1-1-1975

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JOURNAL OF FUNCTIONAL ANALYSIS 20, 179–207 (1975)

The Spherical Bochner Theorem on Semisimple Lie Groups*

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Communicated by the Editors

Received December 30, 1973; revised October 30, 1974

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Denote (i) Harish-Chandra's Schwartz spaces by $\mathscr{C}^p(G)(0 < p \leqslant 2)$, (ii) the K-biinvariant elements in $\mathscr{C}^p(G)$ by $\mathscr{I}^p(G)$, (iii) the positive definite (zonal) spherical functions by $\mathscr{P}$, and (iv) the spherical transform on $\mathscr{C}^p(G)$ by $\varphi \to \hat{\varphi}$. For T a positive definite distribution on G it is established that (i) T extends uniquely onto $\mathscr{C}^1(G)$, (ii) there exists a unique measure μ of polynomial growth on $\mathscr{P}$ such that $T[\varphi] = \int_{\mathscr{P}} \hat{\varphi} d\mu$ for all $\varphi \in \mathscr{I}^1(G)$, (iii) all measures μ of polynomial growth on $\mathscr{P}$ are obtained in this way, and (iv) T may be extended to a particular $\mathscr{I}^p(G)$ space $(1 \leqslant p \leqslant 2)$ if and only if the support of μ lies in a certain easily defined subset of $\mathscr{P}$. These results generalize a well-known theorem of Godement, and the proofs rely heavily on the recent harmonic analysis results of Trombi and Varadarajan.

1. INTRODUCTION

Suppose T is a positive definite distribution on $\mathbb{R}^n$. In [15] Schwartz proves the following sequence of facts about T.

(i) T can be expressed as a finite sum of derivatives of bounded functions;

(ii) T is a tempered distribution; and

(iii) T is the Fourier transform of some unique tempered measure on $\mathbb{R}^n$ (the Bochner theorem).

In this paper we generalize the above procedure to prove a Bochner theorem for the spherical transform on a connected semisimple Lie group with finite center.

* These results comprise the main part of the author's thesis. The author is deeply indebted to his adviser, Professor S. Helgason of M.I.T., for the initial suggestion of the problem and for the guidance and encouragement given during the period of its solution.

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Suppose G is a separable unimodular Lie group. In [3] we have shown that any positive definite distribution T on G can be written as a finite sum

$$T = \sum_j D^j E^j f_j, \tag{1.1}$$

where, for each j, f_j is a bounded function and D^j (resp. E^j) is a left (resp. right) invariant differential operator on G. This generalizes (i).

To obtain (ii) is more difficult since the notion of a "rapidly decreasing function" on an arbitrary Lie group is not easily definable. Harish-Chandra has shown in [8] that Schwartz spaces do exist for G a connected semisimple Lie group with finite center; in fact there exists a whole family of such spaces $\mathscr{C}^p(G) \subset L^p(G)$ $(0 < p \leqslant 2)$, where $\mathscr{C}^p(G) \subset \mathscr{C}^q(G)$ when $p \leqslant q$. From (1.1) it is an easy matter to show that each positive definite distribution extends uniquely to a continuous linear functional on $\mathscr{C}^1(G)$. This generalizes (ii).

In the $\mathbb{R}^n$ case, proving (iii) from (ii) relies on the isomorphism of the Euclidean Schwartz space $\mathscr{S}(\mathbb{R}^n)$ onto itself given by the Fourier transform $f \to \hat{f}$. Unfortunately this is not totally available to us, for although it is possible to define a Fourier transform on the $\mathscr{C}^p(G)$ spaces, it is as yet unknown what the image spaces are, or even whether the mapping is injective. The only known facts for this problem are given in [1], where positive results for $\mathscr{C}^2(G)$, G of real rank one, are obtained.

We must instead consider the spaces $\mathscr{I}^p(G)$, where, for K some fixed maximal compact subgroup of G, $\mathscr{I}^p(G)$ equals all K-biinvariant elements of $\mathscr{C}^p(G)$. For each $\epsilon > 0$ we define spaces $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ consisting of rapidly decreasing functions on certain sets $\mathscr{F}^\epsilon$ of elementary spherical functions. In [16], Trombi and Varadarajan prove, generalizing earlier known results, that the spherical transform $\varphi \to \hat{\varphi}$ is a topological isomorphism of $\mathscr{I}^p(G)$ onto $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ for $\epsilon = (2/p) - 1$. Hence any positive definite distribution gives rise to a unique element in $(\mathscr{I}^1(G))'$ to which harmonic analysis may be applied via the spherical transform. For a general positive definite distribution T it can only be hoped that this procedure will yield an integral formulation for $T[\varphi]$ where φ is K-biinvariant. It does, however, yield the full result when T itself is K-biinvariant.

We apply the Trombi–Varadarajan result to the partial Bochner theorem of Godement [6]. Godement's theorem gives, for T a positive definite distribution on G, the existence of a unique measure μ supported on the positive definite spherical functions $\mathscr{P}$ such that

$$T[\varphi * \psi] = \int_{\mathscr{P}} \hat{\varphi}\hat{\psi}\, d\mu$$

for all K-biinvariant test functions φ and ψ. There is no property of "temperedness" of μ given here, and the heart of proving an analog of (iii) is in defining a notion that a measure on $\mathscr{P}$ is of "polynomial growth," then showing that the above μ is such an object. The theorem we arrive at is the following.

THE SPHERICAL BOCHNER THEOREM. *Suppose T is a positive definite distribution on G. Then T extends uniquely to a continuous linear functional on $\mathscr{C}^1(G)$ and there exists a unique positive regular Borel measure μ of polynomial growth on $\mathscr{P}$ such that*

$$T[\varphi] = \int_{\mathscr{P}} \hat{\varphi}\, d\mu, \qquad \varphi \in \mathscr{I}^1(G).$$

The correspondence between T and μ is bijective when restricted to K-biinvariant distributions, in which case the formula holds for all $\varphi \in \mathscr{C}^1(G)$.

Using the Trombi–Varadarajan theorem we may relate the largest $\mathscr{I}^p(G)$ space to which a positive definite distribution T can be extended with the support of its spherical Bochner measure μ. The result is the following.

THE EXTENSION THEOREM. *Suppose T is a positive definite distribution on G with spherical Bochner measure μ. Then T generates a continuous linear functional on $\mathscr{I}^p(G)$ $(1 \leqslant p \leqslant 2)$ if and only if the support of μ falls in $\mathscr{F}^\epsilon$, $\epsilon = (2/p) - 1$.*

While this is a natural result in view of the Trombi–Varadarajan theorem, the proof is surprisingly complicated. It should be noted that for $p = 2$ this result was first proved by Muta [14] in much the same way as the Euclidean Bochner theorem is proved in Schwartz [15].

2. NOTATION AND PRELIMINARIES

(a) *General notation.* The standard symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ shall be used for the integers, the real numbers, and the complex numbers, respectively; $\mathbb{Z}^+$ is the set of nonnegative integers. If $z \in \mathbb{C}$ then $\bar{z}$ denotes the complex conjugate of z. If S is a set, T a subset, and f a function on S, the restriction of f to T is denoted by $f|_T$.

If S is a topological space, $T \subset S$, then $\mathrm{Cl}(T)$ denotes the closure of T in S, $\mathrm{Int}(T)$ the interior of T, and $\mathrm{Bdry}(T)$ the boundary of T. The space of continuous functions from S to $\mathbb{C}$ is denoted by $C(S)$,

$C_c(S)$ the set of those of compact support. The support of any $f \in C_c(S)$ is denoted by supp f.

Suppose S is a locally compact Hausdorff space. The subspace of $C(S)$ of functions vanishing at infinity is denoted by $C_0(S)$. We call the σ-ring generated by the compact subsets of S the Borel sets of S, and any measure defined on these sets and finite on compacts a Borel measure.

For S a topological vector space, let S' denote the continuous dual.

(b) *Representations.* Let G be a locally compact group which is countable at infinity, and E a locally convex, complete, Hausdorff topological vector space over $\mathbb{C}$. Then a (continuous) representation π of G on E is a homomorphism of G into Aut(E) such that $(g, v) \to \pi(g)v$ of $G \times E \to E$ is continuous. A representation π lifts to a homomorphism of the algebra of Radon measures on G with compact support into the continuous endomorphisms of E by $\pi(\mu)v = \int_G \pi(g)\, v d\mu(g)$, i.e., for each $T \in E'$ we have $T[\pi(\mu)v] = \int_G T[\pi(g)v]\, d\mu(g)$.

(c) *Positive definite functions.* Let G be an arbitrary group with identity e, not necessarily topological. A function f from G to $\mathbb{C}$ is said to be positive definite (written $f \gg 0$) if the inequality

$$\sum_{j,k=1}^{m} \alpha_j \bar{\alpha}_k f(x_j^{-1} x_k) \geqslant 0$$

holds for all subsets $\{x_1, ..., x_m\}$ of elements of G and all sequences $\{\alpha_1, ..., \alpha_m\}$ of complex numbers. For such functions the following properties are true.

$$f(e) \geqslant 0 \quad \text{and} \quad |f(x)| \leqslant f(e) \quad \text{for all} \quad x \in G; \tag{2.1}$$

$$f = f^* \quad \text{where} \quad f^*(x) = \bar{f}(x^{-1}) \quad \text{for all} \quad x \in G. \tag{2.2}$$

(d) *Manifolds.* Let M be a C^∞ manifold countable at infinity. We write $\mathscr{D}(M)$ for the space of C^∞ functions on M of compact support, and for each compact subset H of M we write $\mathscr{D}_H(M)$ for the subspace of functions of $\mathscr{D}(M)$ with support in H. For each H, $\mathscr{D}_H(M)$ is topologized by means of uniform convergence on compacts of functions along with their derivatives, and $\mathscr{D}(M)$ is given the inductive limit topology of the $\mathscr{D}_H(M)$ spaces. $\mathscr{D}'(M)$ denotes the dual space of $\mathscr{D}(M)$, called the space of distributions on M.

Let τ be a diffeomorphism of M onto itself, and take $f \in \mathscr{D}(M)$ and D a differential operator on M. If $f^\tau = f \circ \tau^{-1}$ and $D^\tau f = (Df^{\tau^{-1}})^\tau$, then f^τ is in $\mathscr{D}(M)$ and D^τ is another differential operator on M.

If V is a vector space over $\mathbb{R}$, $\mathscr{S}(V)$ denotes the Schwartz space of rapidly decreasing functions on V with the usual topology.

(e) *Lie groups.* If G is a group and $a \in G$, $L(a)$ denotes the left translation $g \to ag$ and $R(a)$ denotes the right translation $g \to ga^{-1}$ on G. Lie groups will be denoted by Latin capital letters and their Lie algebras by corresponding lower case German letters. The identity of any Lie group is denoted by e.

Let G be a connected semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G, and $\langle\ ,\ \rangle$ the Killing form of $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$. This is an involutive automorphism such that the form $(X, Y) \to -\langle X, \theta Y\rangle$ is strictly positive definite on $\mathfrak{g} \times \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of θ (a Cartan decomposition) and K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. It is known that any maximal compact subgroup K_1 of G is associated in this way with some Cartan decomposition of $\mathfrak{g}$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its dual, and $\mathfrak{a}_c{}^*$ the complexification of $\mathfrak{a}^*$, i.e., the space of $\mathbb{R}$-linear maps of $\mathfrak{a}$ into $\mathbb{C}$. Let $A = \exp \mathfrak{a}$ and let log be the inverse of the map $\exp\colon \mathfrak{a} \to A$. For $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$ then λ is called a (restricted) root and $m_\lambda = \dim(\mathfrak{g}_\lambda)$ is called its multiplicity. If $\lambda, \mu \in \mathfrak{a}_c{}^*$, let $H_\lambda \in \mathfrak{a}_c$ be determined by $\lambda(H) = \langle H_\lambda, H\rangle$ for $H \in \mathfrak{a}$ and put $\langle\lambda, \mu\rangle = \langle H_\lambda, H_\mu\rangle$. Since $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{p}$ we put $|\lambda| = \langle\lambda, \lambda\rangle^{1/2}$ for $\lambda \in \mathfrak{a}^*$ and $|X| = \langle X, X\rangle^{1/2}$ for $X \in \mathfrak{p}$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. The components of $\mathfrak{a}'$ are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+$ and call a (restricted) root α positive if it is positive on $\mathfrak{a}^+$. Let Σ denote the set of restricted roots and Σ^+ the set of positive roots. Let ρ denote half the sum of the positive roots with multiplicity, i.e., $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. Let $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ and let N and $\bar{N}$ denote the corresponding analytic subgroups of G. Let M denote the centralizer of A in K, M' the normalizer of A in K, and W the (finite) factor group M'/M. The group W, called the Weyl group, acts as a group of linear transformations on $\mathfrak{a}$ and also on $\mathfrak{a}_c{}^*$ by $(s\lambda)(H) = \lambda(s^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}_c{}^*$, and $s \in W$. Let s^* be the unique element of W such that $s^*(\Sigma^+) = -\Sigma^+$. Then $s^*\rho = -\rho$.

