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Mathematical Notions of Resilience: The Effects of DisturbanceI in One-Dimensional Nonlinear Systems

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MATHEMATICAL NOTIONS OF RESILIENCE: THE EFFECTS OF DISTURBANCE IN ONE-DIMENSIONAL NONLINEAR SYSTEMS

AN HONORS PAPER FOR THE DEPARTMENT OF MATHEMATICS

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1. Introduction

The goal of this project is to develop ways of understanding and thinking about nonlinear systems of differential equations under of some disturbance. Specifically, I am interested in quantifying the resilience of systems to disturbance. Resilience is qualitatively defined as the capacity of a system to absorb disturbances and maintain the same structure and function [4] [3]. I would like to start with the very basic, and rigorously build from the ground up.

Differential equations describe laws of change. It is often exceedingly difficult to construct closed form functions that give the state of even simple systems as a function of time. It is usually easier to establish the laws of change which govern the physical universe. That's what a differential equation does: it asks for the state of something, and tells you how the state will change at that instant in time. Differential equations can be used to model virtually every physical process from population change, to neuron behavior, to planetary motion, to climate change and weather prediction. Unfortunately, it is impossible to solve most differential equations in a closed form function. Mathematicians and scientists use different methods to understand the processes described by differential equations, including numerical methods of evaluation and mathematical analysis of the behavior of certain types of differential equations, sometimes applied only to specific regions of the domain or on certain small time scales.

Unfortunately, it is impossible to create a perfect model. Many applications modeled by differential equations involve numerous laws of change and many parameters. Population prediction for just one species, for example, must take into account all kinds of factors such as competing species, predation, nutrient availability, and the list goes on. Weather prediction is another example. Despite the assistance of supercomputers and endless quantities of data, the best weather predictions are only accurate up to about a week in the future. Due to the massive number of parameters and the complexity of the universe, models cannot be one hundred percent accurate. Sometimes refining the models which do the predicting with more parameters and more precise laws of change fails to increase the accuracy of the model and ends up only making it even more difficult to find the source of the model error.

Sometimes slight model error isn't that big a deal - so what if it's a degree warmer than predicted in Brunswick today? If slight errors in calculation always produced only slight errors in prediction, model error wouldn't be such a problem. Bigger issues arise in systems which contain multiple stable states with hysterical changes between basins of attraction. This happens in all kinds of systems. For example a small change in an initial condition or a parameter can mean the difference between a species maintaining balance in an ecosystem and that species becoming rapidly extinct. Another example is the very simple Earth energy balance model discussed by Kaper and Engler in their recent book *Mathematics and Climate* [2]. The differential equation for the temperature of the Earth is quite simple:

$$\frac{dT}{dt} = (1 - \alpha(T))\frac{S_0}{4} - \epsilon\sigma T^4$$

where T is the global mean surface temperature, α is the average albedo of the planet, S_0 is insolation from the sun, ϵ represents the greenhouse effect, and σ is the Stefan-Boltzmann constant for black body radiation. With just those simple parameters, Kaper and Engler construct a surprisingly accurate differential equation which has two stable states - one with earth as it is now, and one called "snowball earth," where earth is completely frozen over. It turns out, as Kaper and Engler discuss, that slight changes in the parameters in the energy balance model can produce a bifurcation where the two basins of attraction merge into one, and the current stable state of the earth is catastrophically lost. Because of the threat of bifurcations, model accuracy in many applications is important.

Since adding processes and parameters sometimes only complicates model prediction, rather than improve accuracy, we turn to other methods for understanding models for which we cannot calculate every single parameter and differential equation. One option is to introduce some sort of disturbance to the model and see how it affects predictions. Many physical processes are well represented this way. Take sea level change, for example. We can create a model which predicts the global sea level based on all kinds of slow moving parameters such as temperature, time of year, wind currents, etc. But if our goal were to build a levy to keep the sea from flooding a city, we would need to also consider the changes due to waves or storms. Waves or storms, in this case, could be modeled as a disturbance: an instantaneous change in the state variable which is not reflected in the underlying differential equations. It would be impossible to predict each wave as it came to shore, so instead we can just assume that waves of some given height will occur fairly regularly over time. Perhaps the system has two stable states: one where the city is dry, the other where the waves have crested the levies and the city is under water. The change from one state to the other could be a hysterical bifurcation, and if the builders of the levies only considered global mean sea levels and didn't account for storms and waves, the latter state would be much more likely. Instead, when developing models for the sea levels in the area, the builders should consider the predictions for global sea levels, but also take into account some regular disturbance.

In models with multiple stable states, we are often concerned with the resilience of a given state to changes in the model. Those changes might be parameter changes, or they could come in the form of disturbances. One might ask the question "how resilient is this state to disturbance?" In this project we develop ways to formulate, analyze, and answer that question. Disturbances, of course, can come in many shapes and temporal patterns. We could consider one big disturbance at a certain time, and check how the system has changed. We could consider some distribution of disturbances of varying size. Because I am interested in developing a rigorous understanding of disturbances, I consider a very simple and specific type of disturbance in this project. Disturbances will be manifest as a series of instantaneous kicks, or changes in the state space, all of equal size and evenly spaced in time. These disturbances will be applied to one-dimensional, autonomous differential equations in an effort to better understand the resilience of the system's basins of attraction to disturbances. In so doing, we will develop the notion of a kick-flow system.

2. Kick-flow systems

For this project, we will consider one dimensional, autonomous differential equations of the form

$$\frac{dx}{dt} = \dot{x} = f(x), \text{ where } f : \mathbb{R} \to \mathbb{R} \text{ is } C^1. \tag{1}$$

In addition to the differential equation, the system will be subjected to regular "kicks". A kick is an instantaneous positive change in position x without regard to the underlying differential equation. We define kick-flow systems to follow a kick-then-flow rule; first a kick occurs, then the systems flows undisturbed, according to the underlying differential equation for a given time step. This process is iterated to produce a discrete time dynamical system from the original continuous time system subjected to discrete kicks. So at time $t = 0$, the system is immediately kicked, thus affording an instantaneous change in position which takes place over zero time. The system then flows from this position, following the dynamics of the differential equation until the next kick is applied, in the form of another instantaneous change in position, and the process repeats.

