless of which term is maximal, we have

$$\max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\} \leq \max\{m_1 + m_2 + m_3(g) + 1, m_2 + l_2(ge_{3j})\}$$

So then direct comparison shows $l(g) \leq l(ge_{3j})$.

Case 2: Assume $l(g) = l_\varepsilon(g)$ and $l(ge_{3j}) = l_{(12)}(ge_{3j})$.

First note that $l(g) = l_\varepsilon(g)$ implies $l_{(12)}(g) \geq l_\varepsilon(g)$, i.e.

$$m_2 + \max\{m_1 + m_2 + m_3(g), m_1 + l_1(g)\} \geq m_1 + \max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\}$$

We can see that

$$l(ge_{3j}) = m_2 + \max\{m_1 + m_2 + m_3(ge_{3j}), m_1 + l_1(ge_{3j})\}.$$

Since $m_3(ge_{3j}) = m_3(g) + 1$ and $l_3(ge_{3j}) = l_3(g) = 0$, we can substitute to get

$$l(ge_{3j}) = m_2 + \max\{m_1 + m_2 + m_3(g) + 1, m_1 + l_1(ge_{3j})\}$$

Since $j \neq 3$, we know that $l_1(ge_{3j}) \geq l_1(g)$. Combining the above, we get

$$\begin{aligned} l(g) \leq l_{(12)}(g) &= m_2 + \max\{m_1 + m_2 + m_3(g), m_1 + l_1(g)\} \\ &\leq m_2 + \max\{m_1 + m_2 + m_3(g) + 1, m_1 + l_1(ge_{3j})\} = l(ge_{3j}) \end{aligned}$$

So then $l(g) \leq l(ge_{3j})$.

Case 3: Assume $l(g) = l_{(12)}(g)$ and $l(ge_{3j}) = l_{\varepsilon}(ge_{3j})$.

This case is analogous to case 2, with the permutation (12) substituted for ε and replacing 1 with 2 and 2 with 1 in all subscripts of coordinates.

Case 4: Assume $l(g) = l_{(12)}(g)$ and $l(ge_{3j}) = l_{(12)}(ge_{3j})$.

This case is analogous to case 1, with the permutation (12) substituted for ε and replacing 1 with 2 and 2 with 1 in all subscripts of coordinates.

Regardless of minimal permutation, we have $l(ge_{3j}) \geq l(g)$. □

We now show that e_{13} and e_{23} do not have the property $l(gs) = l(g) - 1$ for any g satisfying the 1 zero condition:

Proposition 4.1.14. *For g satisfying the 1 zero condition, $l(ge_{13}) \geq l(g)$ and $l(ge_{23}) \geq l(g)$.*

Proof. Consider e_{13}. For this generator we have

$$\Pi(ge_{13}) = \big((m_1, l_1(g) - 1), (m_2, l_2), (m_3, 1)\big)$$

We cannot appeal to Proposition 4.1.11, so we enumerate the values of $l_\sigma(ge_{13})$ over all $\sigma \in S_3$ in Table 4.4.

σ	Maximal term in $\Pi(g)$	$l_\sigma(ge_{13})$	$l_\sigma(g)$
ε	$m_1+m_2+m_3$	$m_1+1+m_1+m_2+m_3$	$m_1+m_1+m_2+m_3$
ε	m_2+l_2	$m_1+1+m_2+l_2$	$m_1+m_2+l_2$
(12)	$m_1+m_2+m_3$	$m_2+1+m_1+m_2+m_3$	$m_2+m_1+m_2+m_3$
(12)	$m_1+l_1(g)$	$m_2+1+m_1+l_1(g)-1$ or $m_2+1+m_1+m_2+m_3$	$m_2+m_1+l_1(g)$ or $m_2+m_1+m_2+m_3$
(13)	$m_1+m_2+m_3$	$m_3+l_1(g)-1+m_1+m_2+m_3$	$m_3+l_1(g)+m_1+m_2+m_3$
(13)	m_2+l_2	$m_3+l_1(g)-1+m_2+l_2$	$m_3+l_1(g)+m_2+l_2$
(23)	$m_1+m_2+m_3$	$m_1+l_2+m_1+m_2+m_3$	$m_1+l_2+m_1+m_2+m_3$
(23)	m_3	N/A (m_3 can't be maximal)	N/A
(123)	$m_1+m_2+m_3$	$m_2+l_1(g)-1+m_1+m_2+m_3$	$m_2+l_1(g)+m_1+m_2+m_3$
(123)	m_3	N/A (m_3 can't be maximal)	N/A
(132)	$m_1+m_2+m_3$	$m_3+l_2+m_1+m_2+m_3$	$m_3+l_2+m_1+m_2+m_3$
(132)	$m_1+l_1(g)$	$m_3+l_2+m_1+l_1(g)-1$ or $m_3+l_2+m_1+m_2+m_3$	$m_3+l_2+m_1+l_1(g)$ or $m_3+l_2+m_1+m_2+m_3$

Table 4.4: Values of $l_\sigma(ge_{13})$ and $l_\sigma(g)$, for both possible maximal terms in $\Pi(g)$, over all $\sigma \in S_3$.

From Proposition 4.1.11, we know that for each $\sigma \notin \{\varepsilon, (12)\}$ we have $l(g) < l_\sigma(g)$ and so $l_\sigma(g) - 1 \geq l(g)$. So, for each $\sigma \notin \{\varepsilon, (12)\}$, regardless of maximal term, we can say that

$$l_\sigma(ge_{13}) \geq l_\sigma(g) - 1 \geq l(g)$$

Furthermore $l_\varepsilon(ge_{13}) = l_\varepsilon(g) + 1 > l(g)$ for both maximal terms arising when $\sigma = \varepsilon$, and $l_{(12)}(ge_{13}) \geq l_{(12)}(g) \geq l(g)$ for both maximal terms arising when $\sigma = (12)$. So

$$l(ge_{13}) \geq l(g)$$

Now consider e_{23}. For this generator we have

$$\Pi(ge_{23}) = \big((m_1, l_1(g)+1), (m_2, l_2), (m_3, 1)\big)$$

We cannot appeal to Proposition 4.1.11, so we enumerate the values of $l(ge_{23})$ over all $\sigma \in S_3$ in Table 4.5.

σ	Maximal term in $\Pi(g)$	$l_\sigma(ge_{23})$	$l_\sigma(g)$
ε	$m_1+m_2+m_3$	$m_1+1+m_1+m_2+m_3$	$m_1+m_1+m_2+m_3$
ε	$m_2+l_2(g)$	$m_1+1+m_2+l_2(g)-1$ or $m_1+1+m_1+m_2+m_3$	$m_1+m_2+l_2(g)$ or $m_1+m_1+m_2+m_3$
(12)	$m_1+m_2+m_3$	$m_2+1+m_1+m_2+m_3$	$m_2+m_1+m_2+m_3$
(12)	m_1+l_1	$m_2+1+m_1+l_1$	$m_2+m_1+l_1$
(13)	$m_1+m_2+m_3$	$m_3+l_1+m_1+m_2+m_3$	$m_3+l_1+m_1+m_2+m_3$
(13)	$m_2+l_2(g)$	$m_3+l_1+m_2+l_2(g)-1$ or $m_3+l_1+m_1+m_2+m_3$	$m_3+l_1+m_2+l_2(g)$ or $m_3+l_1+m_1+m_2+m_3$
(23)	$m_1+m_2+m_3$	$m_1+l_2(g)-1+m_1+m_2+m_3$	$m_1+l_2(g)+m_1+m_2+m_3$
(23)	m_3	N/A (m_3 can't be maximal)	N/A
(123)	$m_1+m_2+m_3$	$m_2+l_1+m_1+m_2+m_3$	$m_2+l_1+m_1+m_2+m_3$
(123)	m_3	N/A (m_3 can't be maximal)	N/A
(132)	$m_1+m_2+m_3$	$m_3+l_2(g)-1+m_1+m_2+m_3$	$m_3+l_2(g)+m_1+m_2+m_3$
(132)	m_1+l_1	$m_3+l_2(g)-1+m_1+l_1$	$m_3+l_2(g)+m_1+l_1$

Table 4.5: Values of $l_\sigma(ge_{23})$ and $l_\sigma(g)$, for both possible maximal terms in $\Pi(g)$, over all $\sigma \in S_3$.

Again from Proposition 4.1.11, we know that for each $\sigma \notin \{\varepsilon, (12)\}$ we have $l(g) < l_\sigma(g)$ and so $l_\sigma(g) - 1 \geq l(g)$. So, for each $\sigma \notin \{\varepsilon, (12)\}$, regardless of maximal term, we can say that

$$l_\sigma(ge_{23}) \geq l_\sigma(g) - 1 \geq l(g)$$

Furthermore $l_\varepsilon(ge_{23}) \geq l_\varepsilon(g) \geq l(g)$ for both maximal terms arising when $\sigma = \varepsilon$, and $l_{(12)}(ge_{23}) = l_{(12)}(g) + 1 > l(g)$ for both maximal terms arising when $\sigma = (12)$. So

$$l(ge_{23}) \geq l(g)$$

□

The next proposition enumerates scenarios in which the generators e_{12} and $\bar{e}_{12}$ do not have the property $l(gs) = l(g) - 1$.

Proposition 4.1.15. *If g satisfies the 1 zero condition and is such that $l(g) \neq l_{(12)}(g)$ then $l(ge_{12}) \geq l(g)$ and $l(g\bar{e}_{12}) \geq l(g)$.*

Proof. The following proof is for the generator e_{12}. However, in this case $\Pi(g\bar{e}_{12}) = \Pi(ge_{12})$, so the same argument will work for $\bar{e}_{12}$.

Assume g is such that $l(g) \neq l_{(12)}(g)$.

We can see that $\Pi(g)$ and $\Pi(ge_{12})$ are such that $l_1(ge_{12}) = l_1(g) - 1$ and $l_2(ge_{12}) = l_2(g) + 1$, and every other coordinate is the same.

So $l(g) = l_\varepsilon(g) = m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(g)\} < m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g)\}$, so then $l(g) \leq m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g)\} - 1$.

We divide the problem into cases based on which σ realizes the word length of ge_{12}. Note that, because $l_3(ge_{12}) = l_3(g) = 0$, by Proposition 4.1.11, $\sigma = \varepsilon$ or $\sigma = (12)$.