Let w denote the order of W, $A^+ = \exp \mathfrak{a}^+$; then we have the decompositions

$$G = K\,\mathrm{Cl}(A^+)K \qquad \text{(Cartan decomposition)}, \tag{2.3}$$

$$G = KAN \qquad \text{(Iwasawa decomposition)}. \tag{2.4}$$

Here (2.3) means that each $g \in G$ can be written $g = k_1 A(g)\, k_2$, where $k_1, k_2 \in K$ and $A(g) \in \mathrm{Cl}(A^+)$; $A(g)$ is actually unique. In (2.4) each $g \in G$ can be uniquely written $g = k(g) \exp H(g)\, n(g)$, $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$.

The number $l = \mathrm{Dim}\, \mathfrak{a}$ is called the real rank of G.

(f) *Convolutions and normalization of measures.* With G a connected semisimple Lie group with finite center and K a fixed compact subgroup it is convenient to make some conventions concerning the normalization of certain invariant measures. The Killing form induces Euclidean measures on A, $\mathfrak{a}$, and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-l/2}$, we obtain invariant measures da, dH, and $d\lambda$, and the inversion formula for the Fourier transform

$$\tilde{f}(\lambda) = \int_A f(a)\, e^{-i\lambda(\log a)}\, da, \qquad \lambda \in \mathfrak{a}^*, \qquad f \in \mathscr{S}(A),$$

holds without any multiplicative constant,

$$f(a) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\, e^{i\lambda(\log a)}\, d\lambda.$$

We normalized the Haar measure dk on the compact group K such that the total measure is one. The Haar measures of the nilpotent groups N, $\bar{N}$ are normalized such that

$$\theta(dn) = d\bar{n}, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))}\, d\bar{n} = 1.$$

The Haar measure dx on G can be normalized such that

$$\int_G f(x)\, dx = \int_{KAN} f(kan)\, e^{2\rho(\log a)}\, dk\, da\, dn, \qquad f \in \mathscr{D}(G).$$

Let $\mathscr{I}_c(G)$ and $C(K\backslash G/K)$ be the subspaces of K-biinvariant functions in $\mathscr{D}(G)$ and $C(G)$, respectively. Both are given their respective relative topologies. Defining the convolution of two functions f and g by

$$f * g(y) = \int_G f(x)\, g(x^{-1}y)\, dx, \qquad y \in G,$$

we have that $\mathscr{I}_c(G)$ is commutative under convolution.

For f locally integrable on G we define

$$f^{\natural}(g) = \iint_{K\times K} f(k_1 g k_2)\, dk_1\, dk_2\,, \qquad g \in G.$$

Then $f \to f^{\natural}$ is a continuous linear mapping of $\mathscr{D}(G)$ onto $\mathscr{I}_c(G)$.

(g) *Differential operators.* For G any Lie group let $\mathbb{D}(G)$ denote the algebra of all left invariant differential operators on G. Take $\{X_1, \ldots, X_n\}$ to be any basis of $\mathfrak{g}$. The Birkhoff–Witt theorem gives that $\{X_1^{e_1} \cdots X_n^{e_n} \mid e_j \geqslant 0\}$ is a basis of $\mathbb{D}(G)$ when each X_j is considered as a left invariant vector field on G.

With G and K as in (f) let $\mathbb{D}_0(G)$ denote the set of all $D \in \mathbb{D}(G)$ which are invariant under all right translations from K.

(h) *Spherical functions.* With G and K as in (f) let a nonzero function $\varphi \in C(G)$ be a (zonal) spherical function if it satisfies any of the following equivalent conditions.

(i) $\int_K \varphi(xky)\, dk = \varphi(x)\, \varphi(y)$, $x, y \in G$;

(ii) $\varphi \in C(K\backslash G/K)$, $\varphi(e) = 1$, and $f * \varphi = (\int_G f(g)\, \varphi(g^{-1}) dg)\varphi$, $f \in C(K\backslash G/K)$;

(iii) $\varphi \in C(K\backslash G/K)$ and $L\colon f \to \int_G f(g)\, \varphi(g)\, dg$ is a homomorphism of $C(K\backslash G/K)$ onto $\mathbb{C}$; and

(iv) $\varphi \in C^\infty(K\backslash G/K)$, $\varphi(e) = 1$, and $D\varphi = (D\varphi(e))\varphi$, $D \in \mathbb{D}_0(G)$.

Let $\mathscr{F}^1$ be the set of all bounded spherical functions and $\mathscr{P}$ the subset of all positive definite spherical functions. Giving $\mathscr{P}$ the Godement topology, i.e., the weak* topology as a subset of $L^\infty(G)$, makes $\mathscr{P}$ into a locally compact Hausdorff space.

For any measurable function f on G we define its spherical transform $\hat{f}$ by

$$\hat{f}[\varphi] = \int_G f(g)\, \varphi(g^{-1})\, dg \tag{2.5}$$

for all spherical functions φ for which the integral is defined. In particular, if $f \in L^1(G)$, then $\hat{f}$ is defined on $\mathscr{F}^1$, hence on $\mathscr{P}$, and $\hat{f} \in C_0(\mathscr{P})$ [6, p. 7]. The following properties are quickly verified for $f, g \in L^1(G)$.

$$(f^{\natural})^\wedge = \hat{f} \qquad \text{on } \mathscr{F}^1; \tag{2.6}$$

$$(f^*)^\wedge[\varphi] = \overline{\hat{f}}[\varphi^*] \quad \text{for} \quad \varphi \in \mathscr{F}^1; \quad \text{hence} \quad (f^*)^\wedge = \overline{\hat{f}} \quad \text{on} \quad \mathscr{P}; \tag{2.7}$$

$$(f * g)^\wedge = \hat{f} \cdot \hat{g} \quad \text{on} \quad \mathscr{F}^1 \quad \text{if } f \text{ is right } K\text{-invariant or if } g \text{ is left } K\text{-invariant.} \tag{2.8}$$

Thus if $f \in L^1(G)$ is right K-invariant, then

$$(f*f^*)^\wedge = |\hat{f}|^2 \quad \text{on} \quad \mathscr{P}. \tag{2.9}$$

There exists a basic parametrization and formula for the spherical functions given by Harish–Chandra: the spherical functions are precisely the functions

$$\varphi_\lambda(g) = \int_K \exp(i\lambda - \rho)(H(gk))\, dk, \qquad g \in G, \tag{2.10}$$

where $\lambda \in \mathfrak{a}_c^*$ is arbitrary; moreover, $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = s\mu$ for some $s \in W$. Hence $\mathscr{P}$ and $\mathscr{F}^1$ can be viewed as subsets of $W\backslash\mathfrak{a}_c^*$, or by an obvious abuse of notation, as subsets of $\mathfrak{a}_c^*$. Certain properties are:

(i) $\varphi_{-\lambda}(g^{-1}) = \varphi_\lambda(g), \qquad \lambda \in \mathfrak{a}_c^*, g \in G;$ (2.11)

(ii) $\varphi_{-\lambda}(g) = \bar{\varphi}_{\bar{\lambda}}(g), \qquad \lambda \in \mathfrak{a}_c^*, g \in G;$ (2.12)

(iii) $\lambda \in \mathscr{P}$ implies λ and $\bar{\lambda}$ are Weyl group conjugate;

(iv) If ω is the Casimir operator on G, then

$$\omega\varphi_\lambda = -(\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle)\, \varphi_\lambda, \qquad \lambda \in \mathfrak{a}_c^*;$$

(v) The Helgason–Johnson theorem: Let C_ρ be the convex hull of $\{s\rho \mid s \in W\}$ in $\mathfrak{a}^*$. Then $\mathscr{F}^1 = \mathfrak{a}^* + iC_\rho$;

(vi) The Godement topology on $\mathscr{P}$ is the same as the topology induced by the Euclidean topology of $\mathfrak{a}_c^*$.

Remarks. (i) is [7, Lemma 45]; (ii) follows by easy computation; (iii) follows from (i), (ii), and (2.2); (iv) is [7, p. 271]; (v) is [12, Theorem 2.1]; a proof of (vi) based on (iv) can be found in [2, p. 30].

From (2.11) we can reformulate (2.5) to be

$$\hat{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)\, dg, \qquad \lambda \in \mathfrak{a}_c^*.$$

Then for $f \in \mathscr{I}_c(G)$ it is known that

$$f(g) = w^{-1} \int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_\lambda(g) \mid c(\lambda)|^{-2}\, d\lambda, \qquad g \in G,$$

where w = order of W and

$$c(\lambda) = \int_{\bar{N}} \exp(-(i\lambda + \rho)(H(\bar{n})))\, d\bar{n}, \qquad \lambda \in \mathfrak{a}^*.$$

(i) *Distributions on Lie groups.* Suppose G is a separable Lie group. Then the topology on $\mathscr{D}(G)$ can be described by means of left (or right) invariant differential operators on G [4, Proposition 2]. Let $\varphi, \psi \in \mathscr{D}(G)$ and let D (resp. E) be a left (resp. right) invariant differential operator on G. Then

$$D(\varphi * \psi) = \varphi * (D\psi) \qquad \text{and} \qquad E(\varphi * \psi) = (E\varphi) * \psi.$$

It follows that if $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathscr{D}(G)$, then $\varphi_n * \psi_n \to \varphi * \psi$ in $\mathscr{D}(G)$.

If dx is a left Haar measure on G, then a function f, locally summable with respect to dx, can be identified with the distribution $T_f \in \mathscr{D}'(G)$ defined by

$$T_f[\varphi] = \int_G \varphi(x) f(x)\, dx, \qquad \varphi \in \mathscr{D}(G).$$

For each differential operator D on G we let ${}^tD$ be the adjoint with respect to dx. Then given $T \in \mathscr{D}'(G)$, we define $DT \in \mathscr{D}'(G)$ by

$$DT[\varphi] = T[{}^tD\varphi], \qquad \varphi \in \mathscr{D}(G).$$

Let G be a connected, semisimple Lie group with finite center, and K a maximal compact subgroup. Then $T \in \mathscr{D}'(G)$ is called K-biinvariant if $T[\varphi^{L(k_1)R(k_2)}] = T[\varphi]$ for all $\varphi \in \mathscr{D}(G)$ and $k_1, k_2 \in K$. Notice that T K-biinvariant implies $T[\varphi^{\natural}] = T[\varphi]$, $\varphi \in \mathscr{D}(G)$.

3. The $\mathscr{C}^p(G)$, $\mathscr{I}^p(G)$, and $\mathscr{Z}(\mathscr{F}^\epsilon)$ Spaces

(a) *The spaces $\mathscr{C}^p(G)$.* Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. On G define the two functions

$$\Xi(g) = \int_K \exp(-\rho(H(gk)))\, dk, \qquad g \in G,$$

$$\sigma(g) = |X|, \qquad \text{where} \quad g = k \exp X,\ k \in K,\ X \in \mathfrak{p}.$$

For $0 < p \leqslant 2$ let $\mathscr{C}^p(G)$ be the set of infinitely differentiable functions f such that, for each $m \in \mathbb{Z}^+$ and each left (resp. right) invariant differential operator D (resp. E),

$$\sup_G (1 + \sigma)^m\, \Xi^{-2/p}\, |DEf| < \infty.$$

Topologizing in the obvious manner makes each $\mathscr{C}^p(G)$ into a Fréchet space. Since $|\Xi| \leqslant 1$, it is clear that $p \leqslant q$ implies $\mathscr{C}^p(G) \subset \mathscr{C}^q(G)$, and the existence of $r \geqslant 0$ shown in [8, Lemma 11] such that

$$\int_G \Xi(g)^2(1 + \sigma(g))^{-r}\, dg < \infty$$

shows that $\mathscr{C}^p(G) \subset L^p(G)$.