We make these ideas precise with the following definitions:

Definition 2.1. *A smooth dynamical system on $\mathbb{R}$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $\phi(t, x) = \phi_t(x)$ satisfies*

1. $\phi_0(x) = x, \ \forall x \in \mathbb{R}$ *and*
2. $\phi_t \circ \phi_s = \phi_{t+s}, \ \forall s, t \in \mathbb{R}$

For our purposes, $\phi(t, x)$ will be associated with a differential equation of the form (1) by the following equation:

$$\frac{d}{dt}\bigg|_{t=0} \phi_t(x) = f(x)$$

By the Fundamental Theorem of Calculus we have

$$\phi(t, x) = x + \int_0^t f(x(t))dt \tag{2}$$

So for fixed t, ϕ_t is the time-t map associated to the flow of $\dot{x} = f(x)$. Alternately, one could write $x(t) = x_0 + \int_0^t f(x(t))dt$, where $x(0) = x_0$. Following from this relationship we can describe a dynamical system in terms of trajectories. Given an initial condition $x(0) = x_0$, we have $\phi_{x_0}(t) = x(t)$ where $\phi_{x_0}(t)$ is the particular trajectory passing through $x = x_0$ at time $t = 0$.

Definition 2.2. *Disturbances (K, T) are defined by a kick size K and time interval T.*

T is the time during which the system flows between kicks. Note that in the definition of a kick-flow system below, the kick-flow system is associated with both a differential equation and a disturbance, and the disturbances (K, T) are held fixed; the kick size and flow time interval do not change.

Definition 2.3. *The kick-flow map $\psi : \mathbb{R} \to \mathbb{R}$ for the differential equation (1) with flow ϕ, subjected to disturbance (K, T), is the continuously differentiable function $\psi(x) = \phi(T, x + K)$. The kick-flow system is the discrete dynamical system given by iterating the kick-flow map. The trajectory $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ with initial condition x_0 is given by $x_n = \psi(x_{n-1}) = \psi^n(x_0)$.*

Here, ψ^n represents composition (not multiplication). So $\psi^n(x)$ returns the position of x after n kicks of size K and n flows ϕ over time T spliced together according to the kick-flow fashion. One way to think about the trajectory under ψ is as a series of snapshots of the kick-flow system, each snapshot occurring after a kick and flow, right before the next kick-flow iteration.

Example

Consider the differential equation $\dot{x} = a$ with disturbance $(1, 4)$, representing kicks of size 1 applied every 4 time units. Let's start with initial condition $x_0 = 2$ and find the trajectory for the first few iterations of the kick-flow map. By definition 2.3,

$$x_1 = \psi^1(2) = \phi(4, 2 + 1) = \phi(4, 3)$$

So we need to flow ϕ forward from position $x = 3$ for time $t = 4$. Usually, this would be a difficult or even impossible step to calculate analytically, but since our differential equation $\dot{x} = a$ is so simple, we can actually evaluate ϕ. Recall that $\phi(t, x) = x + \int_0^t f(x)\, dt$. So in this example,

$$x_1 = \phi(4, 3) = 3 + \int_0^4 a\, dt = 3 + 4a$$

By the same procedure

$$x_2 = \psi^2(2) = \phi(4, 3 + 4a + 1) = 4 + 8a$$

and

$$x_n = \psi^n(2) = \phi(4, x_{n-1} + 1) = x_{n-1} + 1 + 4a$$

In this example, if a is positive, then the kick and the flow work 'together' to move points in the positive direction, and the trajectory heads to infinity. If a is negative, then the kick and the flow work in opposite directions, raising the question of which is stronger. If $a = -1/4$ then the kick and the flow exactly balance, and all points are fixed under the kick-flow system: $x_n = x_{n-1} = \ldots x_2 = x_1 = x_0$.

Note that if the kick size K is zero, then the position $x(nT)$ on trajectories under the dynamical system ϕ after time nT has passed is equal to the position $\psi^n(x)$ of trajectories under the kick-flow system ψ after n iterations. So for $\phi(t, x)$ and $\psi^n(x)$, both associated with the same differential equation $\dot{x} = f(x)$ and ψ associated with disturbance $(0, T)$ the following equality holds:

$$\phi(nT, x) = \psi^n(x) \tag{3}$$

Equation (3) follows directly from the definitions of ϕ and ψ.

Choosing whether to kick first or flow first is arbitrary. We can also define a "flow-kick" map associated with the same differential equation and disturbance as the kick-flow system.

Definition 2.4. *The flow-kick map $\sigma : \mathbb{R} \to \mathbb{R}$ for the differential equation (1) with flow ϕ, subjected to disturbance (K, T), is the continuously differentiable function $\sigma(x) = \phi(T, x) + K$. The flow-kick system is the discrete dynamical system given by iterating the flow-kick map. The trajectory $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ with initial condition x_0 is given by $x_n = \sigma(x_{n-1}) = \sigma^n(x_0)$*

In general (unless we are working with a very boring differential equation), $\psi(x) \neq \sigma(x)$. However, we can construct a relationship between the two maps and one between trajectories.

Consider the kick-flow system ψ and flow-kick system σ associated with the differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T). Then the maps ψ and σ are related by

$$\sigma(x + K) = \phi(T, x + K) + K = \psi(x) + K \tag{4}$$

By carefully choosing the initial conditions, we can also construct a relationship between the trajectories of ψ and σ. If the initial condition for σ is one kick size K from the initial condition of ψ, then the value of σ^n will be equal to the value ψ^n plus K. That is to say,

$$\sigma^n(x_0 + K) = \psi^n(x_0) + K \tag{5}$$

where σ^n and ψ^n are flow-kick and kick-flow trajectories respectively.

Kick-flow systems can have fixed points, which are analogous to equilibrium points in dynamical systems. A fixed point of a kick-flow system ψ is a position x_0 such that the trajectory $\psi^n(x_0)$ under ψ is equal to x_0 for all iterations n.

Definition 2.5. *A fixed point of a discrete function $g : \mathbb{R} \to \mathbb{R}$ is point x_0 such that $g^n(x_0) = x_0$ for all iterations n.*

For a point $x = A$ to be a fixed point of a kick-flow system ψ, the trajectory of the flow ϕ from the point one kick size K away from A must reach A at flow time T. So at fixed points, the kick and flow are balanced with each other. This is embodied in the following lemma.

Lemma 2.6. *Given a kick-flow system ψ associated with the differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), then A is a fixed point of ψ iff*

$$\phi(T, A + K) = A$$

where $\phi(t, x)$ is the dynamical system associated with $\dot{x}$.