Case 1: $l(ge_{12}) = l_\varepsilon(ge_{12})$.

So

$$\begin{aligned} l(ge_{12}) &= m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(ge_{12})\} \\ &= m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(g) + 1\} \\ &\geq m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(g)\} \\ &= l(g). \end{aligned}$$

So $l(ge_{12}) \geq l(g)$.

Case 2: $l(ge_{12}) = l_{(12)}(ge_{12})$.

So

$$\begin{aligned} l(ge_{12}) &= m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(ge_{12})\} \\ &= m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g) - 1\}. \end{aligned}$$

Notice that $\max\{m_1 + m_2 + m_3, m_1 + l_1(g)\} - 1 \leq \max\{m_1 + m_2 + m_3, m_1 + l_1(g) - 1\}$.

So

$$\begin{aligned}
l(g) &\leq m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g)\} - 1 \\
&\leq m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g) - 1\} \\
&= l(ge_{12}).
\end{aligned}$$

So $l(ge_{12}) \geq l(g)$. $\square$

We now show a similar result for e_{21} and $\overline{e}_{21}$.

Proposition 4.1.16. *If g satisfies the 1 zero condition and is such that $l(g) \neq l_\varepsilon(g)$ then $l(ge_{21}) > l(g) - 1$ and $l(g\overline{e}_{21}) > l(g) - 1$.*

Proof. The proof of Proposition 4.1.16 follows the reasoning of the proof of Proposition 4.1.15 exactly, with every permutation multiplied by (12) and every projection coordinate subscript and generator subscript permuted by (12). $\square$

Corollary 4.1.17. *The results of Propositions 4.1.13, 4.1.15, and 4.1.16 all hold if $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$ as well.*

Proof. Simply note that none of the proofs of Propositions 4.1.13, 4.1.15, or 4.1.16 relied on m_3 being nonzero. $\square$

We now wish to enumerate exactly when the rest of the generators s have the property $l(gs) = l(g) - 1$. We approach the problem casewise. For g satisfying the 1 zero condition, Proposition 4.1.11 tells us that there are two permutations that can realize $l(g)$. So we consider the problem in three cases: when only ε realizes word length, when only (12) realizes word length, and when both permutations realize word length.

Theorem 4.1.18. *If g satisfies the 1 zero condition and $l(g) = l_\varepsilon(g) = l_{(12)}(g)$ then:*

- *$l(ge_{21}) = l(g) - 1$ and $l(g\bar{e}_{21}) = l(g) - 1$ iff $m_2 + l_2(g) > m_1 + m_2 + m_3$*

- *$l(ge_{12}) = l(g) - 1$ and $l(g\bar{e}_{12}) = l(g) - 1$ iff $m_1 + l_1(g) > m_1 + m_2 + m_3$*

- *$l(g\bar{e}_{13}) = l(g) - 1$*

- *$l(g\bar{e}_{23}) = l(g) - 1$*

- *Any generator s not noted above does not have the property $l(gs) = l(g) - 1$ under any circumstances.*

Proof. Since $l_{\varepsilon}(g) = l_{(12)}(g)$ we know that

$$m_2 + \max\{m_1 + m_2 + m_3(g), m_1 + l_1(g)\} = m_1 + \max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\}$$

First, we can eliminate six generators that we know will not have the property $l(gs) = l(g) - 1$ as follows:

- $l(ge_{32}) \geq l(g)$,

- $l(ge_{31}) \geq l(g)$,

- $l(g\bar{e}_{32}) \geq l(g)$, and

- $l(g\bar{e}_{31}) \geq l(g)$ by Proposition 4.1.13, and

- $l(ge_{13}) \geq l(g)$ and

- $l(ge_{23}) \geq l(g)$ by Proposition 4.1.14.

The remaining 6 generators we consider individually:

Consider $\boldsymbol{e_{21}}$ and $\boldsymbol{\bar{e}_{21}}$. For these generators we have

$$\Pi(ge_{21}) = \Pi(g\bar{e}_{21}) = \big((m_1, l_1(g) + 1), (m_2, l_2(g) - 1), (m_3, 0)\big)$$

so from Proposition 4.1.11 we know that $l(ge_{21})$ and $l(g\overline{e}_{21})$ are minimized by ε or (12). The corresponding expressions are below:

$$l_{(12)}(ge_{21}) = l_{(12)}(g\overline{e}_{21}) = m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g) + 1\} \geq l(g)$$

$$l_{\varepsilon}(ge_{21}) = l_{\varepsilon}(g\overline{e}_{21}) = m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(g) - 1\} \leq l(g)$$

We proceed using e_{21}. The same argument works for $\overline{e}_{21}$.

If (12) realizes $l(ge_{21})$ then $l(ge_{21}) \geq l(g)$.

But if ε realizes $l(ge_{21})$ then $l(ge_{21}) = l(g) - 1$ only if $m_2 + l_2(g) > m_1 + m_2 + m_3$.

Notice that if $m_2 + l_2(g) > m_1 + m_2 + m_3$ then $l_{\varepsilon}(ge_{21}) < l(g) \leq l_{(12)}(ge_{21})$, and so ε realizes $l(ge_{21})$.

So

$$m_2 + l_2(g) > m_1 + m_2 + m_3 \implies \varepsilon \text{ realizes } l(ge_{21})$$

and

$$\varepsilon \text{ realizes } l(ge_{21}) \implies l(ge_{21}) = l(g) - 1 \text{ only if } m_2 + l_2(g) > m_1 + m_2 + m_3$$

So if $m_2 + l_2(g) > m_1 + m_2 + m_3$ then $l(ge_{21}) = l(g) - 1$ and $l(g\overline{e}_{21}) = l(g) - 1$.

Consider $\boldsymbol{e_{12}}$ and $\boldsymbol{\overline{e}_{12}}$. For these generators we have

$$\Pi(ge_{12}) = \Pi(g\overline{e}_{12}) = \big((m_1, l_1(g) - 1), (m_2, l_2(g) + 1), (m_3, 0)\big)$$

so from Proposition 4.1.11 we know that $l(ge_{12})$ and $l(g\overline{e}_{12})$ are minimized by ε or (12). The corresponding expressions are below:

$$l_{\varepsilon}(ge_{12}) = l_{\varepsilon}(g\overline{e}_{12}) = m_1 + \max\{m_1 + m_2 + m_3, m_2 + l_2(g) + 1\} \geq l(g)$$

$$l_{(12)}(ge_{12}) = l_{(12)}(g\overline{e}_{12}) = m_2 + \max\{m_1 + m_2 + m_3, m_1 + l_1(g) - 1\} \leq l(g)$$

We proceed using e_{12}. The same argument works for $\overline{e}_{12}$.

If ε realizes $l(ge_{12})$ then $l(ge_{12}) \geq l(g)$.

But if (12) realizes $l(ge_{12})$ then $l(ge_{12}) = l(g) - 1$ only if $m_1 + l_1(g) > m_1 + m_2 + m_3$.

Notice that if $m_1 + l_1(g) > m_1 + m_2 + m_3$ then $l_{(12)}(ge_{12}) < l(g) \leq l_\varepsilon(ge_{12})$, and so (12) realizes $l(ge_{12})$.

So

$$m_1 + l_1(g) > m_1 + m_2 + m_3 \implies (12) \text{ realizes } l(ge_{12})$$

and

$$(12) \text{ realizes } l(ge_{12}) \implies l(ge_{12}) = l(g) - 1 \text{ only if } m_1 + l_1(g) > m_1 + m_2 + m_3$$

So if $m_1 + l_1(g) > m_1 + m_2 + m_3$ then $l(ge_{12}) = l(g) - 1$ and $l(g\overline{e}_{12}) = l(g) - 1$.

Consider $\overline{e}_{13}$. For this generator we have

$$\Pi(g\overline{e}_{13}) = \big((m_1, l_1(g) - 1), (m_2, l_2), (m_3(g) - 1, 0)\big)$$

Note that

$$l_{(12)}(g\overline{e}_{13}) = m_2 + \max\{m_1 + m_2 + m_3(g) - 1, m_1 + l_1(g) - 1\} = l(g) - 1$$

so by Lemma 4.1.1, $\overline{e}_{13}$ decreases word length.

Consider $\overline{e}_{23}$. For this generator we have

$$\Pi(g\overline{e}_{23}) = \big((m_1, l_1), (m_2, l_2(g) - 1), (m_3(g) - 1, 0)\big)$$

Note that

$$l_\varepsilon(g\overline{e}_{23}) = m_1 + \max\{m_1 + m_2 + m_3(g) - 1, m_2 + l_2(g) - 1\} = l(g) - 1$$

So by Lemma 4.1.1, $\overline{e}_{23}$ decreases word length.

We have now considered every generator. $\square$

Corollary 4.1.19. *If $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$ and $l(g) = l_\varepsilon(g) = l_{(12)}(g)$ then*

- $l(ge_{21}) = l(g) - 1$ *and* $l(g\overline{e}_{21}) = l(g) - 1$ *iff* $m_2 + l_2(g) > m_1 + m_2 + m_3$
- $l(ge_{12}) = l(g) - 1$ *and* $l(g\overline{e}_{12}) = l(g) - 1$ *iff* $m_1 + l_1(g) > m_1 + m_2 + m_3$
- $l(ge_{13}) = l(g\overline{e}_{13}) = l(g) - 1$
- $l(ge_{23}) = l(g\overline{e}_{23}) = l(g) - 1$
- *Any generator s not noted above does not have the property $l(gs) = l(g) - 1$ under any circumstances.*

Proof. For the generators $e_{21}, \overline{e}_{21}, e_{12}$, and $\overline{e}_{12}$, we can simply refer to the proof of Theorem 4.1.18, as these generators did not rely on m_3 being nonzero.

Similarly for $e_{32}, e_{31}, \overline{e}_{32}$, and $\overline{e}_{31}$, we can simply refer to the proof of Theorem 4.1.18, as these generators did not rely on m_3 being nonzero.

Next we will show $l(ge_{13}) = l(g) - 1$ and $l(g\overline{e}_{13}) = l(g) - 1$.