Define two maps L and R from G into Aut $C^\infty(G)$ by $L(g)f = f^{L(g)}$ and $R(g)f = f^{R(g)}$. Then for each $0 < p \leqslant 2$ we have that both L and R are differentiable representations of G on $\mathscr{C}^p(G)$. Moreover, $\mathscr{D}(G)$ is dense in each $\mathscr{C}^p(G)$, and each is a convolution algebra. The proofs of these facts do not differ significantly from those given in [17, Section 8.3.7] for the $p = 2$ case.

(b) *The spaces* $\mathscr{I}^p(G)$. Define $\mathscr{I}^p(G)$ to be the space of K-biinvariant elements of $\mathscr{C}^p(G)$; we give $\mathscr{I}^p(G)$ the relative topology as a subset of $\mathscr{C}^p(G)$. It is a consequence of [17, Lemma 2, p. 164] that only left invariant differential operators need be considered in defining the topology of $\mathscr{I}^p(G)$. Since $\mathscr{D}(G)$ is dense in $\mathscr{C}^p(G)$ it is not hard to show that $\mathscr{I}_c(G)$ is dense in $\mathscr{I}^p(G)$. But the following slightly stronger result is available to us.

Proposition 3.1. *Suppose $f \in \mathscr{I}^p(G)$ for all $p > p_0$ for some fixed $p_0 > 0$. Then there exists a sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{I}_c(G)$ such that $\varphi_j \to f$ in $\mathscr{I}^p(G)$ for all $p > p_0$.*

The proof of this result follows the same lines as the proof that $\mathscr{I}_c(G)$ is dense in $\mathscr{I}^2(G)$ given in [10, p. 571].

The continuous linear functionals on $\mathscr{I}^p(G)$, $\mathscr{C}^p(G)$, and $\mathscr{D}(G)$ are related in the following manner.

Proposition 3.2. (i) *$T \in (\mathscr{C}^p(G))' \Rightarrow T|_{\mathscr{D}(G)} \in \mathscr{D}'(G)$. This correspondence is one-to-one.*

(ii) *$T \in (\mathscr{C}^p(G))' \Rightarrow T|_{\mathscr{I}^p(G)} \in (\mathscr{I}^p(G))'$. Moreover, when restricted to the K-biinvariant elements in $(\mathscr{C}^p(G))'$ this correspondence becomes one-to-one and onto.*

Proof. (i) is obvious, the one-to-one property coming from the density of $\mathscr{D}(G)$ in $\mathscr{C}^p(G)$. The first part of (ii) is also obvious from the definition of the $\mathscr{I}^p(G)$ spaces. To complete the proof we first show that $f \to f^\natural$ defines a continuous endomorphism of $\mathscr{C}^p(G)$ onto $\mathscr{I}^p(G)$.

For each $x \in G$ let $S_x \in (\mathscr{C}^p(G))'$ be defined by $S_x[\varphi] = \varphi(x)$, $\varphi \in \mathscr{C}^p(G)$. Then Section 2(b) gives that

$$f^{\natural}(x) = \iint_{K\times K} S_x(L(k_1)\, R(k_2) f)\, dk_1\, dk_2$$

$$= S_x \left[\iint_{K\times K} L(k_1)\, R(k_2) f\, dk_1\, dk_2\right]$$

$$= L(dk)\, R(dk) f(x)$$

whenever $f \in \mathscr{C}^p(G)$. This proves the desired continuity of $f \to f^{\natural}$.

Thus, given $S \in (\mathscr{I}^p(G))'$, we may define $T \in (\mathscr{C}^p(G))'$ by $T[f] = S[f^{\natural}]$, $f \in \mathscr{C}^p(G)$. This proves the onto property of the mapping in (ii), and the uniqueness of the K-biinvariant T follows quickly. ∎

(c) *The spaces* $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$. Let C_ρ be the closed, convex hull in $\mathfrak{a}^*$ of the finite set $\{s\rho \mid s \in W\}$, and for each $\epsilon \geqslant 0$ let $\mathscr{F}^\epsilon = \mathfrak{a}^* + i\epsilon C_\rho$ in $\mathfrak{a}_c^*$. Since C_ρ is W-invariant, then $C_\rho = s^* C_\rho = -C_\rho$, and hence $-\mathscr{F}^\epsilon = \mathscr{F}^\epsilon$ for each $\epsilon \geqslant 0$. Moreover, each $\mathscr{F}^\epsilon$ is convex and $\operatorname{Int} \mathscr{F}^\epsilon = \bigcup_{0<\epsilon'<\epsilon} \mathscr{F}^{\epsilon'}$ [16, Lemma 3.2.2].

Define $\mathscr{Z}(\mathscr{F}^0) = \mathscr{S}(\mathfrak{a}^*)$ and for any $\epsilon > 0$ define $\mathscr{Z}(\mathscr{F}^\epsilon)$ to be the space of all $\mathbb{C}$-valued functions Φ such that (i) Φ is defined on $\operatorname{Int} \mathscr{F}^\epsilon$ and holomorphic there, and (ii) for each holomorphic differential operator D with polynomial coefficients we have $\sup_{\operatorname{Int} \mathscr{F}^\epsilon} | D\Phi | < \infty$. Each $\mathscr{Z}(\mathscr{F}^\epsilon)$ is an algebra and, when topologized in the obvious manner, becomes a Fréchet space with multiplication being jointly continuous. For each D as in (ii) and each $f \in \mathscr{Z}(\mathscr{F}^\epsilon)$ it can be shown that Df extends to a continuous function on all of $\mathscr{F}^\epsilon$ [16, p. 278]. We define $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ to be the closed subalgebra of W-invariant elements of $\mathscr{Z}(\mathscr{F}^\epsilon)$. The main result of the subject is the following.

THEOREM 3.3 (Trombi–Varadarajan). *Let* $0 < p \leqslant 2$ *and* $\epsilon = (2/p) - 1$. *Then the spherical transform* $f \to \hat{f}$ *is a linear topological isomorphism of* $\mathscr{I}^p(G)$ *onto* $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ *which preserves the algebraic structure.*

Remarks. (1) This theorem was proved by Trombi and Varadarajan in [16, p. 298], generalizing earlier results by Ehrenpreis and Mautner, Harish-Chandra, and Helgason. The most difficult part of the proof is in proving subjectivity. (2) At this stage we could proceed to state and prove a number of needed technical lemmas about the $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ spaces. We will refrain from doing so, however, and instead shall place them in the final sixth section of the paper, citing their use in Sections 4 and 5 when needed. This enables us to get quickly to the main problem at hand while also motivating the need for the various detailed results that are proved in Section 6.

4. The Spherical Bochner Theorem

Let G be a separable unimodular Lie group. A distribution T on G is said to be positive definite (written $T \gg 0$) if $T[\varphi * \varphi^*] \geqslant 0$ for all $\varphi \in \mathscr{D}(G)$. Then for any $f \in C(G)$ we have $f \gg 0$ if and only if $T_f \gg 0$. The following result is proved in [3, Section 5].

THEOREM 4.1. *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$. Then T can be expressed as a finite sum*

$$T = \sum_j D^j E^j f_j ,$$

where, for each j, f_j is a bounded function and D^j (resp. E^j) is a left (resp. right) invariant differential operator.

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Then Godement has proved the following partial Bochner theorem (notation as in Section 2(h)).

THEOREM 4.2. *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$. Then there exists a unique positive regular Borel measure μ on $\mathscr{P}$ such that*

(i) $\hat{\varphi} \in L^2(\mu)$, $\varphi \in \mathscr{D}(G)$; *and*

(ii) $T[\varphi * \psi] = \int_{\mathscr{P}} \hat{\varphi}\hat{\psi}\, d\mu$, $\varphi, \psi \in \mathscr{I}_c(G)$.

Remarks. Theorem 4.2 was first presented in [6]; a slightly different proof may be found in [2]. The measure μ is called the spherical Bochner measure of T. Our procedure now will be to extend Theorem 4.2 by the use of Theorem 4.1 and the Trombi–Varadarajan result.

LEMMA 4.3. *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$ with spherical Bochner measure μ. Then T extends uniquely to an element in $(\mathscr{C}^1(G))'$ and*

$$T[\varphi * \psi] = \int_{\mathscr{P}} \hat{\varphi}\hat{\psi}\, d\mu$$

for all $\psi \in \mathscr{I}^1(G)$ and all $\varphi \in \mathscr{I}_c(G)$ such that $\hat{\varphi} \in L^1(\mu)$.

Proof. That T extends uniquely into $(\mathscr{C}^1(G))'$ is clear from Theorem 4.1 and the properties of $\mathscr{C}^1(G)$. Pick $\{\psi_n\}_{n=1}^{\infty}$ in $\mathscr{I}_c(G)$ such that $\psi_n \to \psi$ in $\mathscr{I}^1(G)$. The Godement theorem gives

$$T[\varphi*\psi_n] = \int_{\mathscr{P}} \hat{\varphi}\hat{\psi}_n \, d\mu,$$

and since $\varphi*\psi_n \to \varphi*\psi$ in $\mathscr{I}^1(G)$, then $T[\varphi*\psi_n] \to T[\varphi*\psi]$. We are left with showing $\int_{\mathscr{P}} \hat{\varphi}\hat{\psi}_n \, d\mu \to \int_{\mathscr{P}} \hat{\varphi}\hat{\psi} \, d\mu$ as $n \to \infty$.

From the Helgason–Johnson theorem (Section 2(h)) we see that $\mathscr{P} \subset \mathscr{F}^1$, and hence $\int_{\mathscr{P}} \hat{\varphi}\hat{\psi} \, d\mu$ is well defined since $\hat{\psi}$ is extendable to all of $\mathscr{F}^1$ as a continuous function. Theorem 3.3 gives $\hat{\psi}_n \to \hat{\psi}$ in $\bar{\mathscr{Z}}(\mathscr{F}^1)$, and hence, in particular, $\{\hat{\psi}_n\}_{n=1}^{\infty}$ is uniformly bounded on $\mathscr{F}^1$. Dominated convergence then gives $\int_{\mathscr{P}} \hat{\varphi}\hat{\psi}_n \, d\mu \to \int_{\mathscr{P}} \hat{\varphi}\hat{\psi} \, d\mu$. ∎

Definition. A positive regular Borel measure μ on $\mathscr{P}$ is said to be of polynomial growth if there exists a holomorphic polynomial Q on $\mathfrak{a}_c^*$ such that $\int_{\mathscr{P}} (d\mu/|Q|) < \infty$.

Lemma 4.4. *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$ with spherical Bochner measure μ. Then μ is of polynomial growth.*

Proof. By Lemma 4.3 we have $T \in (\mathscr{C}^1(G))'$. Hence the Trombi–Varadarajan theorem gives that $\hat{T} \in (\bar{\mathscr{Z}}(\mathscr{F}^1))'$, where $\hat{T}[\hat{\psi}] = T[\psi]$ for all $\psi \in \mathscr{I}^1(G)$. Let $\epsilon_1, \ldots, \epsilon_l$ be any basis of $\mathfrak{a}^*$ and for each $\lambda \in \mathfrak{a}_c^*$ determine $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ by $\lambda = \sum_j \lambda_j \epsilon_j$. For each $m, t \in \mathbb{Z}^+$ define the continuous seminorm $\sigma_m{}^t$ on $\bar{\mathscr{Z}}(\mathscr{F}^1)$ by

$$\sigma_m{}^t(\Phi) = \sup_{|M| \leqslant m, \lambda \in \operatorname{Int} \mathscr{F}^1} (1 + \|\lambda\|^2)^t \, |(d/d\lambda)^M \Phi(\lambda)|,$$

where $M = (m_1, \ldots, m_l)$, $|M| = m_1 + \cdots + m_l$, $(d/d\lambda)^M = (\partial/\partial\lambda_1)^{m_1} \cdots (\partial/\partial\lambda_l)^{m_l}$, and $\|\xi + i\eta\|^2 = |\xi|^2 + |\eta|^2$ for all $\xi, \eta \in \mathfrak{a}^*$.