Proof. First, we show that if $\phi(T, A + K) = A$ then A is a fixed point.

We want to show

$$\phi(T, A + K) = A \implies \psi^n(A) = A \ \forall n$$

We use induction. The base case $\psi^0(A) = A$ is given by Definition (2.3). The inductive step

$$\psi^n(A) = A \implies \psi^{n+1}(A) = A$$

also follows from Definition (2.3)

$$\psi^{n+1}(A) = \phi(T, \psi^n(A) + K) = \phi(T, A + K)$$

where, by assumption,

$$\phi(T, A + K) = A$$

Second, we show that if A is a fixed point, then $\phi(T, A + K) = A$.

We want to show

$$\psi^n(A) = A \ \forall n \implies \phi(T, A + K) = A$$

By Definition (2.3)

$$\phi(T, \psi^n(A) + K) = \psi^{n+1}(A)$$

But since A is a fixed point, $\psi^{n+1}(A) = \psi^n(A) = A$, and hence

$$\phi(T, A + K) = A$$

□

Flow-kick systems can also have fixed points. In this paper, we will mostly discuss and prove our results for kick-flow systems, though parallels in flow-kick systems will also be mentioned and used. The following lemma for a flow-kick system is analogous to the above lemma for kick-flow systems.

Lemma 2.7. *Given a flow-kick system σ associated with the differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), then B is a fixed point of σ iff*

$$\phi(T, B) + K = B$$

where $\phi(t, x)$ is the dynamical system associated with $\dot{x}$.

The proof of Lemma (2.7) is almost identical to the proof of Lemma (2.6).

Now that we have some notation for fixed points, we can begin to explore the relationship between kick-flow systems ψ and flow-kick systems σ. According to our notion of kick-flow systems and informed by Lemma (2.6), when A is a fixed point of ψ, the system kicks a distance K from A and then flows back a distance of $-K$ in T time. The following lemma shows how every fixed point A of a kick-flow ψ system is associated with a fixed point B of the associated flow-kick system σ.

Lemma 2.8. *Given a kick-flow system ψ and a flow-kick system σ associated with the differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), then A is a fixed point of ψ iff $B = A + K$ is a fixed point of σ.*

Proof. Let $B = A + K$.

First, we show that if A is a fixed point of ψ, then B is a fixed point of σ.

Invoking lemmas (2.6) and (2.7) we want to show that

$$\phi(T, A + K) = A \implies \phi(T, B) + K = B$$

Substituting for A in the LHS we obtain

$$\phi(T, B - K + K) = \phi(T, B) = B - K$$

so

$$\phi(T, B) + K = B$$

which is what we wanted.

Second, we show that if B is a fixed point of σ, then A is a fixed point of ψ.

We want

$$\phi(T, B) + K = B \implies \phi(T, A + K) = A$$

Substituting for B on the LHS we obtain

$$\phi(T, A + K) + K = A + K$$

so

$$\phi(T, A + K) = A$$

which is again what we wanted. □

Lemma (2.7) shows that A can be a fixed point of a kick-flow system associated with a certain differential equation subjected to disturbance if and only if $B = A + K$ is a fixed point for the flow-kick system associated with the same differential equation and disturbance. So fixed points come in pairs separated by the kick size. Corollary (2.9) below shows what this implies for trajectories.

Corollary 2.9. *If A is a fixed point of a kick-flow system ψ and $B = A + K$ is a fixed point of a flow-kick system σ where ψ and σ are associated with the same differential equation subjected to disturbance, then we have*

$$\psi(A) + K = A + K = B = \sigma(B)$$

Note that Corollary (2.9) implies that as the kick K grows small, the fixed point A of the kick-flow system approaches the fixed point B of the flow-kick system. We could write

$$\lim_{K \to 0} (A - B) = 0$$

Furthermore, as K approaches 0 the dynamics of the kick-flow system itself increasingly resemble the dynamics of the flow-kick system. In fact, by definitions (2.3) and (2.4) when $K = 0$

$$\psi(x) = \phi(T, x) = \sigma(x)$$

where ψ and σ are the kick-flow and flow-kick systems respectively, associated with the same differential equation and disturbance.

Like equilibrium points of continuous differential equations, fixed points can be sinks, sources, or a mix of the two. Sinks are fixed points to which nearby trajectories converge as n increases. Sources behave like kick-flow sinks, only for decreasing n values. Since this project is primarily concerned with understanding the resilience of basins of attraction, we will not dwell on sources. Some fixed points behave like sinks when approached from the left, and sources when approached from the right, or vice versa. Note that sinks and sources are sometimes referred to as stable fixed points and unstable fixed points respectively.

Definition 2.10. *A fixed point x_0 of a function f is a <u>left hand sink</u> iff there exists $\epsilon > 0$ such that $\forall x \in (x_0 - \epsilon, x_0)$, $\lim_{n\to\infty} f^n(x) = x_0$.*

So a fixed point x_0 is a left hand sink if and only if there is a neighborhood to the left of x_0 within which initial conditions produce kick-flow trajectories ψ which converge to x_0 as n increases.

Definition 2.11. *A fixed point x_0 is a <u>right hand sink</u> iff there exists $\epsilon > 0$ such that $\forall x \in (x_0, x_0 + \epsilon)$, $\lim_{n\to\infty} f^n(x) = x_0$*

The only difference between left hand and right hand sinks is that left hand sinks require a neighborhood to the left of x_0 to contain initial conditions whose trajectories converge to x_0 while for right hand sinks, that neighborhood must be to the right of x_0. Next we define a sink, which is simply a fixed point which is both a left hand and a right hand sink.

Definition 2.12. *A fixed point x_0 is a <u>sink</u> iff there exists $\epsilon > 0$ such that $\forall x \in (x_0 - \epsilon, x_0 + \epsilon)$, $\lim_{n\to\infty} \psi^n(x) = x_0$*

Now that we have developed some language and notation for studying differential equations subjected to regular disturbance, we focus on the special case of linear differential equations in the next section, and use the insights we gain there to generalize to nonlinear differential equations in Section 4.

3. The linear case

In an effort to understand kick-flow systems from a ground-up perspective, we begin by examining a simple, linear system. We begin by proving the existence and uniqueness of fixed points in kick-flow systems associated with linear differential equations subjected to disturbance.