Notice that

$$\Pi(ge_{13}) = \Pi(g\overline{e}_{13}) = \big((m_1, l_1 - 1), (m_2, l_2), (0, 1)\big)$$

So

$$
\begin{aligned}
l_{(13)}(ge_{13}) &= l_{(13)}(g\bar{e}_{13}) \\
&= l_1 - 1 + \max\{m_1 + m_2, m_2 + l_2\} \\
&= l_{(13)}(g) - 1 \\
&= l_{(12)}(g) - 1 \text{ by Corollary 4.1.12} \\
&= l(g) - 1 \text{ by the assumptions of the corollary.}
\end{aligned}
$$

So $l(ge_{13}) = l(g\bar{e}_{13}) = l(g) - 1$ by Lemma 4.1.1.

Finally we show $l(ge_{23}) = l(g) - 1$ and $l(g\bar{e}_{23}) = l(g) - 1$.

Notice that

$$\Pi(ge_{23}) = \Pi(g\bar{e}_{23}) = \big((m_1, l_1), (m_2, l_2 - 1), (0, 1)\big)$$

So

$$
\begin{aligned}
l_{(132)}(ge_{23}) &= l_{(132)}(g\bar{e}_{23}) \\
&= l_2 - 1 + \max\{m_1 + m_2, m_1 + l_1\} \\
&= l_{(132)}(g) - 1 \\
&= l_{\varepsilon}(g) - 1 \text{ by Corollary 4.1.12} \\
&= l(g) - 1 \text{ by the assumptions of the corollary.}
\end{aligned}
$$

So $l(ge_{23}) = l(g\bar{e}_{23}) = l(g) - 1$ by Lemma 4.1.1. $\square$

Theorem 4.1.20. *If g satisfies the 1 zero condition and $l(g) \neq l_{(12)}(g)$ then:*

- $l(ge_{21}) = l(g) - 1$ *and* $l(g\bar{e}_{21}) = l(g) - 1$ *iff* $m_2 + l_2(g) > m_1 + m_2 + m_3$
- $l(g\bar{e}_{13}) = l(g) - 1$ *iff* $m_2 + l_2 < m_1 + m_2 + m_3(g)$

- $l(g\overline{e}_{23}) = l(g) - 1$

- *Any generator s not noted above does not have the property $l(gs) = l(g) - 1$ under any circumstances.*

Proof. By Proposition 4.1.13:

$$l(g) = l_{\varepsilon}(g) = m_1 + \max\{m_1 + m_2 + m_3(g), m_2 + l_2(g)\}$$

First, we can eliminate eight generators that we know will not have the property $l(gs) = l(g) - 1$ as follows:

- $l(ge_{32}) \geq l(g)$,

- $l(ge_{31}) \geq l(g)$,

- $l(g\overline{e}_{32}) \geq l(g)$, and

- $l(g\overline{e}_{31}) \geq l(g)$ by Proposition 4.1.13,

- $l(ge_{13}) \geq l(g)$ and

- $l(ge_{23}) \geq l(g)$ by Proposition 4.1.14, and

- $l(ge_{12}) \geq l(g)$ and

- $l(g\overline{e}_{12}) \geq l(g)$ by Proposition 4.1.15.

So we have 4 generators left to investigate:

Consider $\boldsymbol{e_{21}}$, $\boldsymbol{\overline{e}_{21}}$, and $\boldsymbol{\overline{e}_{23}}$. For these generators we can apply the same argument from the proof of Theorem 4.1.18. So $\overline{e}_{23}$ decreases word length under any conditions, if $m_2+l_2(g) > m_1+m_2+m_3$ then e_{21} decreases word length, if $m_2+l_2(g) > m_1+m_2+m_3$ then $\overline{e}_{21}$ decreases word length, and no other generator s on this list has the property $l(gs) = l(g)-1$.

Consider $\bar{e}_{13}$. For this generator we have

$$\Pi(g\bar{e}_{13}) = \big((m_1, l_1(g)-1), (m_2, l_2), (m_3(g)-1, 0)\big)$$

so from Proposition 4.1.11 we know that $l(g\bar{e}_{13})$ is minimized by ε or (12). The corresponding expressions are below:

$$l_{(12)}(g\bar{e}_{13}) = m_2 + \max\{m_1 + m_2 + m_3(g) - 1, m_1 + l_1(g) - 1\} = l_{(12)}(g) - 1 \geq l(g)$$

$$l_{\varepsilon}(g\bar{e}_{13}) = m_1 + \max\{m_1 + m_2 + m_3(g) - 1, m_2 + l_2\} \leq l_{\varepsilon}(g) = l(g)$$

So if (12) realizes $l(g\bar{e}_{13})$ then $l(g\bar{e}_{13}) \geq l(g)$.
But if ε realizes $l(g\bar{e}_{13})$ then $l(g\bar{e}_{13}) = l(g) - 1$ only if $m_2 + l_2(g) < m_1 + m_2 + m_3$.
Notice that if $m_2 + l_2(g) < m_1 + m_2 + m_3$ then $l_{\varepsilon}(g\bar{e}_{13}) < l(g) \leq l_{(12)}(g\bar{e}_{13})$, and so ε realizes $l(g\bar{e}_{13})$.
So

$$m_2 + l_2(g) < m_1 + m_2 + m_3 \implies \varepsilon \text{ realizes } l(g\bar{e}_{13})$$

and

$$\varepsilon \text{ realizes } l(g\bar{e}_{13}) \implies l(g\bar{e}_{13}) = l(g) - 1 \text{ only if } m_2 + l_2(g) < m_1 + m_2 + m_3$$

So if $m_2 + l_2(g) < m_1 + m_2 + m_3$ then $\bar{e}_{13}$ decreases word length.

We have now considered every generator. $\square$

Corollary 4.1.21. *If $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$ and $l(g) \neq l_{(12)}(g)$ then*

- *$l(ge_{21}) = l(g) - 1$ and $l(g\bar{e}_{21}) = l(g) - 1$ iff $m_2 + l_2(g) > m_1 + m_2$*

- *$l(ge_{13}) = l(g) - 1$ and $l(g\bar{e}_{13}) = l(g) - 1$ iff $m_1 + l_1(g) > m_1 + m_2$*

- *$l(ge_{23}) = l(g\bar{e}_{23}) = l(g) - 1$*

- *Any generator s not noted above does not have the property $l(gs) = l(g) - 1$ under any circumstances.*

Proof. For the generators $e_{21}, \bar{e}_{21}, e_{12}$, and $\bar{e}_{12}$, we can simply refer to the proof of Theorem 4.1.20, as these generators did not rely on m_3 being nonzero.

Similarly for $e_{32}, e_{31}, \bar{e}_{32}$, and $\bar{e}_{31}$, we can simply refer to the proof of Theorem 4.1.20, as these generators did not rely on m_3 being nonzero.

It follows from Corollary 4.1.12 that $l(g) = l_\varepsilon(g) = l_{(132)}(g)$.

First we will show $l(ge_{13}) = l(g) - 1$ and $l(g\bar{e}_{13}) = l(g) - 1$ iff $m_1 + l_1(g) > m_1 + m_2$ using Table 4.6.

σ	Maximal term in $\Pi(g)$	$l_\sigma(ge_{13}) = l_\sigma(g\bar{e}_{13})$	$l_\sigma(g)$
ε	$m_1 + m_2$	$m_1 + 1 + m_1 + m_2$	$m_1 + m_1 + m_2$
ε	$m_2 + l_2$	$m_1 + 1 + m_2 + l_2$	$m_1 + m_2 + l_2$
(12)	$m_1 + m_2$	$m_2 + 1 + m_1 + m_2$	$m_2 + m_1 + m_2$
(12)	$m_1 + l_1(g)$	$m_2 + 1 + m_1 + l_1(g) - 1$ or $m_2 + 1 + m_1 + m_2$	$m_2 + m_1 + l_1(g)$ or $m_2 + m_1 + m_2$
(13)	$m_1 + m_2$	$l_1(g) - 1 + m_1 + m_2$	$l_1(g) + m_1 + m_2$
(13)	$m_2 + l_2$	$l_1(g) - 1 + m_2 + l_2$	$l_1(g) + m_2 + l_2$
(23)	$m_1 + m_2$	$m_1 + l_2 + m_1 + m_2$	$m_1 + l_2 + m_1 + m_2$
(23)	$m_3 = 0$	N/A (0 can't be maximal)	N/A
(123)	$m_1 + m_2$	$m_2 + l_1(g) - 1 + m_1 + m_2$	$m_2 + l_1(g) + m_1 + m_2$
(123)	$m_3 = 0$	N/A (0 can't be maximal)	N/A
(132)	$m_1 + m_2$	$l_2 + m_1 + m_2$	$l_2 + m_1 + m_2$
(132)	$m_1 + l_1(g)$	$l_2 + m_1 + l_1(g) - 1$ or $l_2 + m_1 + m_2$	$l_2 + m_1 + l_1(g)$ or $l_2 + m_1 + m_2$

Table 4.6: Values of $l_\sigma(ge_{13}), l_\sigma(g\bar{e}_{13})$, for either maximal term in $\Pi(g)$, over all $\sigma \in S_3$.

From Corollary 4.1.12 and from the assumptions of the corollary, we know that for each $\sigma \notin \{\varepsilon, (132)\}$ we have $l(g) < l_\sigma(g)$ and so $l_\sigma(g) - 1 \geq l(g)$. So, for $\sigma \neq \varepsilon, (132)$, regardless of maximal term, we have

$$l_\sigma(ge_{13}) = l_\sigma(g\bar{e}_{13}) \geq l_\sigma(g) - 1 \geq l(g)$$

Furthermore, $l_\varepsilon(ge_{13}) = l_\varepsilon(g\bar{e}_{13}) = l_\varepsilon(g) + 1 = l(g) + 1$.

Finally, we can see that $l_{(132)}(ge_{13}) = l_{(132)}(g\bar{e}_{13}) = l(g) - 1$, and therefore $l(ge_{13}) = l(g\bar{e}_{13}) = l(g) - 1$ by Lemma 4.1.1, only under the condition that $m_1 + l_1(g) > m_1 + m_2$.

So we can say that $l(ge_{13}) = l(g) - 1$ and $l(g\bar{e}_{13}) = l(g) - 1$ iff $m_1 + l_1(g) > m_1 + m_2$.