Since $\hat{T}$ is continuous on $\bar{\mathscr{Z}}(\mathscr{F}^1)$ there exists $m, t \in \mathbb{Z}^+$ such that $|\hat{T}[\Phi]| \leqslant \sigma_m{}^t(\Phi)$ for all $\Phi \in \bar{\mathscr{Z}}(\mathscr{F}^1)$. Thus Lemma 4.3 gives

$$\left| \int_{\mathscr{P}} \hat{\varphi}\hat{\psi} \, d\mu \right| \leqslant \sigma_m{}^t(\hat{\varphi}\hat{\psi}) \tag{4.1}$$

for all $\psi \in \mathscr{I}^1(G)$ and all $\varphi \in \mathscr{I}_c(G)$ such that $\hat{\varphi} \in L^1(\mu)$.

Take δ_j to be an approximation of the identity in $\mathscr{D}(G)$ such that $\int_G |\delta_j(g)| \, dg = 1$ for each j. Then $\hat{\delta}_j \to 1$ pointwise on $\mathfrak{a}_c^*$ and $|\hat{\delta}_j| \leqslant 1$ on $\mathscr{P}$. Defining $\varphi_j = \delta_j{}^\natural*(\delta_j{}^*)^\natural$ we see by Section 2(h) that $\hat{\varphi} = |\hat{\delta}_j|^2 \leqslant 1$ on $\mathscr{P}$ and $\hat{\varphi}_j \to 1$ pointwise on $\mathfrak{a}_c^*$. By the Godement theorem $\hat{\varphi}_j \in L^1(\mu)$.

Let Q be a W-invariant, holomorphic polynomial on $\mathfrak{a}_c^*$ of degree $\geqslant 2$ which is uniformly bounded away from zero on $\mathscr{F}^1$ and which is positive on $\mathscr{P}$ (Proposition 6.1). For each j define $\Psi_j = \hat{\varphi}_j/Q$. Then $\Psi_j \in \bar{\mathscr{Z}}(\mathscr{F}^1)$ (Proposition 6.1), and hence there exists a unique $\psi_j \in \mathscr{S}^1(G)$ such that $\Psi_j = \hat{\psi}_j$ by the Trombi–Varadarajan theorem. We claim that with m and t as specified in (4.1) we have only to show $\{\sigma_m{}^t(\hat{\varphi}_j\hat{\psi}_j)\}_{j=1}^{\infty}$ is uniformly bounded with respect to j.

For suppose there exists $c > 0$ such that $\sigma_m{}^t(\hat{\varphi}_j\hat{\psi}_j) < c$ for all j. Then (4.1) gives

$$\left| \int_{\mathscr{P}} (\hat{\varphi}_j{}^2/Q)\, d\mu \right| < c \qquad \text{for all } j \in \mathbb{Z}^+.$$

But on $\mathscr{P}$ we have that $Q > 0$, $0 \leqslant \hat{\varphi}_j \leqslant 1$ and $\hat{\varphi}_j \to 1$ pointwise. Thus, applying monotone convergence to $\int_{\mathscr{P}} \inf_{k \geqslant j} (\hat{\varphi}_k{}^2/Q)\, d\mu$ gives $\int_{\mathscr{P}} (d\mu/Q) < c$, or that μ is of polynomial growth.

To show that $\{\sigma_m{}^t(\hat{\varphi}_j\hat{\psi}_j)\}_{j=1}^{\infty}$ is uniformly bounded, first note that each term $(d/d\lambda)^M(\hat{\varphi}_j{}^2/Q)$ may be expanded out into the form $\sum_{N+R=M} C_N (d/d\lambda)^N(1/Q)(d/d\lambda)^R(\hat{\varphi}_j{}^2)$. Hence

$$\sigma_m{}^t(\hat{\varphi}_j\hat{\psi}_j) \leqslant C \sum_{|N+R| \leqslant m} Q_N \sup_{\lambda \in \operatorname{Int}\mathscr{F}^1} |(d/d\lambda)^R(\hat{\varphi}_j{}^2)|$$

for

$$Q_N = \sup_{\lambda \in \operatorname{Int}\mathscr{F}^1} (1 + \| \lambda \|^2)^t \, |(d/d\lambda)^N(1/Q)|.$$

We first claim that $Q_N < \infty$ for each N. For note that $(d/d\lambda)^N(1/Q)$ is a rational function on $\mathscr{F}^1$ with the order of the denominator minus order of the numerator being $\geqslant$ order $Q \geqslant 2t$. Hence, since Q is strictly bounded away from zero on $\mathscr{F}^1$, we must have that $(1 + \| \lambda \|^2)^t \, |(d/d\lambda)^N(1/Q)|$ is a bounded function on $\mathscr{F}^1$.

We have therefore only to prove that $\sup_{\lambda \in \operatorname{Int}\mathscr{F}^1} |(d/d\lambda)^R(\hat{\varphi}_j{}^2)|$ is uniformly bounded in j for each R. To do so it is sufficient to show that $\bigcup_{j=1}^{\infty} \operatorname{supp}(\varphi_j * \varphi_j)$ is relatively compact and that $\int_G |\varphi_j * \varphi_j(g)|\, dg$ is uniformly bounded in j (Proposition 6.4). But these both follow easily from the definition of φ_j. ∎

Remark. The proof of Lemma 4.4 is based on the proof for the Euclidean case in [15, p. 242].

THEOREM 4.5 (The Spherical Bochner Theorem). *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$. Then T extends uniquely to an element in $(\mathscr{C}^1(G))'$*

and there exists a unique positive regular Borel measure μ of polynomial growth on $\mathscr{P}$ such that

$$T[\varphi] = \int_{\mathscr{P}} \hat{\varphi}\, d\mu, \qquad \varphi \in \mathscr{I}^1(G).$$

The correspondence between T and μ is bijective when restricted to K-biinvariant distributions, in which case the formula holds for all $\varphi \in \mathscr{C}^1(G)$.

Proof. Suppose we are given μ a positive regular Borel measure of polynomial growth on $\mathscr{P}$. We first claim that $\mathscr{Z}(\mathscr{F}^1) \subset L^1(\mu)$. For take $\Phi \in \mathscr{Z}(\mathscr{F}^1)$. Since $\mathscr{P} \subset \mathscr{F}^1$ by the Helgason–Johnson theorem, then Φ is defined on the support of μ. Moreover, there exists some holomorphic polynomial P on $\mathfrak{a}_c{}^*$ such that $\int_{\mathscr{P}} (d\mu/|P|) < \infty$, and thus

$$\int_{\mathscr{P}} |\Phi|\, d\mu \leqslant (\sup_{\mathscr{F}^1} |P\Phi|) \int_{\mathscr{P}} (d\mu/|P|) < \infty,$$

proving the claim.

Hence the linear functional $\hat{T}$ on $\mathscr{Z}(\mathscr{F}^1)$ given by $\hat{T}[\Phi] = \int_{\mathscr{P}} \Phi\, d\mu$ is well defined. Suppose $\Phi_n \to 0$ in $\mathscr{Z}(\mathscr{F}^1)$. Then

$$|\hat{T}[\Phi_n]| \leqslant \sup_{\mathscr{F}^1} |P(\Phi_n)| \int_{\mathscr{P}} (d\mu/|P|)$$

which converges to zero as $n \to \infty$, proving $\hat{T}$ is continuous.

Proposition 3.2 and the Trombi–Varadarajan theorem allow us to define $T \in (\mathscr{C}^1(G))'$ by $T[\psi] = \hat{T}[\hat{\psi}]$, $\psi \in \mathscr{C}^1(G)$. Hence $T[\psi] = \int_{\mathscr{P}} \hat{\psi}\, d\mu$ and we have only to show $T \gg 0$. Take any $\varphi \in \mathscr{D}(G)$. Then $(\varphi * \varphi^*)^\wedge(\lambda) = \varphi_{-\lambda}[\varphi * \varphi^*] \geqslant 0$ for all $\lambda \in \mathscr{P}$ since $\varphi_{-\lambda} \gg 0$. Hence $T[\varphi * \varphi^*] = \int_{\mathscr{P}} (\varphi * \varphi^*)^\wedge\, d\mu \geqslant 0$, so that $T \gg 0$.

Conversely, suppose $T \in \mathscr{D}'(G)$, $T \gg 0$. Then by the Godement theorem there exists a unique positive regular Borel measure μ on $\mathscr{P}$ such that $T[\varphi * \psi] = \int_{\mathscr{P}} \hat{\varphi}\hat{\psi}\, d\mu$ for all $\varphi, \psi \in \mathscr{I}_c(G)$. From Lemma 4.4 the measure μ is of polynomial growth on $\mathscr{P}$, and the above construction shows that we can define $T_0 \in \mathscr{D}'(G)$ by $T_0[\varphi] = \int_{\mathscr{P}} \hat{\varphi}\, d\mu$. But $\{\varphi * \psi \mid \varphi, \psi \in \mathscr{I}_c(G)\}$ is dense in $\mathscr{I}_c(G)$ so that T and T_0 must agree on $\mathscr{I}_c(G)$, hence also on $\mathscr{I}^1(G)$. If T is K-biinvariant, then they must agree on all of $\mathscr{C}^1(G)$, proving the asserted bijection. ∎

5. The Extension Theorem

Using the spherical Bochner theorem and the Trombi–Varadarajan theorem we can deduce which $(\mathscr{I}^p(G))'$ space a given positive definite distribution lies in by examining the support of its spherical Bochner measure. The relationship is $T \in (\mathscr{I}^p(G))'$ if and only if $\operatorname{supp} \mu \subset \mathscr{F}^\epsilon$, where $1 \leqslant p \leqslant 2$ and $\epsilon = (2/p) - 1$ (Theorem 5.5). This is a natural result in light of the Trombi–Varadarajan theorem, but while the underlying idea of the proof is straightforward, the details are surprisingly complicated. The desired result will be a consequence of the first lemma after some rather intricate measure theory and geometry on $\mathfrak{a}_c^*$.

Since $\mathfrak{a}_c^* = \bigcup_{\epsilon>0} \mathscr{F}^\epsilon$ and $\operatorname{Int} \mathscr{F}^\epsilon = \bigcup_{\epsilon'<\epsilon} \mathscr{F}^{\epsilon'}$ we have that each $\alpha \in \mathfrak{a}_c^*$ must lie on the boundary of some unique $\mathscr{F}^\epsilon$, $\epsilon \geqslant 0$. Let $\epsilon(\alpha)$ be defined as that particular ϵ. Notice that $\alpha \to \epsilon(\alpha)$ is continuous. Also, for each $\alpha \in \mathfrak{a}_c^*$ and each $R > 0$ define $\mathscr{B}_R(\alpha) = B_R(\alpha) \cap \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}$, where $B_R(\alpha)$ is the open ball of radius R about α. Note that $\alpha \notin \mathscr{B}_R(\alpha)$.

LEMMA 5.1. *Suppose $T \in \mathscr{D}'(G)$, $T \gg 0$, such that $T \in (\mathscr{I}^{p_0}(G))'$ for some $p_0 \geqslant 1$. Let μ be the spherical Bochner measure of T, and let α_0 be any point in $\mathscr{P}$ outside of $\mathscr{F}^{\epsilon_0}$, where $\epsilon_0 = (2/p_0) - 1$. Then for each nonzero integer m there exists a compact neighborhood U of α_0, an $R > 0$, and an $M < \infty$ such that*

$$\int_{\mathscr{B}_R(\alpha)} \| \lambda - \alpha \|^{-2m} \, d\mu(\lambda) \leqslant M \qquad \text{for all} \quad \alpha \in U.$$

Remark. This lemma shows that the measure μ is in some sense "rapidly decreasing" near every point outside of $\mathscr{F}^{\epsilon_0}$. We eventually wish to show that μ is indeed zero near every such point.