Theorem 3.1. *The kick-flow system ψ associated with a linear differential equation of the form $\dot{x} = -mx$ subjected to disturbance (K, T) has a unique fixed point given by $A = \frac{K}{e^{mT}-1}$.*

Proof. By Lemma (2.6), a point x_0 is a fixed point of ψ iff

$$\phi(T, x_0 + K) = x_0$$

where $\phi(t, x)$ is the dynamical system associated with $\dot{x}$. Note $\phi(t, x)$ is given by

$$\phi(t, x) = xe^{-mt}$$

Evaluating the dynamical system and solving for x_0

$$\begin{aligned} \phi(T, x_0 + K) &= (x_0 + K)e^{-mT} \\ x_0 &= \frac{K}{e^{mT}-1} \end{aligned}$$

Thus

$$A = \frac{K}{e^{mT} - 1} \tag{6}$$

is the only fixed point in kick-flow systems associated with differential equation (8). □

Remark. We could just as easily have proved Theorem (3.1) for a fixed point B of a flow-kick system, only we would have found that

$$B = \frac{K}{e^{mT} - 1} + K \tag{7}$$

which is consistent with Corollary (2.9).

Since the motivation for this project is to understand the resilience of stable systems to disturbance, we will primarily focus on kick-flow systems which have sinks, not sources. In that vein, we limit our discussion of the linear case to kick-flow systems associated with a differential equation of the form

$$\dot{x} = -mx \quad \text{where} \quad m > 0 \tag{8}$$

We will not spend much time on differential equations of this form with $m \leq 0$ since the fixed point in those systems is a source. We next prove that the fixed point in the kick-flow systems associated with differential equation (8) subjected to disturbance (K, T) is a sink.

Theorem 3.2. *If $m > 0$, then the fixed point A of the kick-flow system ψ associated with a differential equation of the form $\dot{x} = -mx$ subjected to disturbance (K, T) is a sink.*

Proof. The dynamical system ϕ associated with $\dot{x} = -mx$ is given by

$$\phi(t, x) = xe^{-mt}$$

Consider the trajectory of the associated kick-flow system, $\psi^n(x)$, with initial condition x_0, defined in Definition (2.3) as

$$\begin{aligned}\psi^0(x_0) &= x_0\\ \psi^{n+1}(x_0) &= \phi(T, \psi^n(x_0)+K)\\ &= (\psi^n(x_0)+K)e^{-mT}\end{aligned}$$

We wish to show that $A = \frac{K}{e^{mT}-1}$ is a sink for $\psi^n(x)$. In fact, we will show that all trajectories converge to A, independent of the choice of initial condition x_0. So by Definition (2.12) we want to show

$$\lim_{n\to\infty} \psi^n(x_0) = A = \frac{K}{e^{mT}-1}$$

Let's examine the sequence $\{x_0, x_1, x_2, \dots, x_n, \dots\}$ defined by the trajectory of $\psi^n(x_0)$.

$$\begin{aligned}\psi^0(x_0) &= x_0\\ \psi^1(x_0) &= (x_0+K)e^{-mT}\\ \psi^2(x_0) &= ((x_0+K)e^{-mT}+K)e^{-mT}\\ \psi^3(x_0) &= (((x_0+K)e^{-mT}+K)e^{-mT}+K)e^{-mT}\end{aligned}$$

Distributing the terms in the above equations we obtain

$$\begin{aligned}\psi^0(x_0) &= x_0\\ \psi^1(x_0) &= x_0e^{-mT}+Ke^{-mT}\\ \psi^2(x_0) &= x_0e^{-2mT}+Ke^{-2mT}+Ke^{-mT}\\ \psi^3(x_0) &= x_0e^{-3mT}+Ke^{-3mT}+Ke^{-2mT}+Ke^{-mT}\end{aligned}$$

Thus we see that the general terms for ψ^n and ψ^{n+1} are given by

$$\psi^n(x_0) = (x_0+K)e^{-nmT}+Ke^{-(n-1)mT}+Ke^{-(n-2)mT}+...+Ke^{-mT} \tag{9}$$

and

$$\psi^{n+1}(x_0) = (x_0+K)e^{-(n+1)mT}+Ke^{-nmT}+Ke^{-(n-1)mT}+...+Ke^{-mT} \tag{10}$$

We are interested in the difference between $\psi^n(x_0)$ and $\psi^{n+1}(x_0)$. Subtracting equation (9) from equation (10)

$$\psi^{n+1}(x_0)-\psi^n(x_0) = (x_0+K)e^{-(n+1)mT}-x_0e^{-nmT} \tag{11}$$

we find that only the first term of equation (10) minus a term from (9) remains. Note that since m and T are both positive, $e^{-(n+1)mT}$ is always less then one and grows small for large values of n. The same is true of $-x_0e^{-nmT}$. In fact, if we take the limit as n goes to infinity we find that the difference between $\psi^n(x_0)$ and $\psi^{n+1}(x_0)$ given in equation (11) approaches zero.

$$\lim_{n\to\infty}\psi^{n+1}(x_0)-\psi^n(x_0) = \lim_{n\to\infty}(x_0+K)e^{-(n+1)mT}-x_0e^{-nmT} = 0 \tag{12}$$

What equation (12) shows is that as n increases to infinity, the difference between $\psi^n(x_0)$ and $\psi^{n-1}(x_0)$ becomes infinitesimally small. We can use this result to find $\lim_{n\to\infty} \psi^n(x_0)$.
Recall

$$\psi^{n+1}(x_0) = (\psi^n(x_0) + K)e^{-mT}$$

Rearranging we obtain

$$\psi^{n+1}(x_0)e^{mT} - \psi^n(x_0) = K$$

Factoring $\psi^n(x_0)$ out

$$\psi^n(x_0)(\frac{\psi^{n+1}(x_0)}{\psi^n(x_0)}e^{mT} - 1) = K$$

But by equations (9), (10), and (12)

$$\lim_{n\to\infty} \frac{\psi^{n+1}(x_0)}{\psi^n(x_0)} = 1$$

So we have

$$\lim_{n\to\infty} \psi^n(x_0)(e^{mT} - 1) = K$$

And, as desired

$$\lim_{n\to\infty} \psi^n(x_0) = \frac{K}{e^{mT} - 1}$$

Thus $A = \frac{K}{e^{mT}-1}$ is a kick-flow sink for ψ. □

Remark. As with Theorem (3.1), we could have used a fixed point B of the flow-kick system σ instead. We would have found that σ always has a sink given by

$$B = \frac{K}{e^{mT} - 1} + K$$

This result is also consistent with Lemma (2.8).