Finally we must show that $l(ge_{23}) = l(g\bar{e}_{23}) = l(g) - 1$, but this argument will be identical to the argument for the same claim from the proof of Corollary 4.1.19. □

Theorem 4.1.22. *If g satisfies the 1 zero condition and $l(g) \neq l_\varepsilon(g)$ then:*

- $l(ge_{12}) = l(g) - 1$ *and* $l(g\bar{e}_{12}) = l(g) - 1$ *iff* $m_1 + l_1(g) > m_1 + m_2 + m_3$
- $l(g\bar{e}_{23}) = l(g) - 1$ *iff* $m_2 + l_2 < m_1 + m_2 + m_3(g)$
- $l(g\bar{e}_{13}) = l(g) - 1$
- *Any generator s not noted above does not have the property $l(gs) = l(g) - 1$ under any circumstances.*

Proof. By Proposition 4.1.13:

$$l(g) = l_{(12)}(g) = m_2 + \max\{m_1 + m_2 + m_3(g), m_1 + l_1(g)\}$$

First, we can eliminate eight generators that we know will not have the property $l(gs) = l(g) - 1$ as follows:

- $l(ge_{32}) \geq l(g)$,
- $l(ge_{31}) \geq l(g)$,
- $l(g\overline{e}_{32}) \geq l(g)$, and
- $l(g\overline{e}_{31}) \geq l(g)$ by Proposition 4.1.13,
- $l(ge_{13}) \geq l(g)$ and
- $l(ge_{23}) \geq l(g)$ by Proposition 4.1.14, and
- $l(ge_{21}) \geq l(g)$ and
- $l(g\overline{e}_{21}) \geq l(g)$ by Proposition 4.1.16.

So we have 4 generators left to investigate:

Consider $\boldsymbol{e_{12}}$, $\boldsymbol{\overline{e}_{12}}$, and $\boldsymbol{\overline{e}_{13}}$. For these generators we can apply the same argument from the proof of Theorem 4.1.18. So $\overline{e}_{13}$ decreases word length under any conditions, if $m_1+l_1(g) > m_1+m_2+m_3$ then e_{12} decreases word length, if $m_1+l_1(g) > m_1+m_2+m_3$ then $\overline{e}_{12}$ decreases word length, and no other generator s on this list has the property $l(gs) = l(g)-1$.

Consider $\boldsymbol{\overline{e}_{23}}$. For this generator we have

$$\Pi(g\overline{e}_{23}) = \big((m_1, l_1), (m_2, l_2(g)-1), (m_3(g)-1, 0)\big)$$

so from Proposition 4.1.11 we know that $l(g\overline{e}_{23})$ is minimized by ε or (12). The corresponding expressions are below:

$$l(g\overline{e}_{23}) = l_\varepsilon(g\overline{e}_{23}) = m_1 + \max\{m_1+m_2+m_3(g)-1, m_2+l_2(g)-1\} = l_\varepsilon(g) - 1 \geq l(g)$$

$$l_{(12)}(g\overline{e}_{23}) = m_2 + \max\{m_1+m_2+m_3(g)-1, m_1+l_1\} \leq l(g)$$

So if ε realizes $l(g\bar{e}_{23})$ then $l(g\bar{e}_{23}) \geq l(g)$.

But if (12) realizes $l(g\bar{e}_{23})$ then $l(g\bar{e}_{23}) = l(g) - 1$ only if $m_1 + l_1(g) < m_1 + m_2 + m_3$.

Notice that if $m_1 + l_1(g) < m_1 + m_2 + m_3$ then $l_{(12)}(g\bar{e}_{23}) < l(g) \leq l_{(12)}(g\bar{e}_{23})$, and so (12) realizes $l(g\bar{e}_{23})$.

So

$$m_1 + l_1(g) < m_1 + m_2 + m_3 \implies (12) \text{ realizes } l(g\bar{e}_{23})$$

and

$$(12) \text{ realizes } l(g\bar{e}_{23}) \implies l(g\bar{e}_{23}) = l(g) - 1 \text{ only if } m_1 + l_1(g) < m_1 + m_2 + m_3$$

So if $m_1 + l_1(g) < m_1 + m_2 + m_3$ then $\bar{e}_{23}$ decreases word length.

We have now considered every generator. $\square$

Corollary 4.1.23. *If* $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$ *and* $l(g) \neq l_\varepsilon(g)$ *then*

- $l(ge_{12}) = l(g) - 1$ *and* $l(g\bar{e}_{12}) = l(g) - 1$ *iff* $m_1 + l_1(g) > m_1 + m_2$
- $l(ge_{23}) = l(g\bar{e}_{23}) = l(g) - 1$ *iff* $m_2 + l_2(g) > m_1 + m_2$
- $l(ge_{13}) = l(g\bar{e}_{13}) = l(g) - 1$
- *Any generator* s *not noted above does not have the property* $l(gs) = l(g) - 1$ *under any circumstances.*

Proof. For the generators $e_{21}, \bar{e}_{21}, e_{12}$, and $\bar{e}_{12}$, we can simply refer to the proof of Theorem 4.1.20, as these generators did not rely on m_3 being nonzero.

Similarly for $e_{32}, e_{31}, \bar{e}_{32}$, and $\bar{e}_{31}$, we can simply refer to the proof of Theorem 4.1.20, as these generators did not rely on m_3 being nonzero.

It follows from Corollary 4.1.12 that $l(g) = l_{(12)}(g) = l_{(13)}(g)$.

First we will show $l(ge_{23}) = l(g) - 1$ and $l(g\bar{e}_{23}) = l(g) - 1$ iff $m_2 + l_2(g) > m_1 + m_2$ in Table 4.7

σ	Maximal term in $\Pi(g)$	$l_\sigma(ge_{23}) = l_\sigma(g\bar{e}_{23})$	$l_\sigma(g)$
ε	$m_1 + m_2$	$m_1 + 1 + m_1 + m_2$	$m_1 + m_1 + m_2$
ε	$m_2 + l_2(g)$	$m_1 + 1 + m_2 + l_2(g) - 1$ or $m_1 + 1 + m_1 + m_2$	$m_1 + m_2 + l_2(g)$ or $m_1 + m_1 + m_2$
(12)	$m_1 + m_2$	$m_2 + 1 + m_1 + m_2$	$m_2 + m_1 + m_2$
(12)	$m_1 + l_1$	$m_2 + 1 + m_1 + l_1$	$m_2 + m_1 + l_1$
(13)	$m_1 + m_2$	$l_1 + m_1 + m_2$	$l_1 + m_1 + m_2$
(13)	$m_2 + l_2(g)$	$l_1 + m_2 + l_2(g) - 1$ or $l_1 + m_1 + m_2$	$l_1 + m_2 + l_2(g)$ or $l_1 + m_1 + m_2$
(23)	$m_1 + m_2$	$m_1 + l_2(g) - 1 + m_1 + m_2$	$m_1 + l_2(g) + m_1 + m_2$
(23)	$m_3 = 0$	N/A (0 can't be maximal)	N/A
(123)	$m_1 + m_2$	$m_2 + l_1 + m_1 + m_2$	$m_2 + l_1 + m_1 + m_2$
(123)	$m_3 = 0$	N/A (0 can't be maximal)	N/A
(132)	$m_1 + m_2$	$l_2(g) - 1 + m_1 + m_2$	$l_2(g) + m_1 + m_2$
(132)	$m_1 + l_1$	$l_2(g) - 1 + m_1 + l_1$	$l_2(g) + m_1 + l_1$

Table 4.7: Values of $l_\sigma(ge_{23}), l_\sigma(g\bar{e}_{23})$, for either maximal term in $\Pi(g)$, over all $\sigma \in S_3$.

From Corollary 4.1.12 and from the assumptions of the corollary, we know that for each $\sigma \notin \{(12), (13)\}$ we have $l(g) < l_\sigma(g)$ and so $l_\sigma(g) - 1 \geq l(g)$. So, for $\sigma \neq (12), (13)$, regardless of maximal term, we have

$$l_\sigma(ge_{23}) = l_\sigma(g\bar{e}_{23}) \geq l_\sigma(g) - 1 \geq l(g)$$

Furthermore, $l_{(12)}(ge_{23}) = l_{(12)}(g\bar{e}_{23}) = l_{(12)}(g) + 1 = l(g) + 1$.

Finally, we can see that $l_{(13)}(ge_{23}) = l_{(13)}(g\bar{e}_{23}) = l(g) - 1$, and therefore $l(ge_{23}) = l(g\bar{e}_{23}) = l(g) - 1$ by Lemma 4.1.1, only under the condition that $m_2 + l_2(g) > m_1 + m_2$.

So we can say that $l(ge_{23}) = l(g) - 1$ and $l(g\bar{e}_{23}) = l(g) - 1$ iff $m_2 + l_2(g) > m_1 + m_2$.

Finally we must show that $l(ge_{13}) = l(g\bar{e}_{13}) = l(g) - 1$, but this argument will be identical to the argument for the same claim from the proof of Corollary 4.1.19. $\square$

4.1.3 No-Zero Case

Assume we have $g \in \Gamma_3(2)$ such that $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$ where $l_i > 0$ for $i \in \{1, 2, 3\}$. We have shown that

$$l(g) = \min_{\sigma \in S_3} \{m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}\} \tag{4.2}$$

First note that, when $l_i > 0$ for $i \in \{1, 2, 3\}$, for any $i, j \in \{1, 2, 3\}$ we have $\Pi(ge_{ij}) = \Pi(g\bar{e}_{ij})$. So we will omit the bar in this case, and everything we say of a generator e_{ij} will also hold for the corresponding $\bar{e}_{ij}$.

We now show that, in this case, if we have a group element g and a generator e_{ij} such that $l(g) - 1 = l(ge_{ij})$ then $l(g)$ and $l(ge_{ij})$ can be realized by the same permutation.

Proposition 4.1.24. *Suppose e_{ij} is such that $l(ge_{ij}) = l(g) - 1$ and $l(ge_{ij}) = l_\tau(ge_{ij})$. Then $l(g) = l_\tau(g)$.*

Proof. We can see that

$$\begin{aligned} l(ge_{ij}) =& l_\tau(ge_{ij}) \\ =& m_{\tau(1)}(ge_{ij}) + l_{\tau(3)}(ge_{ij}) + \max\{m_1(ge_{ij}) + m_2(ge_{ij}) + m_3(ge_{ij}), \\ & \qquad m_{\tau(2)}(ge_{ij}) + l_{\tau(2)}(ge_{ij})\} \end{aligned}$$

Claim. $|l(ge_{ij}) - l_\tau(g)| \leq 1$.