Proof. Let $\varphi \in \mathscr{I}_c(G)$ be such that $|\hat{\varphi}(\alpha_0)| > 0$ and let V be a compact neighborhood of α_0 such that V and $\mathscr{F}^{\epsilon_0}$ are disjoint, and $|\hat{\varphi}(\alpha)| \geqslant c > 0$ for all $\alpha \in V$. Now for each $\alpha \in \mathfrak{a}_c^*$ construct a W-invariant holomorphic polynomial P_α on $\mathfrak{a}_c^*$ such that the degree of P_α is uniformly bounded in α, and

(i) $P_\alpha(\alpha) = 0$ for all $\alpha \in \mathfrak{a}_c^*$;

(ii) $P_{\alpha^n} \to P_\alpha$ uniformly on compact sets as $\alpha^n \to \alpha$ in $\mathfrak{a}_c^*$; and

(iii) given U a compact set in $\mathfrak{a}_c^*$ and $\epsilon > 0$ such that $\mathscr{F}^\epsilon$ and U are disjoint, then there exists a $c > 0$ such that $|P_\alpha(\lambda)| \geqslant c$ for all $\alpha \in U$ and $\lambda \in \mathscr{F}^\epsilon$.

Such a collection of polynomials exists by Proposition 6.2. For each $\alpha \in \mathfrak{a}_c^* - \mathscr{F}^{\epsilon_0}$ we define

$$\Psi_\alpha = \hat{\varphi}/P_\alpha^{\,m} \qquad \text{on} \qquad \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}. \tag{5.1}$$

Then Proposition 6.3 gives that $\Psi_\alpha \in \bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ for all $\epsilon < \epsilon(\alpha)$, and

$$\alpha \to \Psi_\alpha \text{ is continuous from } V \text{ into } \bar{\mathscr{Z}}(\mathscr{F}^{\epsilon_0}). \tag{5.2}$$

The Trombi–Varadarajan theorem gives the existence of functions ψ_α which are in $\mathscr{I}^p(G)$ for all $p > p(\alpha)$, where $p(\alpha) = 2/(\epsilon(\alpha) - 1)$, and are such that $\Psi_\alpha = \hat{\psi}_\alpha$ on Int $\mathscr{F}^{\epsilon(\alpha)}$. Thus from (5.2) we have

$$\alpha \to \psi_\alpha \text{ is continuous from } V \text{ into } \mathscr{I}^{p_0}(G), \tag{5.3}$$

and by assumption on T,

$$\alpha \to T[\psi_\alpha * \psi_\alpha^*] \text{ is continuous from } V \text{ into } \mathbb{C}. \tag{5.4}$$

For each $\alpha \in V$ take $\{\psi_{\alpha,n}\}_{n=1}^\infty \subset \mathscr{I}_c(G)$ such that $\psi_{\alpha,n} \to \psi_\alpha$ in $\mathscr{I}^p(G)$ for all $p > p(\alpha)$ (Proposition 3.1). The spherical Bochner theorem then gives

$$T[\psi_{\alpha,n} * \psi_{\alpha,n}^*] = \int_{\mathscr{P}} |\hat{\psi}_{\alpha,n}|^2 \, d\mu. \tag{5.5}$$

But the right-hand side of (5.5) is greater than or equal to $\int_{\mathscr{F}^\epsilon} |\hat{\psi}_{\alpha,n}|^2 \, d\mu$ for each $\epsilon < \epsilon(\alpha)$, and these latter quantities tend toward $\int_{\mathscr{F}^\epsilon} |\hat{\psi}_\alpha|^2 \, d\mu$ as $n \to \infty$ since the measure $\nu_\epsilon(E) = \mu(E \cap \mathscr{F}^\epsilon)$, $\epsilon < \epsilon(\alpha)$, defines a continuous linear functional $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ (see proof of Theorem 4.5). Hence (5.5) gives

$$T[\psi_\alpha * \psi_\alpha^*] \geqslant \int_{\mathscr{F}^\epsilon} |\hat{\psi}_\alpha|^2 \, d\mu, \qquad \alpha \in V, \quad \epsilon < \epsilon(\alpha),$$

and monotone convergence then implies

$$T[\psi_\alpha * \psi_\alpha^*] \geqslant \int_{\mathrm{Int}\,\mathscr{F}^{\epsilon(\alpha)}} |\hat{\psi}_\alpha|^2 \, d\mu, \qquad \alpha \in V. \tag{5.6}$$

Consider the function $F\colon \mathfrak{a}_c^* \times \mathfrak{a}_c^* \to \mathbb{R}$ defined by $F(\alpha, \lambda) = |P_\alpha(\lambda)|/\|\lambda - \alpha\|$ if $\lambda \neq \alpha$, and zero otherwise. Pick $R > 0$ and U a compact neighborhood of α_0 such that $\bigcup_{\alpha \in U} B_R(\alpha) \subset V$. Since F is bounded on compacts (Proposition 6.5) there exists $M_0 < \infty$ such that

$$|P_\alpha(\lambda)| \leqslant M_0 \|\lambda - \alpha\|, \qquad \alpha \in U, \quad \lambda \in B_R(\alpha).$$

Using this in (5.6) we obtain

$$T[\Psi_\alpha * \Psi_\alpha^*] \geqslant \int_{\mathscr{B}_R(\alpha)} (|\hat{\varphi}(\lambda)|^2/(M_0^{2m} \|\lambda - \alpha\|^{2m})) \, d\mu(\lambda).$$

Taking $M = (M_0^{2m}/c^2)(\sup_{\alpha \in U} T[\Psi_\alpha * \Psi_\alpha^*])$, which is finite from (5.4), proves the lemma. ∎

Remark. To prove that $\mu = 0$ outside of $\mathscr{F}^{\epsilon_0}$ we must examine the sets $\mathscr{B}_r(\alpha)$ for all $\alpha \in U$ and r small. There are two problems to be dealt with: (1) The sets $\mathscr{B}_r(\alpha)$ are irregular in shape, and (2) $\alpha \notin \mathscr{B}_r(\alpha)$. These will be circumvented by Proposition 5.2, and then Lemma 5.1 can be put into the more useful form of Lemma 5.3.

For each nonzero $\alpha \in \mathfrak{a}_c^*$ and $r \geqslant 0$ define $\alpha_r \in \mathfrak{a}_c^*$ by $\alpha_r = (1 - (r/2 \| \alpha \|))\alpha$. This is simply translation of α by a distance of $r/2$ toward the origin.

PROPOSITION 5.2. *For each compact set U in $\mathfrak{a}_c^*$, disjoint from the origin, there exists $0 < c < \frac{1}{2}$ such that*

$$B_{cr}(\alpha_r) \subset \mathscr{B}_r(\alpha)$$

for all $\alpha \in U$ and $r \leqslant R_0 = \inf_{\alpha \in U} \| \alpha \|$.

Proof. Since α_r is a convex combination of α and zero when $r \leqslant R_0$, and since α is a boundary point of the convex set $\mathscr{F}^{\epsilon(\alpha)}$ and zero is an interior point, then α_r is also an interior point. Hence, since α_r is trivially in $B_r(\alpha)$, for each $\alpha \in U$ and $r \leqslant R_0$ there exists a constant $c(r, \alpha)$ such that $B_{cr}(\alpha_r) \subset \mathscr{B}_r(\alpha)$. The nontrivial part of the proposition is that c can be taken independently of both α and r for $a \in U$ and $r \leqslant R_0$.

Let $S(r, \alpha) = \sup\{s \mid B_s(\alpha_r) \subset \mathscr{B}_r(\alpha)\}$. Then $S(r, \alpha) > 0$ for each $\alpha \in U$ and $0 < r \leqslant R_0$ from the above. We thus have only to show the existence of $0 < c < 1$ such that $c < S(r, \alpha)/r$ for all $\alpha \in U$ and $0 < r \leqslant R_0$.

(i) First we claim that $S(r, \alpha)/r$ is nondecreasing for each fixed α as r goes down to zero. For suppose $0 < r_1 < r$. Then

$$S(r_1, \alpha)/r_1 = \sup_s \{s/r_1 \mid B_s(\alpha_{r_1}) \subset \mathscr{B}_{r_1}(\alpha)\}.$$

We alter the coordinate system on $\mathfrak{a}_c^*$ by shifting α to zero. Then $\operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}$ becomes some open convex set C with zero on the boundary, and we have

$$S(r_1, \alpha)/r_1 = \sup_s \{s/r_1 \mid B_s(\alpha_{r_1} - \alpha) \subset B_{r_1}(0) \cap C\}.$$

But $(\alpha_{r_1} - \alpha) = (r_1/r)(\alpha_r - \alpha)$ and $B_s(t\alpha) = tB_{s/t}(\alpha)$ so that

$$\begin{aligned} S(r_1, \alpha)/r_1 &= \sup_s\{s/r_1 \mid (r_1/r)\, B_{rs/r_1}(\alpha_r - \alpha) \subset ((r_1/r)\, B_r(0)) \cap C\} \\ &= \sup_s\{s/r_1 \mid B_{rs/r_1}(\alpha_r - \alpha) \subset B_r(0) \cap (r/r_1)C\} \\ &= \sup_s\{s/r \mid B_s(\alpha_r - \alpha) \subset B_r(0) \cap (r/r_1)C\}. \end{aligned}$$

But since $(r/r_1) > 1$ and C is an open convex set with zero on the boundary we get that $C \subset (r/r_1)C$, hence

$$\begin{aligned} S(r_1, \alpha)/r_1 &\geqslant \sup_s\{s/r \mid B_s(\alpha_r - \alpha) \subset B_r(0) \cap C\} \\ &= S(r, \alpha)/r. \end{aligned}$$

This of course proves (i).

(ii) Our second claim is that $\alpha \to S(r, \alpha)/r$ is continuous from U to $\mathbb{R}$ for each fixed $0 < r \leqslant R_0$. For suppose $\alpha, \beta \in U$ such that $\| \alpha - \beta \| \leqslant \delta$ for $\delta > 0$. Take any s such that $B_s(\alpha_r) \subset \mathscr{B}_r(\alpha)$ and note that $B_{s-\delta}(\alpha_r) \subset B_r(\beta) \cap \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}$ for δ small. Now $\| \alpha - \beta \| < \delta \Rightarrow \| \alpha_r - \beta_r \| < \delta$ when $r \leqslant R_0$ by simple verification, and hence

$$B_{s-2\delta}(\beta_r) \subset B_r(\beta) \cap \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)} \tag{5.7}$$

for δ small. Define

$$\begin{aligned} \Delta &= \sup_\lambda\{d(\lambda, \operatorname{Int} \mathscr{F}^{\epsilon(\beta)}) \mid \lambda \in \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}\} \\ &\quad + \sup_\lambda\{d(\lambda, \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}) \mid \lambda \in \operatorname{Int} \mathscr{F}^{\epsilon(\beta)}\}, \end{aligned}$$

where $d(\lambda, E)$ is the distance of λ from the set E. (Notice that one of the two terms defining Δ always has to be zero.) Then from (5.7) we find that

$$B_{s-2\delta-\Delta}(\beta_r) \subset B_r(\beta) \cap \operatorname{Int} \mathscr{F}^{\epsilon(\beta)} = \mathscr{B}_r(\beta)$$

for small δ and Δ. Since everything done was symmetric with respect to α and β, we find that

$$| S(r, \alpha) - S(r, \beta)| \leqslant 2\delta + \Delta$$

for $\alpha, \beta \in U$ and $\| \alpha - \beta \| \leqslant \delta$, δ small. Our desired continuity will be established once we show $\Delta \to 0$ as $\delta \to 0$ for each fixed α.

Without loss of generality suppose $\epsilon(\alpha) > \epsilon(\beta)$. If $\lambda \in \operatorname{Int} \mathscr{F}^{\epsilon(\alpha)}$ then

$$d(\lambda, \operatorname{Int} \mathscr{F}^{\epsilon(\beta)}) = \inf_{\lambda'}\{\| \lambda - \lambda' \| \mid \lambda' \in \operatorname{Int} \mathscr{F}^{\epsilon(\beta)}\}$$

$$= \inf_{\eta'}\{\| \eta - \eta' \| \mid \eta' \in \epsilon(\beta) \operatorname{Int} C_\rho\}$$

where $\lambda = \xi + i\eta$, $\eta \in \epsilon(\alpha) \operatorname{Int} C_\rho$. But C_ρ is norm bounded in $\mathfrak{a}^*$, say by M_2. Hence $\| \eta - (\epsilon(\beta)/\epsilon(\alpha))\eta \| \leqslant | 1 - (\epsilon(\beta)/\epsilon(\alpha))| M_2$, and this gives $\Delta \leqslant | 1 - (\epsilon(\beta)/\epsilon(\alpha))| M_2 \to 0$ as $\delta \to 0$ since $\eta' = (\epsilon(\beta)/\epsilon(\alpha))\eta \in \epsilon(\beta) \operatorname{Int} C_\rho$ and $\alpha \to \epsilon(\alpha)$ is continuous. This proves (ii).