Theorems (3.1) and (3.2) show that the kick-flow system ψ associated with differential equation (8) subjected to disturbance (K, T) always has a sink A which we can find using K, T, and m.

We have two more concepts to define. The first is disturbance rate.

Definition 3.3. *The disturbance rate C for a kick-flow system $\psi(x)$ (or flow-kick system $\sigma(x)$) with disturbance (K, T) is given by $C = K/T$.*

The disturbance rate represents the average rate of disturbance accumulation over time. It is found by dividing the kick size by the time interval. The cumulative disturbance - the sum of all the kicks in some amount of time - can be found by multiplying the disturbance rate by the run time (defined below). Cumulative disturbance is interesting because there are many applications for which we have a good estimate of the cumulative disturbance over a certain amount of time. Take rainfall, for example. We have plenty of data indicating that annual rainfall in some areas is very steady, yet we are still bad at predicting when the rain will come. Developing an understanding of the effects of the disturbance rate might allow us to understand the effects of that rainfall without necessitating improved accuracy of climate models. Notice that the disturbance rate stays the same as long as the ratio between kick size and flow time remains constant. That should make sense since if we cut the kick size in half and apply it twice as often, the cumulative disturbance over time should remain the same.

Definition 3.4. *The run time $R(n)$ for a kick-flow system $\psi^n(x)$ (or flow-kick system $\sigma^n(x)$) iterated n times is given by $R(n) = nT$, where T is the flow time interval.*

The run time represents the amount of time that passes while a kick-flow system flows from x_0 to $\psi^n(x_0)$. Since zero time passes while the kick is applied, the run time is just the sum of n time intervals of length T. Run time can be used to compare the state of a kick-flow system with that of its associated dynamical system, and is useful when comparing the disturbance rate of kick-flow systems associated with the same differential equation but different disturbances. We now look at what happens to kick-flow systems when the disturbance rate is held constant.

Lemma 3.5. *If the disturbance rate C is held constant, then*

1. *A is monotone decreasing in K and T.*
2. *B is monotone increasing in K and T.*

where A is the fixed point of a kick-flow system ψ and B is the fixed point of a flow-kick system σ and ψ and σ are associated with differential equation (8) subjected to disturbance (K, T).

Proof. By Theorem (3.1) A is given by

$$A = \frac{K}{e^{mT} - 1}$$

By Definition (3.3)

$$C = \frac{K}{T}$$

Thus $K = TC$. Substituting TC for K gives

$$A = C\frac{T}{e^{mT} - 1}$$

Taking the derivative of A with respect to T

$$\frac{dA}{dT} = \frac{C}{(e^{mT} - 1)^2}(e^{mT} - mTe^{mT} - 1)$$

We proceed by showing that $\frac{dA}{dT}$ is always negative. Note that

$$\frac{C}{(e^{mT} - 1)^2} > 0$$

for $C > 0$, so we can disregard this term. Consider the remaining term

$$g(T) = e^{mT} - mTe^{mT} - 1$$

Note that $g(0) = 0$, and

$$g'(T) = -m^2Te^{mT}$$

Since T and K are always positive, $g'(T)$ is always negative, so $g(T)$ is decreasing for positive T. Thus, since $g(0) = 0$, $g(T) < 0$ for all $T > 0$,

$$\frac{dA}{dT} < 0$$

for all $T > 0$ as required, and A is monotone decreasing.

We now proceed to show that B is monotone decreasing in K and T. By Theorem (3.1) and Corollary (2.9) B is given by

$$B = \frac{K}{e^{mT} - 1} + K$$

Substituting TC for K and taking the derivative of B with respect to T we obtain

$$\begin{aligned} B &= \tfrac{TC}{e^{mT}-1} + TC \\ \tfrac{dB}{dT} &= \tfrac{Ce^{mT}}{(e^{mT}-1)^2}(e^mT - mT - 1) \end{aligned}$$

We proceed by showing that $\frac{dB}{dT}$ is always positive. Note that

$$\frac{Ce^{mT}}{(e^{mT}-1)^2} > 0$$

for $C > 0$, so we can disregard this term. Consider the remaining term

$$h(T) = e^{mT} - mT - 1$$

As before, $h(0) = 0$, and

$$h'(T) = m(e^{mT} - 1)$$

Since m is positive, $h'(T)$ is always positive, so $h(T)$ is increasing for positive T. Thus, since $h(0) = 0$, $h(T) > 0$ for all $T > 0$,

$$\frac{dB}{dT} > 0$$

for all $T > 0$ as required, and B is monotone increasing. □

Lemma (3.5) shows that given a kick-flow system associated with differential equation (8) subjected to disturbance (K, T) and a flow-kick system associated with the same differential equation and disturbance, the fixed point A of the kick-flow system and the fixed point B of the flow-kick system not only move closer to each other as K and T simultaneously decrease while maintaining a fixed disturbance rate C, but they do so in such a way that for each successively smaller K and T values, the new fixed points A and B are nested between the previous fixed points. As K and T converge to zero, the nested intervals [A,B] converge to a single point, which we can find as follows:

Lemma 3.6. *If the disturbance rate C is held constant, then*

$$\lim_{K,T\to 0} A = \frac{C}{m}$$

where A is the fixed point of the kick-flow system ψ associated with a linear differential equation subjected to disturbance (K, T).

Proof. Consider the kick-flow system ψ associated with a linear differential equation given by

$$\dot{x} = -mx$$

By Theorem (3.1) ψ has a fixed point A is given by

$$A = \frac{K}{e^{mT} - 1}$$

By Definition (3.3)

$$C = \frac{K}{T}$$

Thus $K = TC$. Substituting TC for K we have

$$\lim_{T\to 0} A = \lim_{T\to 0} \frac{TC}{e^{mT} - 1}$$

Both the numerator and the denominator head towards infinity as T goes to zero so we can employ L'Hospital's Rule and take the T derivative of the numerator and the denominator

$$\lim_{T\to 0} \frac{TC}{e^{mT}-1} = \lim_{T\to 0} \frac{C}{me^{mT}}$$

Evaluating the limit and substituting $\frac{K}{T}$ back in for C we find

$$\lim_{T\to 0} \frac{C}{me^{mT}} = \frac{C}{m}$$

as desired. □

Remark. Since the proof of Lemma (3.6) relies only on Theorem (3.1) and does not depend on the $m > 0$ constraint, Lemma (3.6) holds for any linear differential equation.