Proof. Consider $l_\tau(g) = m_{\tau(1)}(g) + l_{\tau(3)}(g) + \max\{m_1(g) + m_2(g) + m_3(g), m_{\tau(2)}(g) + l_{\tau(2)}(g)\}$. Because no values in $\Pi(g)$ are zero, we know that for $k \neq i, j$ we have $l_k(ge_{ij}) = l_k(g)$. Furthermore, for each $i \in \{1, 2, 3\}$ we have $m_i(g) = m_i(ge_{ij})$. Finally, we know

$l_i(g) = l_i(ge_{ij}) + 1$ and $l_j(g) = l_j(ge_{ij}) - 1$.

So we can write the difference of the word lengths as follows:

$$
\begin{aligned}
l(ge_{ij}) - l(g) = {} & [l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g)] \\
& + [\max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(ge_{ij})\} - \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(g)\}]
\end{aligned}
$$

We have used square brackets to divide the expression into the sum of two differences. For ease of reference, we let $l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g) = A$ and $\max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(ge_{ij})\} - \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(g)\} = B$. Notice that $|l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g)| \leq 1$ by the way we defined our generators. So $|A| \leq 1$. We break the problem into cases based on the value of A.

Case 1: $A = -1$.

So $l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g) = -1$.

Then, because $i \neq j$, we have $0 \leq l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) \leq 1$. So we have two possible subcases for the second difference.

Case 1a: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = 0$

Then $B = \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(ge_{ij})\} - \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(g)\}$

$= 0$, and so $A + B = -1$.

Case 1b: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = 1$

Then $B = \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(ge_{ij})\} - \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(g)\}$

$= 1$, and so $A + B = 0$.

Case 2: $A = 0$.

So $l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g) = 0$.

Then, because $i \neq j$, we have $|l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g)| = 1$. So we have two possible subcases for the second difference.

Case 2a: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = -1$

Then $B = \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(ge_{ij})\} - \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(g)\}$

$= -1$, and so $A+B=-1$.

Case 2b: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = 1$

Then $B = \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(ge_{ij})\} - \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(g)\}$

$= 1$, and so $A+B=1$.

Case 3: $A = 1$.

So $l_{\tau(3)}(ge_{ij}) - l_{\tau(3)}(g) = 1$.

Then, because $i \neq j$, we have $-1 \leq l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) \leq 0$. So we have two possible subcases for the second difference.

Case 3a: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = -1$

Then $B = \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(ge_{ij})\} - \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(g)\}$

$= -1$, and so $A+B=0$.

Case 3b: $l_{\tau(2)}(ge_{ij}) - l_{\tau(2)}(g) = 0$

Then $B = \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(ge_{ij})\} - \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(g)\}$

$= 0$, and so $A+B=1$.

In any case, $|A+B| \leq 1$, so we can conclude that

$$\begin{aligned}&|l_{\tau(3)}(ge_{ij}) + \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(ge_{ij})\}\\ &\quad - l_{\tau(3)}(g) - \max\{m_1+m_2+m_3, m_{\tau(2)}+l_{\tau(2)}(g)\}| \leq 1\end{aligned}$$

Therefore $|l(ge_{ij}) - l_\tau(g)| \leq 1$, as desired. $\square$

So then $l_\tau(g) = l_\tau(ge_{ij}) + \varepsilon$ where $\varepsilon \in \{-1, 0, 1\}$.

First assume $\varepsilon \leq 0$. So $l_\tau(g) \leq l(ge_{ij})$, and $l(g) \leq l_\tau(g)$ by our definition of the l function, and $l(ge_{ij}) < l(g)$ by our choice of e_{ij}. Combining these, we get $l(g) < l(g)$, a contradiction. So we must have $\varepsilon = 1$. So $l_\tau(g) = l_\tau(ge_{ij}) + 1 = l(g) - 1 + 1$, so $l_\tau(g) = l(g)$, as desired. $\square$

For any generator e_{ij} such that $l(ge_{ij}) = l_\tau(ge_{ij}) = l(g) - 1$, Proposition 4.1.24 tells us that:

$$\begin{aligned} l(ge_{ij}) &= m_{\tau(1)} + l_{\tau(3)}(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(ge_{ij})\} \\ &= m_{\tau(1)} + l_{\tau(3)}(g) + \max\{m_1 + m_2 + m_3, m_{\tau(2)} + l_{\tau(2)}(g)\} - 1 = l(g) - 1 \end{aligned} \tag{4.3}$$

We will enumerate which generators s have the property $l(gs) = l(g) - 1$ in three cases based on which term in the $l(g)$ expression realizes the maximum. In each proposition, we will show which generators s have the property $l(gs) = l(g) - 1$ based on the particular σ that realizes word length for both g and gs.

Proposition 4.1.25. *Assume g is such that $l(g) = l_\sigma(g)$ and $m_1 + m_2 + m_3 > m_{\sigma(2)} + l_{\sigma(2)}(g)$, and assume $i, j \in \{1, 2, 3\}$ are such that $i \neq j$. Then*

$$l(ge_{ij}) = l(g) - 1 \textit{ iff } i = \sigma(3)$$

Proof. Assume that g, i, and j are as above.

($\Longrightarrow$) Assume $l_\sigma(ge_{ij}) = l(ge_{ij}) = l(g) - 1$. Because $m_1 + m_2 + m_3 \geq m_{\sigma(2)} + l_{\sigma(2)}(g) + 1$ we can eliminate all common terms from Equation 4.3 and get $l_{\sigma(3)}(ge_{ij}) = l_{\sigma(3)}(g) - 1$. We know $l_i(ge_{ij}) = l_i(g) - 1$ and no other l_x value has this property. So $l_{\sigma(3)} = l_i$, and therefore $\sigma(3) = i$, as desired.

($\Longleftarrow$) Assume $i = \sigma(3)$. We know $l_i(ge_{ij}) = l_i(g) - 1$ and $l_j(ge_{ij}) = l_j(g) + 1$. Then

$$\begin{aligned} l_\sigma(ge_{ij}) &= m_{\sigma(1)} + l_i(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\ &= m_{\sigma(1)} + l_i(g) - 1 + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\ &= m_{\sigma(1)} + l_i(g) - 1 + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\}, \text{ as } m_1 + m_2 + m_3 > m_{\sigma(2)} \\ &\qquad + l_{\sigma(2)}(g) \\ &= l_\sigma(g) - 1 \\ &= l(g) - 1 \end{aligned}$$

So $l_\sigma(ge_{ij}) = l(g) - 1$, and by Lemma 4.1.1 $l(ge_{ij}) = l(g) - 1$ $\square$

Proposition 4.1.26. *Assume g is such that $l(g) = l_\sigma(g)$ and $m_1+m_2+m_3 = m_{\sigma(2)}+l_{\sigma(2)}(g)$, and assume $i, j \in \{1,2,3\}$ are such that $i \neq j$. Then*

$$l(ge_{ij}) = l(g) - 1 \textit{ iff } i = \sigma(3) \textit{ and } j = \sigma(1)$$

Proof. Assume that g, i, and j are as above.

($\Longrightarrow$) Assume $l_\sigma(ge_{ij}) = l(ge_{ij}) = l(g) - 1$.

Assume by way of contradiction that $\sigma(1) = i$. Then $m_{\sigma(1)} + l_{\sigma(3)}(ge_{ij}) \geq m_{\sigma(1)} + l_{\sigma(3)}(ge_{ij})$ and $\max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \geq \max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\}$. So then $l(ge_{ij}) \geq l(g)$, contradicting our assumption. So $\sigma(1) \neq i$.

Assume by way of contradiction that $\sigma(2) = i$. Then $m_\sigma(1) + l_\sigma 3(ge_{ij}) \geq m_\sigma(1) + l_\sigma 3(ge_{ij})$ and $\max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} = \max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\}$. So then $l(ge_{ij}) \geq l(g)$, contradicting our assumption. So $\sigma(2) \neq i$.

We have now shown that $\sigma(3) = i$.

Assume by way of contradiction that $\sigma(2) = j$. Then

$$\max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\} + 1 = \max\{m_1+m_2+m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\}$$

So we can eliminate common terms from Equation 4.3 and get $l_{\sigma(3)}(ge_{ij}) + 1 = l_{\sigma(3)}(g) - 1$. But this is a contradiction, as $l_i(ge_{ij}) = l_i(g) - 1$, $l_j(ge_{ij}) = l_j(g) + 1$, and every other coordinate is identical between $\Pi(g)$ and $\Pi(ge_{ij})$. So $\sigma(2) \neq j$. Therefore $\sigma(1) = j$.

So $\sigma(3) = i$ and $\sigma(1) = j$, as desired.

($\Longleftarrow$) Assume $i = \sigma(3)$ and $j = \sigma(1)$.

We know $l_i(ge_{ij}) = l_i(g) - 1$ and $l_j(ge_{ij}) = l_j(g) + 1$. Then

$$\begin{aligned}
l_\sigma(ge_{ij}) &= m_{\sigma(1)} + l_i(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\
&= m_j + l_i(g) - 1 + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\
&= m_j + l_i(g) - 1 + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\}, \text{ as } l_{\sigma(2)}(ge_{ij}) = l_{\sigma(2)}(g) \\
&= l_\sigma(g) - 1 \\
&= l(g) - 1
\end{aligned}$$

So $l_\sigma(ge_{ij}) = l(g) - 1$, and by Lemma 4.1.1 $l(ge_{ij}) = l(g) - 1$

□

Proposition 4.1.27. *Assume g is such that $l(g) = l_\sigma(g)$ and $m_1+m_2+m_3 < m_{\sigma(2)}+l_{\sigma(2)}(g)$, and assume $i, j \in \{1, 2, 3\}$ are such that $i \neq j$. Then*

$$l(ge_{ij}) = l(g) - 1 \textit{ iff } j = \sigma(1)$$

Proof. Assume that g, i, and j are as above.

($\Longrightarrow$) Assume $l_\sigma(ge_{ij}) = l(ge_{ij}) = l(g) - 1$.