(iii) We claim (i) and (ii) prove the proposition, since $0 < r \leqslant R_0$ and $\alpha \in U$ imply

$$S(r, \alpha)/r \geqslant S(R_0, \alpha)/R_0 \qquad \text{from (i)}$$

$$\geqslant \inf_{\alpha \in U} S(R_0, \alpha)/R_0 > 0 \qquad \text{from (ii).} \ \blacksquare$$

LEMMA 5.3. *Suppose T, μ, p_0, ϵ_0, and α_0 as in Lemma 5.1, and let Λ be lebesgue measure on $\mathfrak{a}_c^*$. Then there exists a compact neighborhood U_1 of α_0 and $R_1 > 0$ such that $\mu(B_r(\alpha)) \leqslant \Lambda(B_r(\alpha))r$ for all $r \leqslant R_1$ and $\alpha \in U_1$.*

Proof. Let $l = \dim \mathfrak{a}^*$ and apply Lemma 5.1 with $m = l + 1$. Then there exists a compact neighborhood U of α_0, $R > 0$ and $M < \infty$ such that

$$\int_{\mathscr{B}_r(\alpha)} \| \lambda - \alpha \|^{-2l-2} \, d\mu(\lambda) \leqslant M, \qquad \alpha \in U, \quad r \leqslant R.$$

Hence

$$\mu(\mathscr{B}_r(\alpha)) \leqslant M r^{2l+2}, \qquad \alpha \in U, \quad r \leqslant R. \tag{5.8}$$

But from Proposition 5.2 we find there exists $0 < c < 1$ such that

$$B_r(\alpha_{(r/c)}) \subset \mathscr{B}_{(r/c)}(\alpha), \qquad \alpha \in U, 0 < r \text{ small.} \tag{5.9}$$

Hence (5.8) and (5.9) combined yield

$$\mu(B_r(\alpha_{(r/c)})) \leqslant (M/c^{2l+2})\, r^{2l+2}, \qquad \alpha \in U, \quad 0 < r \text{ small.}$$

Since $\Lambda(B_r(\alpha))$ is proportional to r^{2l}, then there exists $M_0 > 0$ such that $\mu(B_r(\alpha_{(r/c)})) \leqslant M_0 \Lambda(B_r(\alpha_{(r/c)}))\, r^2$, or

$$\mu(B_r(\alpha_{(r/c)})) \leqslant \Lambda(B_r(\alpha_{(r/c)}))r, \qquad \alpha \in U, \quad 0 < r \text{ small.} \tag{5.10}$$

To show (5.10) implies the lemma we have only to show that there exists a compact subneighborhood U_1 of α_0 and some sufficiently small R_1 such that for each $\alpha \in U_1$ and $r \leqslant R_1$ we can find a $\beta \in U$ for which $\alpha = \beta_{(r/c)}$. To do so simply define $U_1 = \mathrm{Cl}(B_{R_2}(\alpha_0))$ for some $R_2 > 0$ such that $B_{2R_2}(\alpha_0) \subset U$. Then for each $\alpha \in U_1$ and $r > 0$ the equation $\alpha = \beta_{(r/c)}$ determines β to be $(1 + (r/(2c \| \alpha \|)))\alpha$ by simple verification. Again, simple verification yields that we have only to restrict r so that $r \leqslant 2cR_2$ to have $\beta \in U$. The desired R_1 thus equals $2cR_2$. ▮

Remark. Our extension theorem will now result from the above lemma and the following general covering theorem, a variant of material found in [5, Theorems 2.8.4 and 2.8.7].

PROPOSITION 5.4. *Suppose C is a compact set in $\mathbb{R}^n$, Λ the lebesgue measure on $\mathbb{R}^n$, and μ a Borel measure on $\mathbb{R}^n$ with the property that for every $\epsilon > 0$ there exists $r_\epsilon > 0$ such that $\mu(B_r(x)) \leqslant \epsilon\Lambda(B_r(x))$ for all $x \in C$ and all $r \leqslant r_\epsilon$. Then $\mu = 0$ on C.*

Proof. If $S = B_r(x)$ then define $\hat{S} = B_{3r}(x)$ and for each $\epsilon > 0$ define $\mathscr{S}_\epsilon = \{B_r(x) \mid x \in C \text{ and } r \leqslant r_\epsilon/3\}$. We claim that $\mathscr{S}_\epsilon$ has a disjoint subfamily $\mathscr{G}_\epsilon$ with the property that for each $T \in \mathscr{S}_\epsilon$ there exists $S \in \mathscr{G}_\epsilon$ such that $T \cap S \neq \varnothing$ and $T \subset \hat{S}$. To prove this let Ω_ϵ be the collection of disjoint subfamilies $\mathscr{H}$ of $\mathscr{S}_\epsilon$ such that for each $T \in \mathscr{S}_\epsilon$ either

(i) for all $S \in \mathscr{H}$, $T \cap S = \varnothing$, or

(ii) for some $S \in \mathscr{H}$, $T \cap S \neq \varnothing$ and $T \subset \hat{S}$.

Notice that (i) implies Ω_ϵ is nonempty since $\{\varnothing\} \in \Omega_\epsilon$. Partially order Ω_ϵ by inclusion; then every chain in Ω_ϵ has an upper bound which is also in Ω_ϵ. Hence Zorn's lemma gives the existence of a maximal subfamily $\mathscr{G}_\epsilon$ in Ω_ϵ, and to show $\mathscr{G}_\epsilon$ is the collection desired we have only to show that each $T \in \mathscr{S}_\epsilon$ satisfies (ii) relative to $\mathscr{G}_\epsilon$, not (i). Hence we must show $\mathscr{K} = \varnothing$ where

$$\mathscr{K} = \{T \in \mathscr{S}_\epsilon \mid T \cap S = \varnothing \text{ for all } S \in \mathscr{G}_\epsilon\}.$$

Suppose $\mathscr{K} \neq \varnothing$. Then there exists $W \in \mathscr{K}$ such that $2\Lambda(W) \geqslant \sup_{T \in \mathscr{K}} \Lambda(T)$ since this supremum must be finite. But we then claim that $\mathscr{G}_\epsilon \cup \{W\} \in \Omega_\epsilon$. For take any $T \in \mathscr{S}_\epsilon$. If (ii) holds for T relative to $\mathscr{G}_\epsilon$ it also clearly holds for T relative to $\mathscr{G}_\epsilon \cup \{W\}$. Thus suppose

(i) holds for T relative to $\mathscr{G}_\epsilon$, i.e., $T \in \mathscr{K}$. Then $2\Lambda(W) \geqslant \Lambda(T)$ by definition of W, which implies

$$2 \text{ radius}(W) \geqslant \text{radius}(T). \tag{5.11}$$

There are only two cases to consider here: $T \cap W$ empty or nonempty. If $T \cap W$ empty, then (i) holds for T relative to $\mathscr{G}_\epsilon \cup \{W\}$. If $T \cap W$ nonempty, then $T \subset \hat{W}$ from (5.11) and hence (ii) holds for T relative to $\mathscr{G}_\epsilon \cup \{W\}$. We have thus verified that $\mathscr{G}_\epsilon \cup \{W\} \in \Omega_\epsilon$, which is a contradiction to the maximality of $\mathscr{G}_\epsilon$. Hence $\mathscr{K} = \varnothing$ and our claim is proved.

We use the subfamily $\mathscr{G}_\epsilon$ to show that $\mu \equiv 0$ on C. Let $C_0 = \{x \in \mathbb{R}^n \mid d(x, C) \leqslant r_\epsilon \text{ where } \epsilon = 1\}$. We then see that

$$C \subset \bigcup_{T \in \mathscr{S}_\epsilon} T \subset \bigcup_{S \in \mathscr{G}_\epsilon} \hat{S} \subset C_0 \text{ for } \epsilon < 1,$$

where without loss of generality we assume $r_\epsilon \leqslant r_1$ when $\epsilon \leqslant 1$. Hence $\sum_{S \in \mathscr{G}_\epsilon} \Lambda(S) \leqslant \Lambda(C_0) < \infty$, and therefore $\mathscr{G}_\epsilon$ is countable since $\Lambda(S) > 0$ for all $S \in \mathscr{G}_\epsilon$. Thus for each $\epsilon > 0$ we have

$$\mu(C) \leqslant \mu\Big(\bigcup_{S \in \mathscr{G}_\epsilon} \hat{S}\Big) \leqslant \sum_{S \in \mathscr{G}_\epsilon} \mu(\hat{S}) \leqslant \sum_{S \in \mathscr{G}_\epsilon} \epsilon \Lambda(\hat{S})$$

$$= 3^n \epsilon \sum_{S \in \mathscr{G}_\epsilon} \Lambda(S) \leqslant 3^n \epsilon \Lambda(C_0).$$

Hence $\mu(C) = 0$ since $\Lambda(C_0) < \infty$. ∎

THEOREM 5.5 (The Extension Theorem). *Suppose T is a positive definite distribution with spherical Bochner measure μ. Then $T \in (\mathscr{I}^p(G))'$ if and only if* $\operatorname{supp} \mu \subset \mathscr{F}^\epsilon$, *where* $1 \leqslant p \leqslant 2$ *and* $\epsilon = (2/p) - 1$. *In such a case* $T[\varphi] = \int_{\mathscr{P}} \hat{\varphi}\, d\mu$ *for all* $\varphi \in \mathscr{I}^p(G)$.

Proof. Suppose $\operatorname{supp} \mu \subset \mathscr{F}^\epsilon$ for some $0 \leqslant \epsilon \leqslant 1$. Then we easily see that the linear functional $\hat{T}$ on $\mathscr{Z}(\mathscr{F}^\epsilon)$ defined by $\hat{T}[\Phi] = \int_{\mathscr{P}} \Phi\, d\mu$ is continuous (same procedure as in the proof of Theorem 4.5 for the $\epsilon = 1$ case). Hence we can extend T to $\mathscr{I}^p(G)$ by $T[\varphi] = \hat{T}[\hat{\varphi}] = \int_{\mathscr{P}} \hat{\varphi}\, d\mu$.

Suppose $T \in (\mathscr{I}^p(G))'$ for some $1 \leqslant p \leqslant 2$. Then by Lemma 5.3 and Proposition 5.4 we have that $\operatorname{supp} \mu \subset \mathscr{F}^\epsilon$. ∎

Remarks. When considered for $p = 2$, Theorem 5.5 becomes simply the Bochner theorem for K-biinvariant positive definite tempered distributions (i.e., distributions which lie in $(\mathscr{C}^2(G))'$), which was first proved by Muta [14] in much the same fashion as the euclidean Bochner theorem is proved in Schwartz [15, p. 275]. It is of interest to note that Muta's definition of a tempered K-biinvariant positive definite distribution differs from ours, while the spherical Bochner theorem indirectly proves them equal. Muta defines a tempered K-biinvariant distribution T to be positive definite if $T[\varphi * \varphi^*] \geqslant 0$ for all $\varphi \in \mathscr{I}_c(G)$, which on the surface is a less restrictive definition than ours. We prove the equivalence directly in the following manner. Suppose T is positive definite in Muta's sense, and take $\varphi \in \mathscr{D}(G)$. Then $(\varphi * \varphi^*)^\wedge(\lambda) \geqslant 0$ for all $\lambda \in \mathscr{P}$ since $\varphi_{-\lambda} \gg 0$ for all $\lambda \in \mathscr{P}$. But from the proof of [15, Theorem XVIII, p. 276] there exists a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset \mathscr{S}(\mathfrak{a}^*) = \bar{\mathscr{Z}}(\mathscr{F}^0)$ such that $|\Phi_n|^2 \to (\varphi * \varphi^*)^\wedge$ in $\mathscr{S}(\mathfrak{a}^*)$. Noting Schwartz's construction we see that these Φ_n may be taken as W-invariant, and thus $|\Phi_n|^2 \to (\varphi * \varphi^*)^\wedge$ in $\bar{\mathscr{Z}}(\mathscr{F}^0)$. But there exists $\varphi_n \in \mathscr{I}^2(G)$ such that $\hat{\varphi}_n = \Phi_n$, and hence $\varphi_n * \varphi_n^* \to (\varphi * \varphi^*)^\natural$ in $\mathscr{I}^2(G)$. Therefore $T[\varphi_n * \varphi_n^*] \to T[\varphi * \varphi^*]$ in $\mathbb{C}$, and it is easy to see that $T[\varphi * \varphi^*] \geqslant 0$ for all $\varphi \in \mathscr{I}^2(G)$. Hence T is positive definite in our sense.