When we shrink the kick size and time interval down infinitesimally small yet maintain a constant disturbance rate C, we are really just spreading the disturbance evenly throughout the kick-flow system. So it would make sense that in so doing we are actually reverting to a continuous dynamical system defined by some differential equation. We can in fact define a dynamical system η without any disturbance whose behavior over time is the same as that of our kick-flow system with infinitesimally small K and T. The differential equation for η is given by

$$\dot{x} = -m(x - \frac{C}{m}) = -mx + C \tag{13}$$

Differential equation (13) has an equilibrium point given by $x = \frac{C}{m}$. Consider the kick-flow system ψ associated with differential equation (8) subjected to disturbance (K, T). By Lemma (3.6), when we hold disturbance rate constant and take the limit as K and T approach zero, the fixed point is equal to $\frac{C}{m}$. It comes as no surprise to find that this fixed point is also the position of the equilibrium point in differential equation (13). But we can make an even stronger statement. Not only does differential equation (13) have an equilibrium point equal to A but it turns out that the smooth dynamical η associated with differential equation (13) has the same dynamics as ψ. We show this in Theorem (3.9), which makes use of a few lemmas.

Lemma 3.7. *Given the kick-flow system ψ associated with differential equation (8) subjected to disturbance (K, T), and the dynamical system η associated with the differential equation $\dot{x} = -mx + C$,*

$$\psi^n(x_0) \leq \eta_{x_0}(R(n))$$

where $R(n)$ is the run time and C is the disturbance rate for ψ and n is an integer.

Proof. The dynamical system ϕ associated with ψ is given by

$$\phi(t, x) = xe^{-mt}$$

and so by Definition (2.3), $\psi^n(x_0)$ is iteratively defined as

$$\psi^n(x_0) = (\psi^{n-1}(x_0) + K)e^{-mT} \tag{14}$$

where

$$\psi^0(x_0) = x_0$$

The dynamical system η is given by

$$\eta(t, x) = (x - \frac{C}{m})e^{-mt} + \frac{C}{m}$$

Using Definition (3.4) for $R(n)$

$$\eta_{x_0}(R(n)) = (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}$$

So we want to show

$$\psi^n(x_0) \leq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}$$

We use induction. Plugging $n = 0$ into the RHS yields x_0, which proves the base case since by Definition (2.3)

$$\psi^0(x_0) = x_0$$

To prove the inductive step we must show that if

$$\psi^n(x_0) \leq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \tag{15}$$

then

$$\psi^{n+1}(x_0) \leq (x_0 - \frac{C}{m})e^{-m(n+1)T} + \frac{C}{m}$$

Let

$$h(mT) = e^{mT} - mT - 1$$

Note $h(0) = 0$, and

$$h'(mT) = e^{mT} - 1$$

Since m and T are positive, $h'(T)$ is always positive, so $h(T)$ is increasing for positive T. Thus, since $h(0) = 0$, $h(T) > 0$ for all $T > 0$ and using a bit of algebra

$$\begin{aligned} 0 &\leq e^{mT} - mT - 1 \\ 1 &\leq e^{mT} - mT \\ \tfrac{1}{mT} &\leq \tfrac{1}{mT}e^{mT} - 1 \\ \tfrac{K}{mT} &\leq \tfrac{K}{mT}e^{mT} - K \\ \tfrac{C}{m} &\leq \tfrac{C}{m}e^{mT} - K \end{aligned}$$

and finally

$$(x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \leq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}e^{mT} - K \tag{16}$$

Note that the LHS of equation (16) is equal to the RHS of inequality (15) So we can write

$$\psi^n(x_0) \leq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \leq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}e^{mT} - K$$

and so employing some more algebra

$$\begin{aligned} \psi^n(x_0) &\leq (x_0 - \tfrac{C}{m})e^{-mnT} + \tfrac{C}{m}e^{mT} - K \\ \psi^n(x_0) + K &\leq (x_0 - \tfrac{C}{m})e^{-mnT} + \tfrac{C}{m}e^{mT} \\ (\psi^n(x_0) + K)e^{-mT} &\leq (x_0 - \tfrac{C}{m})e^{-m(n+1)T} + \tfrac{C}{m} \end{aligned}$$

But from equation (14)

$$\psi^{n+1}(x_0) = (\psi^n(x_0) + K)e^{-mT}$$

And so

$$\psi^{n+1}(x_0) \le (x_0 - \frac{C}{m})e^{-m(n+1)T} + \frac{C}{m}$$

satisfying the inductive step. And so

$$\psi^n(x_0) \le \eta_{x_0}(R(n)),\ \forall n$$

as desired. $\square$

Lemma (3.7) shows that the dynamical system η associated with differential equation (13) is bounded below by the the kick-flow system associated with differential equation (8) subjected to disturbance (K, T) for all time and for any kick size or flow time. Lemma (3.8) below creates an upper bound for η.

Lemma 3.8. *Given the flow-kick system σ associated with differential equation (8) subjected to disturbance (K, T), and the dynamical system η associated with the differential equation $\dot{x} = -mx + C$,*

$$\sigma^n(x_0 + K) \ge \eta_{x_0}(R(n))$$

where $R(n)$ is the run time and C is the disturbance rate for σ and n is an integer.

Proof. The dynamical system ϕ associated with σ is given by

$$\phi(t, x) = xe^{-mt}$$

and so by Definition (2.4), $\sigma^n(x_0 + K)$ is iteratively defined as

$$\sigma^n(x_0 + K) = \sigma^{n-1}(x_0 + K)e^{-mT} + K \tag{17}$$

where

$$\sigma^0(x_0 + K) = x_0 + K$$

As before, the dynamical system η is given by

$$\eta(t, x) = (x - \frac{C}{m})e^{-mt} + \frac{C}{m}$$

Using Definition (3.4) for $R(n)$

$$\eta_{x_0}(R(n)) = (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}$$

So we want to show

$$\sigma^n(x_0 + k) \ge (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}$$