Assume by way of contradiction that $\sigma(2) = j$. Then $\max\{m_1+m_2+m_3, m_{\sigma(2)}+l_{\sigma(2)}(g)\}+1 = \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\}$. So we can eliminate common terms from Equation 4.3 and get $l_{\sigma(3)}(ge_{ij}) + 1 = l_{\sigma(3)}(g) - 1$. But this is a contradiction, as $l_i(ge_{ij}) = l_i(g) - 1$, $l_j(ge_{ij}) = l_j(g) + 1$, and every other coordinate is identical between $\Pi(g)$ and $\Pi(ge_{ij})$. So $\sigma(2) \neq j$.

Now assume by way of contradiction that $\sigma(3) = j$. Then we can eliminate common terms from Equation 4.3 and get $\max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} + 1 = \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\} - 1$, but this is a contradiction, as $l_i(ge_{ij}) = l_i(g) - 1$, $l_j(ge_{ij}) = l_j(g) + 1$, and every other coordinate is identical between $\Pi(g)$ and $\Pi(ge_{ij})$. So $\sigma(3) \neq j$.

We conclude that $\sigma(1) = j$, as desired.

($\Longleftarrow$) Assume $j = \sigma(1)$.

We know $l_i(ge_{ij}) = l_i(g) - 1$ and $l_j(ge_{ij}) = l_j(g) + 1$. We now must consider the problem in cases, based on which value σ maps to i:

Case 1: $\sigma(2) = i$. Then

$$\begin{aligned}
l_\sigma(ge_{ij}) &= m_{\sigma(1)} + l_{\sigma(3)}(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_i + l_i(ge_{ij})\} \\
&= m_j + l_{\sigma(3)}(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_i + l_i(ge_{ij})\} \\
&= m_j + l_{\sigma(3)}(g) + \max\{m_1 + m_2 + m_3, m_i + l_i(g) - 1\}, \text{ as } l_{\sigma(3)}(ge_{ij}) = l_{\sigma(3)}(g) \\
&= m_j + l_{\sigma(3)}(g) + \max\{m_1 + m_2 + m_3, m_i + l_i(g)\} - 1, \text{ as } m_1 + m_2 + m_3 < m_{\sigma(2)} + l_{\sigma(2)}(g) \\
&= l_\sigma(g) - 1 \\
&= l(g) - 1
\end{aligned}$$

So $l_\sigma(ge_{ij}) = l(g) - 1$, and by Lemma 4.1.1 $l(ge_{ij}) = l(g) - 1$

Case 2: $\sigma(3) = i$. Then

$$\begin{aligned}
l_\sigma(ge_{ij}) &= m_{\sigma(1)} + l_i(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\
&= m_j + l_i(ge_{ij}) + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(ge_{ij})\} \\
&= m_j + l_i(g) - 1 + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}(g)\}, \text{ as } l_{\sigma(2)}(ge_{ij}) = l_{\sigma(2)}(g) \\
&= l_\sigma(g) - 1 \\
&= l(g) - 1
\end{aligned}$$

So $l_\sigma(ge_{ij}) = l(g) - 1$, and by Lemma 4.1.1 $l(ge_{ij}) = l(g) - 1$ $\square$

4.2 Summary of Sections 4.1.1, 4.1.2, and 4.1.3

The findings of sections 4.1.1, 4.1.2, and 4.1.3 are summarized in Figure 4.1. Note that the 28 rows of the figure represent, up to permutation, every possible value for the projection of an element of $\Gamma_3(2)$.

	$\Pi(g)$	Min. σ	Maximal term in $l_\sigma(g)$	$s \in S$ such that $l(gs) = l(g) - 1$
1	$((m_1, l_1), (m_2, 0), (m_3, 0))$	Any	Any	$\overline{e}_{12}, \overline{e}_{13}$
2	$((0, l_1), (m_2, 0), (m_3, 0))$	Any	Any	$\overline{e}_{12}, \overline{e}_{13}$
3	$((m_1, l_1), (0, 0), (m_3, 0))$	Any	Any	$e_{12}, \overline{e}_{12}, \overline{e}_{13}$
4	$((m_1, l_1), (m_2, 0), (0, 0))$	Any	Any	$e_{13}, \overline{e}_{12}, \overline{e}_{13}$
5	$((0, l_1), (0, 0), (m_3, 0))$	Any	Any	$\overline{e}_{13}$
6	$((0, l_1), (m_2, 0), (0, 0))$	Any	Any	$\overline{e}_{12}$
7	$((m_1, l_1), (0, 0), (0, 0))$	Any	Any	$e_{12}, \overline{e}_{12}, e_{13}, \overline{e}_{13}$
8	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	$\varepsilon, (12)$	$m_1 + m_2 + m_3 \geq m_1 + l_1$ $m_1 + m_2 + m_3 \geq m_2 + l_2$	$\overline{e}_{23}, \overline{e}_{13}$
9	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	$\varepsilon, (12)$	$m_1 + m_2 + m_3 < m_1 + l_1$	$\overline{e}_{23}, \overline{e}_{13}, e_{12}, \overline{e}_{12}$
10	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	$\varepsilon, (12)$	$m_1 + m_2 + m_3 < m_2 + l_2$	$\overline{e}_{23}, \overline{e}_{13}, e_{21}, \overline{e}_{21}$
11	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	ε	$m_1 + m_2 + m_3 = m_2 + l_2$	$\overline{e}_{23}$
12	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	ε	$m_1 + m_2 + m_3 > m_2 + l_2$	$\overline{e}_{23}, \overline{e}_{13}$
13	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	ε	$m_1 + m_2 + m_3 < m_2 + l_2$	$\overline{e}_{23}, e_{21}, \overline{e}_{21}$
14	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	(12)	$m_1 + m_2 + m_3 = m_1 + l_1$	$\overline{e}_{13}$
15	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	(12)	$m_1 + m_2 + m_3 > m_1 + l_1$	$\overline{e}_{13}, \overline{e}_{23}$
16	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3, 0))$	(12)	$m_1 + m_2 + m_3 < m_1 + l_1$	$\overline{e}_{13}, e_{12}, \overline{e}_{12}$
17*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132), (12), (13)$	$m_1 + l_1 \leq m_1 + m_2,$ $m_2 + l_2 \leq m_1 + m_2$	$e_{13}, \overline{e}_{13}, e_{23}, \overline{e}_{23}$
18	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132), (12), (13)$	$m_1 + m_2 < m_1 + l_1$	$e_{12}, \overline{e}_{12}, e_{13}, \overline{e}_{13}, e_{23}, \overline{e}_{23}$
19	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132), (12), (13)$	$m_1 + m_2 < m_2 + l_2$	$e_{21}, \overline{e}_{21}, e_{13}, \overline{e}_{13}, e_{23}, \overline{e}_{23}$
20	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132)$	$m_1 + l_1 \leq m_1 + m_2 < m_2 + l_2$	$e_{21}, \overline{e}_{21}, e_{23}, \overline{e}_{23}$
21*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132)$	$m_1 + l_1 \leq m_1 + m_2,$ $m_2 + l_2 \leq m_1 + m_2$	$e_{23}, \overline{e}_{23}$
22*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$\varepsilon, (132)$	$m_2 + l_2 \leq m_1 + m_2 < m_1 + l_1$	$e_{13}, \overline{e}_{13}, e_{23}, \overline{e}_{23}$
23	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$(12), (13)$	$m_2 + l_2 \leq m_1 + m_2 < m_1 + l_1$	$e_{12}, \overline{e}_{12}, e_{13}, \overline{e}_{13}$
24*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$(12), (13)$	$m_1 + l_1 \leq m_1 + m_2,$ $m_2 + l_2 \leq m_1 + m_2$	$e_{13}, \overline{e}_{13}$
25*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (0, 0))$	$(12), (13)$	$m_1 + l_1 \leq m_1 + m_2 < m_2 + l_2$	$e_{23}, \overline{e}_{23}, e_{13}, \overline{e}_{13}$
26	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3 \text{ or } 0, l_3))$	σ	$m_1 + m_2 + m_3 > m_{\sigma(2)} + l_{\sigma(2)}$	$e_{\sigma(3)\sigma(1)}, \overline{e}_{\sigma(3)\sigma(1)}, e_{\sigma(3)\sigma(2)}, \overline{e}_{\sigma(3)\sigma(2)}$
27*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3 \text{ or } 0, l_3))$	σ	$m_1 + m_2 + m_3 = m_{\sigma(2)} + l_{\sigma(2)}$	$e_{\sigma(3)\sigma(1)}, \overline{e}_{\sigma(3)\sigma(1)}$
28*	$((m_1 \text{ or } 0, l_1), (m_2 \text{ or } 0, l_2), (m_3 \text{ or } 0, l_3))$	σ	$m_1 + m_2 + m_3 < m_{\sigma(2)} + l_{\sigma(2)}$	$e_{\sigma(2)\sigma(1)}, \overline{e}_{\sigma(2)\sigma(1)}, e_{\sigma(3)\sigma(1)}, \overline{e}_{\sigma(3)\sigma(1)}$

Figure 4.1: Table showing all generators s that have the property $l(gs) = l(g) - 1$ for any given scenario. The cells with "any" indicate that the generators listed have the property $l(gs) = l(g) - 1$ regardless of the value of that cell. The rows marked with * are those for which we can produce no fixed-length path.

4.3 Triangles in $\mathbf{DL}_3(2)$

In this section we enumerate every instance in which two elements of the form gs_0, gs_1 for some $s_0, s_1 \in S$ are connected by an edge in $\mathrm{DL}_3(2)$. These represent the triangles in $\mathrm{DL}_3(2)$, and these triangles will be used to connect most of the elements identified in the Figure 4.1.