It is by no means obvious that this equivalence of definitions holds for nontempered K-biinvariant positive definite distributions. This would seem to depend on the truth of the following

Conjecture. Given $\Psi \in \bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$, $\Psi \geqslant 0$ on $\mathscr{P} \cap \mathscr{F}^\epsilon$, then there exists a sequence Φ_n in $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$ such that, with $\Psi_n(\lambda) = \Phi_n(\lambda)\, \bar{\Phi}_n(\bar{\lambda})$ for each $\lambda \in \mathscr{F}^\epsilon$, we have $\Psi_n \to \Psi$ in $\bar{\mathscr{Z}}(\mathscr{F}^\epsilon)$. This is certainly a questionable statement except for the case $\epsilon = 0$.

COROLLARY 5.6. *Consider $T \in \mathscr{D}'(G)$, $T \gg 0$, K-biinvariant. Then if the spherical Bochner measure μ is supported in $\mathfrak{a}^* \cup C$, where C is a compact subset of $\mathfrak{a}_c^*$, then $T = S + f$, where S is a tempered K-biinvariant positive definite distribution and f is a continuous K-biinvariant positive definite function. In particular, this is always true in the real rank one case.*

Proof. Let $\mu_1(E) = \mu(E \cap \mathfrak{a}^*)$ and $\mu_2(E) = \mu(E \cap (\mathfrak{a}_c^* - \mathfrak{a}^*))$ for all Borel sets E in $\mathfrak{a}_c^*$. Then by Theorem 5.5 we have that μ_1 gives rise to a tempered K-biinvariant positive definite distribution S. By the spherical Bochner theorem, we have that μ_2 must be finite since it is supported in the compact set C. Hence μ_2 gives rise to a continuous K-biinvariant function f and clearly $T = S + f$.

Suppose that G is of real rank one. Then $\mathscr{P}$ is contained in $\mathfrak{a}^* \cup (i\mathfrak{a}^*)$ since λ and $\bar{\lambda}$ are W-conjugate for all $\lambda \in \mathscr{P}$, and $W = \{s_1, s_2\}$ where

$s_1\lambda = \lambda$, $s_2\lambda = -\lambda$ for all $\lambda \in \mathfrak{a}^*$. Then the Helgason–Johnson theorem gives that $\mathscr{P} \subset \mathfrak{a}^* \cup iC_\rho$, which is the desired form. ∎

6. Technical Results on the $\mathscr{Z}(\mathscr{F}^\epsilon)$ Spaces

Most of the previous results have relied on the existence of certain special elements in the $\mathscr{Z}(\mathscr{F}^\epsilon)$ spaces. In this section we construct these various functions.

Proposition 6.1. *For each $\epsilon > 0$ there exists a nonconstant W-invariant holomorphic polynomial on $\mathfrak{a}_c^*$ which is uniformly bounded away from zero on $\mathscr{F}^\epsilon$. For P any such polynomial and $\Phi \in \mathscr{Z}(\mathscr{F}^\epsilon)$, then $(\Phi/P) \in \mathscr{Z}(\mathscr{F}^\epsilon)$. Moreover, there exists such a polynomial P which is positive on $\mathscr{P}$.*

Proof. Let $\epsilon_1, \ldots, \epsilon_l$ be any basis of $\mathfrak{a}^*$. Then for $\lambda \in \mathfrak{a}_c^*$ let $\lambda = \sum_{j=1}^{l} \lambda_j \epsilon_j$ ($\lambda_j \in \mathbb{C}$), $\lambda = \xi + i\eta$ ($\xi, \eta \in \mathfrak{a}^*$), and for each $c > 0$ let $P_c(\lambda) = c + \sum_{j=1}^{l} \lambda_j^2$. We then have

$$P_c(\lambda) = c + \sum \xi_j^2 - \sum \eta_j^2 + 2i \sum \xi_j \eta_j .$$

But, as a function on $\mathscr{F}^\epsilon$, $|\sum \eta_j^2|$ is uniformly bounded by some finite constant, say c_0. Taking $c = 2c_0$ we obtain $|P_c(\lambda)| \geqslant |2c_0 + \sum \xi_j^2 - \sum \eta_j^2| \geqslant c_0 > 0$ for all $\lambda \in \mathscr{F}^\epsilon$. Then $P(\lambda) = \prod_{s \in W} P_c(\lambda^s)$ satisfies the desired conditions since $\mathscr{F}^\epsilon$ is W-invariant.

Suppose $\Phi \in \mathscr{Z}(\mathscr{F}^\epsilon)$. Then $\Psi = \Phi/P$ on $\operatorname{Int} \mathscr{F}^\epsilon$ is clearly well defined, holomorphic and W-invariant. To prove $\Psi \in \mathscr{Z}(\mathscr{F}^\epsilon)$ there remains only to show that for each holomorphic differential operator D with polynomial coefficients we have

$$\sup_{\operatorname{Int}\mathscr{F}^\epsilon} |D\Psi| < \infty. \tag{6.1}$$

Suppose D is of order zero. Then D equals a holomorphic polynomial Q on $\mathfrak{a}_c^*$ so that

$$\sup_{\operatorname{Int}\mathscr{F}^\epsilon} |D\Psi| \leqslant (1/c) \sup_{\operatorname{Int}\mathscr{F}^\epsilon} |Q\Phi|,$$

where $|P| \geqslant c$ on $\mathscr{F}^\epsilon$. Thus (6.1) holds for such D since $\Phi \in \mathscr{Z}(\mathscr{F}^\epsilon)$.

Suppose (6.1) holds for all D of order $\leqslant k - 1$, all $\Phi \in \mathscr{Z}(\mathscr{F}^\epsilon)$ and all P as specified in the proposition. If D is of order k we may assume without loss of generality that $D = E(\partial/\partial\lambda_1)$, where E is of order $k - 1$. Hence

$$D\Psi = E[(\partial\Phi/\partial\lambda_1)/P] - E[(\partial P/\partial\lambda_1)\Phi/P^2].$$

But $\partial\Phi/\partial\lambda_1$ and $(\partial P/\partial\lambda_1)\Phi$ are $\mathscr{Z}(\mathscr{F}^\epsilon)$, and both P and P^2 satisfy the induction hypothesis. Hence (6.1) holds for D of order k, and hence for all D by induction.

The last statement of the proposition follows by taking $Q(\lambda) = P(\lambda)\,\bar{P}(\bar{\lambda})$, since $\lambda \in \mathscr{P}$ implies λ and $\bar{\lambda}$ are Weyl group conjugate. ▮

Remark. The next two propositions comprise a refinement of Proposition 6.1 which was used in proving the extension theorem.

PROPOSITION 6.2. *For each $\alpha \in \mathfrak{a}_c^*$ there exists a W-invariant holomorphic polynomial P_α on $\mathfrak{a}_c^*$ such that degree P_α is uniformly bounded in α, and*

(i) $P_\alpha(\alpha) = 0$ *for all* $\alpha \in \mathfrak{a}_c^*$;

(ii) $P_{\alpha^n} \to P_\alpha$ *uniformly on compacts as* $\alpha^n \to \alpha$ *in* $\mathfrak{a}_c^*$; *and*

(iii) *Given* U *a compact set in* $\mathfrak{a}_c^*$ *and* $\epsilon > 0$ *such that* $\mathscr{F}^\epsilon \cap U = \varnothing$, *then there exists a* $c > 0$ *such that* $|P_\alpha(\lambda)| \geqslant c$ *for all* $\alpha \in U$ *and* $\lambda \in \mathscr{F}^\epsilon$.

Proof. We let L be any hyperplane in $\mathfrak{a}^*$ which lies on a face of C_ρ, and take a basis $\epsilon_1, ..., \epsilon_l$ of $\mathfrak{a}^*$ such that $\epsilon_1 \in L$ and $\epsilon_2, ..., \epsilon_l$ span a hyperplane parallel to L. This is possible since zero is an interior point of the convex set C_ρ. Then coordinatizing $\mathfrak{a}^*$ by $\eta = \sum \eta_j \epsilon_j$ gives that L is the solution set to the equation $\eta_1 = 1$. Notice that the set of solutions to $\eta_1 = -1$ must also lie on a face of C_ρ since $C_\rho = -C_\rho$, and therefore C_ρ lies between the hyperplanes determined by $\eta_1^2 = 1$.

Parametrize $\mathfrak{a}_c^*$ by $\lambda = \sum \lambda_j \epsilon_j$, $\lambda_j = \xi_j + \eta_j$ and $\alpha = \sum \alpha_j \epsilon_j$, $\alpha_j = \beta_j + i\gamma_j$ for all $\lambda, \alpha \in \mathfrak{a}_c^*$. With $\epsilon(\alpha)$ the unique ϵ such that $\alpha \in \text{Bdry}\, \mathscr{F}^\epsilon$, let Q_α be the holomorphic polynomial in λ given by

$$Q_\alpha(\lambda) = \epsilon(\alpha)^2 + (\lambda_1 - \beta_1)^2.$$

Suppose U is a compact set in $\mathfrak{a}_c^*$ and $\epsilon > 0$ such that $\mathscr{F}^\epsilon \cap U = \varnothing$. Then $\alpha \in U$ implies $\epsilon(\alpha) > \epsilon$ (since $\mathscr{F}^\epsilon = \bigcap_{\epsilon<\epsilon'} \mathscr{F}^{\epsilon'}$) and taking $\lambda \in \mathscr{F}^\epsilon$ arbitrary we obtain

$$|Q_\alpha(\lambda)| = |\epsilon(\alpha)^2 + (\xi_1 - \beta_1)^2 - \eta_1^2 + 2i(\xi_1 - \beta_1)\,\eta_1| \geqslant \epsilon(\alpha)^2 - \eta_1^2.$$

But $\lambda \in \mathscr{F}^\epsilon$ implies $\eta \in \epsilon C_\rho$, which in turn gives $|\eta_1| \leqslant \epsilon$. Hence, letting $c = \inf_{\alpha\in U} \epsilon(\alpha)^2 - \epsilon^2$ we have $|Q_\alpha(\lambda)| \geqslant c$ for all $\alpha \in U$ and $\lambda \in \mathscr{F}^\epsilon$, where $c > 0$ since U is compact. Thus Q_α satisfies (iii). It follows trivially that Q_α also satisfies (ii), at least pointwise, since $\alpha \to \epsilon(\alpha)$ and $\alpha \to \beta_1$ are continuous. The uniform convergence on compacts results because degree $Q_\alpha = 2$ for all $\alpha \in \mathfrak{a}_c^*$.

Let $\{L_1,\ldots,L_t\}$ be all hyperplanes of $\mathfrak{a}^*$ which lie on faces of C_ρ. Then for each L_j we have a set of polynomials $\{P_\alpha^j\}_{\alpha\in\mathfrak{a}_c^*}$ as defined above. For each $\alpha\in\mathfrak{a}_c^*$ we let $R_\alpha=\prod_{j=1}^t P_\alpha^j$. Then (ii) and (iii) still hold for R_α, but we claim (i) also holds from the following. By definition of $\epsilon(\alpha)$ we have $\alpha\in\text{Bdry}\,\mathscr{F}^{\epsilon(\alpha)}$. Thus, with $\alpha=\beta+i\gamma$, we have that γ lies on at least one face $\epsilon(\alpha)L_j$ of $\epsilon(\alpha)\,C_\rho$. Hence, in the coordinate system associated with L_j, we have $\gamma_1=\epsilon(\alpha)$, and if P_α^j is the polynomial associated with L_j we have $P_\alpha^j(\alpha)=\epsilon(\alpha)^2+(\beta_1+i\gamma_1-\beta_1)^2=0$. Hence $R_\alpha(\alpha)=0$.