We use induction. Plugging $n = 0$ into the RHS yields x_0, which proves the base case since by Definition (2.4)

$$\sigma^0(x_0 + K) = x_0 + K$$

To prove the inductive step we must show that if

$$\sigma^n(x_0 + K) \ge (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \tag{18}$$

then

$$\sigma^{n+1}(x_0 + K) \ge (x_0 - \frac{C}{m})e^{-m(n+1)T} + \frac{C}{m}$$

Let

$$h(mT) = e^{mT} + mT - 1$$

Note

$$h'(mT) = e^{mT} + 1$$

Since m and T are positive, $h'(T)$ is always positive, so $h(T)$ is increasing for positive T. Thus, since $h(0) = 0$, $h(T) > 0$ for all $T > 0$ and using a bit of algebra

$$\begin{aligned}
e^{mT} + mT - 1 &\geq 0 \\
e^{mT} &\geq 1 - mT \\
1 &\geq e^{-mT} - mTe^{-mT} \\
\tfrac{1}{mT} &\geq \tfrac{1}{mT}e^{-mT} - e^{-mT} \\
\tfrac{K}{mT} &\geq \tfrac{K}{mT}e^{-mT} - Ke^{-mT} \\
\tfrac{C}{m} &\geq \tfrac{C}{m}e^{-mT} - Ke^{-mT}
\end{aligned}$$

And finally

$$(x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \geq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}e^{-mT} - Ke^{-mT} \tag{19}$$

Note that the LHS of equation (19) is equal to the RHS of inequality (18) So we can write

$$\sigma^n(x_0 + K) \geq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m} \geq (x_0 - \frac{C}{m})e^{-mnT} + \frac{C}{m}e^{-mT} - Ke^{-mT}$$

and so employing some more algebra

$$\begin{aligned}
\sigma^n(x_0 + K) &\geq (x_0 - \tfrac{C}{m})e^{-mnT} + \tfrac{C}{m}e^{-mT} - Ke^{-mT} \\
\sigma^n(x_0 + K)e^{-mT} &\geq (x_0 - \tfrac{C}{m})e^{-m(n+1)T} + \tfrac{C}{m} - K \\
\sigma^n(x_0 + K)e^{-mT} + K &\geq (x_0 - \tfrac{C}{m})e^{-m(n+1)T} + \tfrac{C}{m}
\end{aligned}$$

But from equation (17)

$$\sigma^n(x_0 + K) = \sigma^{n-1}(x_0 + K)e^{-mT} + K$$

And so

$$\sigma^{n+1}(x_0 + K) \geq (x_0 - \frac{C}{m})e^{-m(n+1)T} + \frac{C}{m}$$

satisfying the inductive step. And so

$$\sigma_n(x_0 + K) \geq \eta_{x_0}(R(n))$$

as desired. □

Lemmas (3.7) and (3.8) create bounds for the smooth dynamical system η associated with differential equation (13). The lower bound is the kick-flow system ψ associated with differential equation (8) subjected to disturbance (K, T) and the upper bound is the flow-kick system σ associated with the same differential equation and disturbance. But the reason for lemmas (3.7) and (3.8) is not to learn about the differential equation (13), which is not a kick-flow system and whose solution we can calculate via separation of variables. What we are now equipped to show is that differential equation (13) gives the dynamics of ψ and σ when the cumulative disturbance is held constant but the kick size and flow time approach zero.

Theorem 3.9. *Given a kick-flow system ψ and a flow-kick system σ associated with a linear differential equation of the form $\dot{x} = -mx$ where $m > 0$ subjected to disturbance (K, T), if the cumulative disturbance C is held constant, then*

$$\lim_{K,T\to 0} \psi^n(x_0) = \eta_{x_0}(R(n)) = \lim_{K,T\to 0} \sigma^n(x_0 + K)$$

where η is the dynamical system associated with $\dot{x} = -mx + C$, $R(n)$ is the run time, and n is an integer.

Proof. We know from lemmas (3.7) and (3.8) that

$$\psi^n(x_0) \leq \eta_{x_0}(R(n)) \leq \sigma^n(x_0 + K)$$

For any values of $K, T, m >$. We also know from equation (5) that

$$\psi^n(x_0) + K = \sigma^n(x_0 + K)$$

Thus

$$\lim_{K,T\to 0} \psi^n(x_0) + K = \lim_{K,T\to 0} \sigma^n(x_0 + K)$$

Since K goes to zero, the LHS can be written as

$$\lim_{K,T\to 0} \psi^n(x_0) + K = \lim_{K,T\to 0} \psi^n(x_0)$$

Therefore

$$\lim_{K,T\to 0} \psi^n(x_0) = \lim_{K,T\to 0} \sigma^n(x_0 + K)$$

And since $\eta_{x_0}(R(n))$ can be neither less than $\psi^n(x_0)$ nor greater than $\sigma^n(x_0 + K)$, we must have

$$\lim_{K,T\to 0} \psi^n(x_0) = \eta_{x_0}(R(n)) = \lim_{K,T\to 0} \sigma^n(x_0 + K)$$

as desired. □

The significance of Theorem (3.9) in terms of resilience is that for a kick-flow system ψ associated with a linear differential equation subjected to disturbance, we can construct a nonlinear, but solvable dynamical system η which shares the same dynamics as ψ. In kick-flow systems, resilience is effected by both the flow and the kick. The addition of the kick means that if the systems grows close to the edge of a basin of attraction, it might "kick" out of the basin before reaching the brink. η provides a sort of lower bound for the resilience of a kick-flow system given a steady disturbance rate, because η represents the effects of the disturbances spread evenly over time. As such, if η loses resilience, ψ surely will as well. As we continue into the nonlinear section, we construct upper bounds for the resilience of kick-flow systems, which may sometimes be more difficult to calculate, but provide more useful information.

4. The nonlinear case

Unlike with linear differential equations, we don't have ready access to the solutions for nonlinear differential equations. Because of this, our study of kick-flow systems will often require some stipulation on the differential equation. As with kick-flow systems associated with linear differential equations, we will prove existence and uniqueness of fixed point sinks in nonlinear systems given certain requirements on the differential equation. We begin with a lemma which will be useful for understanding trapping domains (defined below).

Remark Note that in Section (4), we only talk about kick-flow systems. Most of the results hold for flow-kick systems as well given appropriately altered notation.