Theorem 4.3.1. *For any $g \in \Gamma_3(2)$ and $\{i, j, k\} = \{1, 2, 3\}$, there is an edge between the vertices ge_{ij} and ge_{ik} in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label e_{jk}. Similarly, there is an edge between the vertices $g\bar{e}_{ij}$ and ge_{ik} in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label e_{jk}*

Proof. Let $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$.
We will prove the theorem with $i = 1, j = 2$, and $k = 3$. Note that the same argument will work with other values of i, j, and k. So we will first show there is an edge between ge_{12} and ge_{13}, by comparing the projections of each:

$$\Pi(ge_{12}) = \big((m_1, l_1 - 1), (m_2, l_2 + 1), (m_3, l_3)\big)$$
$$\text{or } \Pi(ge_{12}) = \big((m_1 + 1, 0), (m_2, l_2 + 1), (m_3, l_3)\big) \text{ if } l_1 = 0$$

$$\Pi(ge_{13}) = \big((m_1, l_1 - 1), (m_2, l_2), (m_3, l_3 + 1)\big)$$
$$\text{or } \Pi(ge_{13}) = \big((m_1 + 1, 0), (m_2, l_2), (m_3, l_3 + 1)\big) \text{ if } l_1 = 0$$

$$\Pi(ge_{12}e_{23}) = \big((m_1, l_1 - 1), (m_2, l_2), (m_3, l_3 + 1)\big)$$
$$\text{or } \Pi(ge_{12}e_{23}) = \big((m_1 + 1, 0), (m_2, l_2), (m_3, l_3 + 1)\big) \text{ if } l_1 = 0$$

We can see that $\Pi(ge_{12}e_{23}) = \Pi(ge_{13})$, and since both have the same position in the k tree, we can say that $ge_{12}e_{23} = ge_{13}$.
So there is an edge that connects ge_{12} and ge_{13} with label e_{23}.

The same argument produces the same result for $g\overline{e}_{12}$ and ge_{13}. $\square$

Theorem 4.3.2. *For any $g \in \Gamma_3(2)$ and $\{i, j, k\} = \{1, 2, 3\}$, there is an edge between the vertices ge_{ij} and $g\overline{e}_{ik}$ in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label $\overline{e}_{jk}$. Similarly, there is an edge between the vertices $g\overline{e}_{ij}$ and $g\overline{e}_{ik}$ in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label $\overline{e}_{jk}$*

Proof. Let $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$.
We will prove the theorem with $i = 1, j = 2$, and $k = 3$. Note that the same argument will work with other values of i, j, and k. So we will first show there is an edge between ge_{12} and $g\overline{e}_{13}$, by comparing the projections of each:

$$\begin{aligned}
\Pi(ge_{12}) &= \big((m_1, l_1 - 1), (m_2, l_2 + 1), (m_3, l_3)\big) \\
\text{or } \Pi(ge_{12}) &= \big((m_1 + 1, 0), (m_2, l_2 + 1), (m_3, l_3)\big) \text{ if } l_1 = 0
\end{aligned}$$

$$\begin{aligned}
\Pi(g\overline{e}_{13}) &= \big((m_1, l_1 - 1), (m_2, l_2), (m_3, l_3 + 1)\big) \\
\text{or } \Pi(g\overline{e}_{13}) &= \big((m_1 + 1, 0), (m_2, l_2), (m_3, l_3 + 1)\big) \text{ if } l_1 = 0 \\
\text{or } \Pi(g\overline{e}_{13}) &= \big((m_1, l_1 + 1), (m_2, l_2), (m_3 - 1, 0)\big) \text{ if } l_3 = 0 \text{ and } m_3 > 0 \\
\text{or } \Pi(g\overline{e}_{13}) &= \big((m_1 + 1, 0), (m_2, l_2), (m_3 - 1, 0)\big) \text{ if } l_1 = l_3 = 0 \text{ and } m_3 > 0
\end{aligned}$$

$$\begin{aligned}
\Pi(ge_{12}\overline{e}_{23}) &= \big((m_1, l_1 - 1), (m_2, l_2), (m_3, l_3 + 1)\big) \\
\text{or } \Pi(ge_{12}\overline{e}_{23}) &= \big((m_1 + 1, 0), (m_2, l_2), (m_3, l_3 + 1)\big) \text{ if } l_1 = 0 \\
\text{or } \Pi(ge_{12}\overline{e}_{23}) &= \big((m_1, l_1 + 1), (m_2, l_2), (m_3 - 1, 0)\big) \text{ if } l_3 = 0 \text{ and } m_3 > 0 \\
\text{or } \Pi(ge_{12}\overline{e}_{23}) &= \big((m_1 + 1, 0), (m_2, l_2), (m_3 - 1, 0)\big) \text{ if } l_1 = l_3 = 0 \text{ and } m_3 > 0
\end{aligned}$$

We can see that $\Pi(ge_{12}\overline{e}_{23}) = \Pi(g\overline{e}_{13})$, and since both have the same position in the k tree, we can say that $ge_{12}\overline{e}_{23} = g\overline{e}_{13}$.

So there is an edge that connects ge_{12} and $g\overline{e}_{13}$ with label $\overline{e}_{23}$.

The same argument produces the same result for $g\overline{e}_{12}$ and $g\overline{e}_{13}$. $\square$

Theorem 4.3.3. *For any $g \in \Gamma_3(2)$ and $\{i, j, k\} = \{1, 2, 3\}$, there is an edge between the vertices ge_{ij} and ge_{kj} in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label e_{ki} or $\overline{e}_{ki}$. Similarly, there is an edge between the vertices $g\overline{e}_{ij}$ and $g\overline{e}_{kj}$ in the Cayley graph $\Gamma(\Gamma_3(2), S)$ with label e_{ki} or $\overline{e}_{ki}$*

Proof. Let $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$.

We will prove the theorem with $i = 1, j = 2$, and $k = 3$. Note that the same argument will work with other values of i, j, and k. So we will first show there is an edge between ge_{12} and ge_{32}, by comparing the projections of each:

$$\Pi(ge_{12}) = \big((m_1, l_1 - 1), (m_2, l_2 + 1), (m_3, l_3)\big)$$
$$\text{or } \Pi(ge_{12}) = \big((m_1 + 1, 0), (m_2, l_2 + 1), (m_3, l_3)\big) \text{ if } l_1 = 0$$

$$\Pi(ge_{23}) = \big((m_1, l_1), (m_2, l_2 - 1), (m_3, l_3 + 1)\big)$$
$$\text{or } \Pi(ge_{13}) = \big((m_1, l_1), (m_2 + 1, 0), (m_3, l_3 + 1)\big) \text{ if } l_2 = 0$$

$$\Pi(ge_{12}e_{31}) = \big((m_1, l_1), (m_2, l_2 - 1), (m_3, l_3 + 1)\big)$$
$$\text{or } \Pi(ge_{12}e_{31}) = \big((m_1, l_1), (m_2 + 1, 0), (m_3, l_3 + 1)\big) \text{ if } l_2 = 0$$

$$\Pi(ge_{12}\overline{e}_{31}) = \big((m_1, l_1), (m_2, l_2 - 1), (m_3, l_3 + 1)\big)$$
$$\text{or } \Pi(g\overline{e}_{12}e_{31}) = \big((m_1, l_1), (m_2 + 1, 0), (m_3, l_3 + 1)\big) \text{ if } l_2 = 0$$

We can see that $\Pi(ge_{12}e_{31}) = \Pi(ge_{12}\bar{e}_{31}) = \Pi(ge_{32})$, and since one of $ge_{12}e_{31}$ or $ge_{12}\bar{e}_{31}$ has the same position in the k tree as ge_{32}, we can say that $ge_{12}e_{31} = ge_{32}$ or $ge_{12}\bar{e}_{31} = ge_{32}$.
So there is an edge that connects ge_{12} and ge_{32} with label e_{31} or $\bar{e}_{31}$.
The same argument produces the same result for $g\bar{e}_{12}$ and $g\bar{e}_{32}$. $\square$

We can connect most of the points in Figure 4.1 with fixed-length paths constructed entirely from the triangles identified above.
In rows **5, 6, 11, and 14** of Table 4.1 there is no fixed-length path to build, as only one generator decreases word length.
In rows **1, 2, 3, 4, 7, 13, 16, 20, 23, and 26** of Table 4.1, Theorems 4.3.1 and 4.3.2 can be used to construct a fixed-length path between any two points of length $l(g)-1$ that differ from g by a generator.
In rows **8, 12, and 15** of Table 4.1, Theorem 4.3.3 can be used to construct a fixed-length path between any two points of length $l(g)-1$ that differ from g by a generator.
In rows **9, 10, 18, and 19** of Table 4.1, Theorems 4.3.1, 4.3.2, and 4.3.3 can be used to construct a fixed-length path between any two points of length $l(g)-1$ that differ from g by a generator.

4.4 Paths of non-constant length

We have yet to address rows 17, 21, 22, 24, 25, 27, or 28 of Table 4.1. Notice that the family of elements from Section 3.0.3 satisfies the conditions of row 27. For these we will have to produce paths whose length is not fixed, but is nevertheless less than $2l(g)-1$. In each case, the problematic points are the points of the form ge_{ij} and $g\bar{e}_{ij}$. If we can produce such a path, then we can use the triangles from Section 4.3 to connect any of the points identified in these rows. The technique for producing these paths will be to reduce one of the other l-values down to zero, then change the coordinate in tree j, then return the l-value to it's original position.

We first identify a path that can be used to connect points that arise from any g that satisfies the conditions of row 17, 21, or 22 of Table 4.1:

Theorem 4.4.1. *If g is such that $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$, and $l_\varepsilon(g) = l_{(132)}(g) = l(g)$ then there is a path from ge_{23} to $g\overline{e}_{23}$ of length strictly less than $2l(ge_{23})$ that remains inside the ball $B(l(g) - 1)$. This path is of the form $e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}\mathring{e}_{12}^{l_2-1}$, where $\mathring{e}_{12}$ represents either e_{12} or $\overline{e}_{12}$.*

Proof. Begin by observing that $l(g) = l_2 + \max\{m_1 + m_2, m_1 + l_1\}$. We have

$$\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$$

From this it follows that

$$\Pi(ge_{23}) = \big((m_1, l_1), (m_2, l_2 - 1), (0, 1)\big) = \Pi(g\overline{e}_{23})$$

We can see that $l_{(132)}(ge_{23}) = l_2 - 1 + \max\{m_1 + m_2, m_1 + l_1\} = l(g) - 1$, so by Lemma 4.1.1 $l(ge_{23}) = l_{(132)}(ge_{23}) = l(g) - 1$.

The nth intermediate point on the path (with $n \leq l_2 - 1$) has projection:

$$\Pi(ge_{23}e_{21}^n) = \big((m_1, l_1 + n), (m_2, l_2 - n - 1), (0, 1)\big)$$

So $l(ge_{23}e_{21}^n) \leq l_{(132)}(ge_{23}e_{21}^n) = l_2 - n - 1 + \max\{m_1 + m_2, m_1 + l_1 + n\} \leq l(ge_{23}) = l(g) - 1$.