Finally, since $\mathscr{F}^\epsilon$ is W-invariant we have that $P_\alpha=\prod_{s\in W}R_\alpha^s$ satisfies all the desired conditions. ∎

PROPOSITION 6.3. *Suppose* $\{P_\alpha\mid\alpha\in\mathfrak{a}_c^*\}$ *as in Proposition* 6.2, *and fix both* $\epsilon>0$ *and* $\Phi\in\mathscr{Z}(\mathscr{F}^\epsilon)$. *Defining* $\Psi_\alpha=\Phi/P_\alpha$ *on* $\mathscr{F}^\epsilon$ *for each* $\alpha\in\mathfrak{a}_c^*-\mathscr{F}^\epsilon$, *then* $\alpha\to\Psi_\alpha$ *is continuous from* $\mathfrak{a}_c^*-\mathscr{F}^\epsilon$ *into* $\mathscr{Z}(\mathscr{F}^\epsilon)$.

Proof. First note that $\Psi_\alpha\in\mathscr{Z}(\mathscr{F}^\epsilon)$ for all $\alpha\notin\mathscr{F}^\epsilon$ from Proposition 6.1. Let $\alpha^n\to\alpha$ in $\mathfrak{a}_c^*-\mathscr{F}^\epsilon$. We have only to show that if D is a holomorphic differential operator on $\mathfrak{a}_c^*$ with polynomial coefficients, then $\sup_{\text{Int}\mathscr{F}^\epsilon}|D(\Psi_\alpha-\Psi_{\alpha^n})|\to 0$. First notice that $\Psi_\alpha-\Psi_{\alpha^n}=((P_{\alpha^n}-P_\alpha)/P_\alpha P_{\alpha^n})\Phi$ is of the form $(Q_n/R_n)\Phi$, where Q_n and R_n are W-invariant holomorphic polynomials on $\mathfrak{a}_c^*$ such that (a) $|R_n|\geqslant c>0$ on $\mathscr{F}^\epsilon$ for all n, (b) Q_n and R_n converge uniformly on compact sets, $Q_n\to 0$ and $R_n\to R$, where R is some W-invariant holomorphic polynomial on $\mathfrak{a}_c^*$, and (c) the degrees of Q_n and R_n are uniformly bounded in n by some $N<\infty$. We thus set $\Psi_n=(Q_n/R_n)\Phi$ and show that $\sup_{\text{Int}\mathscr{F}^\epsilon}|D\Psi_n|\to 0$ by inducting on the order of D.

(i) Suppose order $D=0$. Then $D=P$ for some holomorphic polynomial P on $\mathfrak{a}_c^*$, and thus

$$\sup_{\text{Int}\mathscr{F}^\epsilon}|D\Psi_n|\leqslant c^{-1}\sup_{\text{Int}\mathscr{F}^\epsilon}|Q_nP\Phi|.$$

But we may express $Q_n(\lambda)$ as $\sum_{|I|\leqslant N}c_I^n\lambda^I$, where $I=(i_1,\ldots,i_l)$, $|I|=i_1+\cdots+i_l$, and $\lambda^I=\lambda_1^{i_1}\cdots\lambda_l^{i_l}$ for $\lambda_1,\ldots,\lambda_l$ some complex coordinate system on $\mathfrak{a}_c^*$. Moreover, for each fixed I, $c_I^n\to 0$ as $n\to\infty$. Thus

$$|D\Psi_n(\lambda)|\leqslant c^{-1}|Q_n(\lambda)\,P(\lambda)\,\Phi(\lambda)|$$

$$\leqslant c^{-1}\left(\sum_{|I|\leqslant N}(\sup_n|c_I^n|)\,|\lambda^I|\right)|P(\lambda)\,\Phi(\lambda)|\qquad\text{for }\ \lambda\in\mathscr{F}^\epsilon,$$

where $\sup_n |c_I^n| < \infty$ for each I. Take any $\delta > 0$. Since $\Phi \in \mathscr{Z}(\mathscr{F}^\epsilon)$ there exists a compact set $C \subset \mathscr{F}^\epsilon$ such that

$$\left(\sum_I (\sup_n |c_I^n|) |\lambda^I|\right) |P(\lambda)\, \Phi(\lambda)| \leqslant (c/2)\delta \qquad \text{for all} \quad \lambda \in \mathscr{F}^\epsilon - C.$$

But, on C, Q_n tends to zero uniformly so there exists n_0 such that $\sup_C |Q_n| < (c/2)\delta \sup_C |P\Phi|$ for all $n \geqslant n_0$. Hence $|D\Psi_n(\lambda)| < \delta$ for all $n \geqslant n_0$ and $\lambda \in \operatorname{Int} \mathscr{F}^\epsilon$, and therefore $D\Psi_n \to 0$ uniformly on $\operatorname{Int} \mathscr{F}^\epsilon$ when D is of order zero.

(ii) Suppose $D\Psi_n \to 0$ uniformly on $\operatorname{Int} \mathscr{F}^\epsilon$ for all D of order less than k and for all Ψ_n of the specified form.

(iii) Take D of order k. Using some complex coordinate system $\lambda_1, \ldots, \lambda_l$ on $\mathfrak{a}_c^*$ we may assume without loss of generality that $D = E(\partial/\partial\lambda_1)$, where E is a differential operator as in (ii). Then

$$\sup_{\operatorname{Int} \mathscr{F}^\epsilon} |D\Psi_n| \leqslant \sup_{\operatorname{Int} \mathscr{F}^\epsilon} |E\{(\partial Q_n/\partial\lambda_1)\Phi/R_n\} - E\{Q_n(\partial R_n/\partial\lambda_1)\Phi/R_n^2\} + E\{Q_n(\partial\Phi/\partial\lambda_1)/R_n\}|$$

and each of the terms satisfies the induction hypothesis. ▮

Remark. We end with two technical results which were used in Sections 4 and 5.

PROPOSITION 6.4. *Suppose $\{\zeta_j\}_{j=1}^\infty$ is a sequence of functions in $C_c(G)$ such that $\bigcup_j \operatorname{supp} \zeta_j$ is relatively compact and $\int_G |\zeta_j(g)|\, dg$ is uniformly bounded in j. Then for D any constant coefficient holomorphic differential operator on $\mathfrak{a}_c^*$, $\{D\hat{\zeta}_j\}_{j=1}^\infty$ is a uniformly bounded set of functions on $\mathscr{F}^\epsilon$ for each $\epsilon > 0$.*

Proof. Let $C = \operatorname{Cl}(\bigcup_j \operatorname{supp} \zeta_j)$. Then

$$D\zeta_j(\lambda) = \int_C \zeta_j(g)\, D_\lambda(\varphi_{-\lambda}(g))\, dg$$

since C is compact. Hence $|D\hat{\zeta}_j(\lambda)| \leqslant c_1 \sup_{g \in C} |D_\lambda(\varphi_{-\lambda}(g))|$ for all $\lambda \in \mathfrak{a}_c^*$ and $j \in \mathbb{Z}^+$, where $c_1 = \sup_j \int_G |\zeta_j(g)|\, dg$. We have only to show that $\sup_{g \in C} |D_\lambda(\varphi_{-\lambda}(g))|$ is uniformly bounded on $\mathscr{F}^\epsilon$ for each $\epsilon > 0$.

Take a basis $\epsilon_1, \ldots, \epsilon_l$ of $\mathfrak{a}^*$, and for $\lambda \in \mathfrak{a}_c^*$ put $\lambda = \sum_j \lambda_j \epsilon_j$ ($\lambda_j \in \mathbb{C}$) and $\lambda_j = \xi_j + i\eta_j$. Let $H_1, \ldots, H_l$ be a dual basis of $\mathfrak{a}$, so that

$\epsilon_i(H_j) = \delta_{ij}$. We write $H(g) = \sum_{j=1}^{l} c_j(g) H_j$, with each c_j a continuous function from G to $\mathbb{R}$. Then $\lambda(H(g)) = \sum_{j=1}^{l} \lambda_j c_j(g)$ so that $D_\lambda e^{-i\lambda(H(g))} = P_D(-ic(g)) e^{-i\lambda(H(g))}$ for P_D some polynomial in l-variables and $c(g) = (c_1(g),..., c_l(g))$. Thus

$$
\begin{aligned}
| D_\lambda(\varphi_{-\lambda}(g))| &= \left| D_\lambda \int_K e^{-(i\lambda+\rho)H(gk)}\, dk \right| \\
&\leqslant \sup_{k \in K} | P_D(-ic(gk))\, e^{-(i\lambda+\rho)H(gk)} | \\
&= \sup_{k \in K} | P_D(-ic(gk))\, e^{(\eta-\rho)H(gk)} |, \qquad (6.2)
\end{aligned}
$$

where $\lambda = \xi + i\eta$. Consider $g \in C$ and $\lambda \in \mathscr{F}^\epsilon$ for some fixed $\epsilon > 0$. Then $gk \in CK$ which is compact in G, and hence $\sup_{k \in K} | P_D(-ic(gk))|$ is uniformly bounded for $g \in C$. But $\lambda \in \mathscr{F}^\epsilon$ if and only if $\lambda = \xi + i\eta$, where $\eta \in \epsilon C_\rho$. Hence $\{\eta - \rho \mid \lambda \in \mathscr{F}^\epsilon\}$ is compact in $\mathfrak{a}^*$ which implies $\{(\eta - \rho) H(gk) \mid \lambda \in \mathscr{F}^\epsilon, g \in C, k \in K\}$ is bounded in $\mathbb{R}$. This proves from (6.2) that $| D_\lambda(\varphi_{-\lambda}(g))|$ is uniformly bounded for $\lambda \in \mathscr{F}^\epsilon$ and $g \in C$. ∎

PROPOSITION 6.5. *Suppose for each $\alpha \in \mathfrak{a}_c^*$ we have a W-invariant holomorphic polynomial P_α on $\mathfrak{a}_c^*$ with degree $\leqslant N < \infty$ for all α such that*

(i) *$P_\alpha(\alpha) = 0$ for all $\alpha \in \mathfrak{a}_c^*$; and*

(ii) *$P_{\alpha_n} \to P_\alpha$ uniformly on compact sets as $\alpha_n \to \alpha$ in $\mathfrak{a}_c^*$.*

Then the function F: $\mathfrak{a}_c^* \times \mathfrak{a}_c^* \to \mathbb{R}$ defined by $F(\alpha, \lambda) = | P_\alpha(\lambda)|/\| \lambda - \alpha \|$ if $\lambda \neq \alpha$, $F = 0$ otherwise, is bounded on compact subsets of $\mathfrak{a}_c^* \times \mathfrak{a}_c^*$.

Proof. Pick a basis $\epsilon_1,..., \epsilon_l$ of $\mathfrak{a}^*$ which is orthonormal with respect to $\langle\ ,\ \rangle$, and for $\lambda \in \mathfrak{a}_c^*$ let $\lambda = \sum \lambda_j \epsilon_j$ ($\lambda_j \in \mathbb{C}$). Then $| \lambda_j | \leqslant \| \lambda \|$ for each $j = 1,..., l$. Since $P_\alpha(\lambda) = \sum_{0<|I|\leqslant N} a_I(\alpha)(\lambda - \alpha)^I$ (notation as in the proof of Proposition 6.3), where a_I: $\mathfrak{a}_c^* \to \mathbb{C}$ is continuous for each I, we then easily see that $P_\alpha(\lambda)/\| \lambda - \alpha \|$ is bounded on compact subsets of $\mathfrak{a}_c^* \times \mathfrak{a}_c^*$. ∎

Remark. The referee has suggested that it may be possible to find analogs of the results of this paper for functions on the homogeneous space G/K which transform according to a fixed representation of K on the left. It is hoped that the formulation and proof of such results would follow in the same manner as for the K-biinvariant functions on G previously given.

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