So $l(ge_{23}e_{21}^n) \leq l(g) - 1$.

The l_1th intermediate point on the path has projection:

$$\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}) = \big((m_1, l_1 + l_2 - 1), (m_2 - 1, 0), (0, 0)\big)$$

So $l(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}) \leq l_{(132)}(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}) = \max\{m_1 + m_2, m_1 + l_1 + l_2 - 1\} \leq l(ge_{23}) = l(g) - 1$.

So $l(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}) \leq l(g) - 1$.

The $(l_1 + 1)$th intermediate point on the path has projection:

$$\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}) = \big((m_1, l_1 + l_2 - 1), (m_2, 0), (0, 1)\big)$$

Notice that $\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}) = \Pi(ge_{23}e_{21}^{l_2-1})$.

So $l(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}) \leq l(g) - 1$.

The remaining edges on the path are of the form e_{12} or $\overline{e}_{12}$. The goal is to reach a point with the same position in tree 2 as ge_{23}. It is clear that there is some sequence of e_{12}s and $\overline{e}_{12}$s that accomplishes this, so we will just write $\mathring{e}_{12}$.

The kth intermediate on the path (with $l_2 + 1 < k < 2l_2$) has projection:

$$\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}\mathring{e}_{12}^{k}) = \big((m_1, l_1 + l_2 - k - 1), (m_2, k), (0, 1)\big)$$

Notice that $\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}\mathring{e}_{12}^{k}) = \Pi(ge_{23}e_{21}^{2l_2-k})$.

So $l(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}\mathring{e}_{12}^{m}) \leq l(g) - 1$

The $2l_2$th intermediate point on the path has projection:

$$\Pi(ge_{23}e_{21}^{l_2-1}\overline{e}_{32}\overline{e}_{23}\mathring{e}_{12}^{l_2-1}) = \big((m_1, l_1), (m_2, l_2 - 1), (0, 1)\big) = \Pi(g\overline{e}_{23})$$

So the given path goes from ge_{23} to $g\overline{e}_{23}$, and remains entirely inside the ball $B(l(g) - 1)$.

Note that $2l(ge_{ij}) = 2(l(g) - 1) = 2(l_2 + \max\{m_1 + m_2, m_1 + l_1\}) - 2 > 2l_2$. So this path is short enough to satisfy minimal almost convexity. $\square$

This path can be used in conjunction with the triangles from Theorem 4.3.3 to connect any of the points that arise from g satisfying the conditions of rows 17 or 22 of Table 4.1.

We now produce a similar path for rows 17, 24, or 25 of Table 4.1:

Theorem 4.4.2. *If g is such that $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (0, 0)\big)$, and $l_{(12)}(g) = l_{(13)}(g) = l(g)$, then there is a path from ge_{13} to $g\overline{e}_{13}$ of length strictly less than $2l(ge_{13})$ that remains inside the ball $B(l(g) - 1)$. This path is of the form $e_{12}^{l_1-1}\overline{e}_{31}\overline{e}_{13}\mathring{e}_{21}^{l_1-1}$, where $\mathring{e}_{21}$ represents either e_{21} or $\overline{e}_{21}$.*

Proof. The proof of this theorem is identical to the argument for Theorem 4.4.1, with the permutation (12) applied to all subscripts. □

This path can be used in conjunction with the triangles from Theorem 4.3.3 to connect any of the points that arise from g satisfying the condition of rows 17 or 25 of Table 4.1.

This leaves rows 27 and 28 of Table 4.1. The following path can be used to connect the points that arise from any g that satisfies the conditions of rows 27 or 28 of Table 4.1.

Theorem 4.4.3. *If g is such that*

- $\Pi(g) = \big((m_1, l_1), (m_2, l_2), (m_3, l_3)\big)$,
- $l_\sigma(g) = l(g)$ *where* $\sigma(1) = j$ *and* $\sigma(3) = i$ *for some* $i \neq j \in \{1, 2, 3\}$, *and*
- $m_1 + m_2 + m_3 \leq m_{\sigma(2)} + l_{\sigma(2)}$,

then there is a path from ge_{ij} to $g\overline{e}_{ij}$ of length strictly less than $2l(ge_{ij})$ that remains inside the ball $B(l(g) - 1)$. This path is of the form $e_{ik}^{l_i-1}\overline{e}_{ji}\overline{e}_{ij}\mathring{e}_{ki}^{l_i-1}$, where $\mathring{e}_{ki}$ represents either e_{ki} or $\overline{e}_{ki}$.

Proof. From Section 4.1.3, we can see that $l(ge_{ij}) = l(g\overline{e}_{ij}) = l_\sigma(ge_{ij}) = l_\sigma(g\overline{e}_{ij}) = l(g) - 1$. Without loss of generality, assume $i = 1, j = 2$, and $k = 3$. So $l(ge_{12}) = l(g\overline{e}_{12}) = l(g) - 1$. The result can be shown in the same way for any other choices of i, j, and k. We can see that

$$\Pi(ge_{12}) = \big((m_1, l_1 - 1), (m_2, l_2 + 1), (m_3, l_3)\big) = \Pi(g\overline{e}_{12})$$

Recall that

$l(g\overline{e}_{12}) = l(ge_{12}) = l_{(123)}(ge_{12}) = m_2 + l_1 - 1 + \max\{m_1 + m_2 + m_3, m_3 + l_3\} = l(g) - 1.$

The nth intermediate point on the path (with $n \leq l_1 - 1$) has projection:

$$\Pi(ge_{12}e_{13}^n) = \big((m_1, l_1 - n - 1), (m_2, l_2 + 1), (m_3, l_3 + n)\big)$$

So

$$\begin{aligned} l(ge_{12}e_{13}^n) \leq l_{(123)}(ge_{12}e_{13}^n) &= m_2 + l_1 - n - 1 + \max\{m_1 + m_2 + m_3, m_3 + l_3 + n\} \\ &\leq l_{(123)}(ge_{12}) = l_{(123)}(ge_{12}) = l(g) - 1 \end{aligned}$$

So $l(ge_{12}e_{13}^n) \leq l(g) - 1$.

The l_1th intermediate point on the path has projection:

$$\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}) = \big((m_1 - 1, 0), (m_2, l_2), (m_3, l_3 + l_1 - 1)\big)$$

So

$$\begin{aligned} l(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}) \leq l_{(123)}(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}) &= m_2 + \max\{m_1 + m_2 + m_3, m_3 + l_3 + l_1 - 1\} \\ &\leq l_{(123)}(ge_{12}) = l_{(123)}(ge_{12}) = l(g) - 1 \end{aligned}$$

So $l(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}) \leq l(g) - 1$.

The $(l_1 + 1)$th intermediate point on the path has projection:

$$\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}) = \big((m_1, 0), (m_2, l_2 + 1), (m_3, l_3 + l_1 - 1)\big)$$

Notice that $\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}) = \Pi(ge_{12}e_{13}^{l_1-1}$.

So $l(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}) \leq l(g) - 1$.

The remaining edges on the path are of the form e_{31} or $\overline{e}_{31}$. The goal is to reach a point with the same position in tree 1 as ge_{12}. It is clear that there is some sequence of e_{31}s and $\overline{e}_{31}$s that accomplishes this. So we will simply write $\mathring{e}_{31}$.

The kth intermediate point on the path (with $l_1 + 1 < k < 2l_1$) has projection:

$$\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}\mathring{e}_{31}^{k}) = \big((m_1, k), (m_2, l_2+1), (m_3, l_3 + l_1 + k - 1)\big)$$

Notice that $\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}\mathring{e}_{31}^{k}) = \Pi(ge_{12}e_{13}^{2l_1-k})$.

So $l(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}\mathring{e}_{31}^{k}) \leq l(g) - 1$

The $2l_1$th intermediate point on the path has projection:

$$\Pi(ge_{12}e_{13}^{l_1-1}\overline{e}_{21}\overline{e}_{12}e_{31}^{l_1-1}) = \big((m_1, l_1 - 1), (m_2, l_2+1), (m_3, l_3)\big) = \Pi(g\overline{e}_{12})$$

So the given path goes from ge_{12} to $g\overline{e}_{12}$, and remains entirely inside the ball.

Note that $2l(ge_{ij}) = 2(l(g)-1) = 2(m_{\sigma(1)} + l_{\sigma(3)} + \max\{m_1 + m_2 + m_3, m_{\sigma(2)} + l_{\sigma(2)}\}) - 2 > 2l_{\sigma(3)}$.
So this path is short enough to satisfy minimal almost convexity. $\square$

Bibliography

[1] Bartholdi, L., Neuhauser, M., and Woess, W. Horocyclic products of trees. *J. Eur. Math. Soc. (JEMS) 10*, 3 (2008), 771–816.

[2] Belk, J., and Bux, K.-U. Thompson's group F is maximally nonconvex. In *Geometric methods in group theory*, vol. 372 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2005, pp. 131–146.

[3] Cannon, J. W. Almost convex groups. *Geom. Dedicata 22*, 2 (1987), 197–210.

[4] Cleary, S., and Riley, T. R. A finitely presented group with unbounded dead-end depth. *Proc. Amer. Math. Soc. 134*, 2 (2006), 343–349.

[5] Cleary, S., and Taback, J. Thompson's group F is not almost convex. *J. Algebra 270*, 1 (2003), 133–149.

[6] Cleary, S., and Taback, J. Dead end words in lamplighter groups and other wreath products. *Q. J. Math. 56*, 2 (2005), 165–178.

[7] Diestel, R., and Leader, I. A conjecture concerning a limit of non-Cayley graphs. *J. Algebraic Combin. 14*, 1 (2001), 17–25.

[8] Eskin, A., Fisher, D., and Whyte, K. Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs. *Ann. of Math. (2) 176*, 1 (2012), 221–260.

[9] Miller, III, C. F., and Shapiro, M. Solvable Baumslag-Solitar groups are not almost convex. *Geom. Dedicata 72*, 2 (1998), 123–127.

[10] Stein, M., and Taback, J. Metric properties of Diestel-Leader groups. *Michigan Math. J. 62*, 2 (2013), 365–386.

[11] Taback, J. An introduction to lamplighter groups. *To appear in: Office Hours with a Geometric Group Theorist, Princeton University Press*.