Lemma 4.1. *If $x > y$, then $\phi_t(x) \geq \phi_t(y)$, where ϕ is a dynamical system.*

Proof. We want to show that if

$$x > y$$

then

$$\phi_t(x) \geq \phi_t(y)$$

Suppose, for contradiction, that

$$\phi_\tau(x) = \phi_\tau(y)$$

for some time τ. Then

$$x = \phi_{-\tau}\phi_\tau(x) = \phi_{-\tau}\phi_\tau(y) = y$$

But this contradicts our assumption that $x > y$. So $\phi_t(x)$ never crosses $\phi_t(y)$. □

Lemma (4.1) shows that for a dynamical system ϕ, if an initial condition x_0 starts to the right of y_0, then $\phi_t(x)$ cannot be to the left of $\phi_t(y)$, and visa versa. Note that in Definition (2.3) the kick-flow system $\psi(x)$ is given by $\phi(T, x + K)$, where T and K are both constants. Thus, by Lemma (4.1):

Corollary 4.2. *If $x > y$, then $\psi(x) \geq \psi(y)$, where ψ is a kick-flow system.*

We are now prepared to define trapping points and domains.

Definition 4.3. *A <u>left hand trapping point</u> of a kick-flow system ψ is a point L such that $\psi(L) \geq L$.*

So if L is a left hand trapping point of the kick-flow system ψ, then by Lemma (4.1) any trajectories with initial condition to the right of L remains to the right of L forever. We also define a right hand trapping point.

Definition 4.4. *A <u>right hand trapping point</u> of a kick-flow system ψ is a point R such that $\psi(R) \leq R$.*

So if R is a right hand trapping point of the kick-flow system ψ, then by Lemma (4.1) any trajectories with initial condition to the left of R remains to the left of R forever. Note that a point x_0 can be both a left hand trapping point, and a right hand trapping point. In that case, x_0 is a fixed point of ψ, since $\psi(x_0) = x_0$. A trapping domain is an interval bounded on the left by a left hand trapping point and bounded on the right by a right hand trapping point.

Definition 4.5. *A <u>trapping domain</u> of a kick-flow system ψ is a closed interval $[L, R]$ such that $\psi(L) \geq L$ and $\psi(R) \leq R$.*

So If the interval $[L, R]$ is a trapping domain of a kick-flow system ψ then trajectories of an initial condition in the trapping domain never leave. In a kick-flow system, if a point x_0 is not a fixed point, then the next iteration of its trajectory must be either to the left or to the right of x_0. We use this fact in the following theorem for the existence of of a fixed point within a trapping domain.

Theorem 4.6. *If a kick-flow system ψ has a trapping domain $[L, R]$, then ψ has a fixed point S such that $S \in [L, R]$.*

Proof. We want to show that for the interval $[L, R]$ if

$$\psi(L) \geq L \text{ and } \psi(R) \leq R$$

then there exists some point $S \in [L, R]$ such that

$$\psi(S) = S$$

Define the continuous function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \psi(x) - x$$

Note that for a point x_0, if $g(x_0) = 0$, then $\psi^n(x_0) = x_0$ and so x_0 must be a fixed point of ψ. So we have

$$g(L) = \psi(L) - L \geq 0 \text{ and } g(R) = \psi(R) - R \leq 0$$

Since g is continuous, by the intermediate value theorem, $g(S) = 0$ for some $S \in [L, R]$. Since $g(S) = 0$, S is a fixed point and we are done. □

Theorem (4.6) proves that a kick-flow system has at least one fixed point on a trapping domain. Assuming not all the points are fixed points, we can show more: that a kick-flow system has at least one sink on a trapping domain. First we prove a lemma on the monotonicity of kick-flow trajectories.

Lemma 4.7. *If x_0 is not a fixed point of kick-flow system ψ, then the trajectory $\psi^n(x_0)$ of the kick-flow system with initial condition x_0 is either monotone increasing or monotone decreasing.*

Proof. By Definition (2.3), the trajectory of a kick-flow system with initial condition x_0 is given by

$$\psi^n(x_0) = \psi(x_{n-1})$$

Since x_0 is not a fixed point, we know that $\psi(x_0) \neq x_0$. This leaves two options. Either $\psi(x_0) > x_0$ or $\psi(x_0) < x_0$. To show the monotonicity of the trajectory $\psi^n(x_0)$, we must show that

$$\psi^1(x_0) > x_0 \implies \psi^n(x_0) > \psi^{n-1}(x_0) \tag{20}$$

and

$$\psi^1(x_0) < x_0 \implies \psi^n(x_0) < \psi^{n-1}(x_0) \tag{21}$$

We can use induction to prove inequality (20). The base case $\psi^1(x_0) > x_0$ for inequality (20) is given by assumption. For the inductive step we want to show

$$\psi^n(x_0) > \psi^{n-1}(x_0) \implies \psi^{n+1}(x_0) > \psi^n(x_0)$$

By Lemma (4.2) we have

$$\psi^n(x_0) > \psi^{n-1}(x_0) \implies \psi(\psi^n(x_0)) > \psi(\psi^{n-1}(x_0))$$

But by Definition (2.3), we have

$$\psi(\psi^n(x_0)) = \psi^{n+1}(x_0)$$

and

$$\psi(\psi^{n-1}(x_0)) = \psi^n(x_0)$$

Thus

$$\psi^n(x_0) > \psi^{n-1}(x_0) \implies \psi^{n+1}(x_0) > \psi^n(x_0)$$

and so the inductive step is satisfied. Thus inequality (20) holds for all n. We prove inequality (21) similarly, by induction. The base case $\psi^1(x_0) < x_0$ for inequality (21) is given by assumption. For the inductive step we want to show

$$\psi^n(x_0) < \psi^{n-1}(x_0) \implies \psi^{n+1}(x_0) < \psi^n(x_0)$$

By Lemma (4.2) we have

$$\psi^n(x_0) < \psi^{n-1}(x_0) \implies \psi(\psi^n(x_0)) < \psi(\psi^{n-1}(x_0))$$

But by Definition (2.3), we have

$$\psi(\psi^n(x_0)) = \psi^{n+1}(x_0)$$

and

$$\psi(\psi^{n-1}(x_0)) = \psi^n(x_0)$$

Thus

$$\psi^n(x_0) < \psi^{n-1}(x_0) \implies \psi^{n+1}(x_0) < \psi^n(x_0)$$

and so the inductive step is satisfied. Thus inequality (21) holds for all n. So the trajectory of ψ with initial condition x_0 is either monotone increasing, or monotone decreasing, as desired. □

Lemma (4.7) shows that trajectories whose initial conditions are not fixed points are monotone. We will use this fact to prove that kick-flow systems have at least one sink on trapping domains.

Theorem 4.8. *If a kick-flow system ψ has a trapping domain $[L, R]$, and not all points in $[L, R]$ are fixed points, then ψ has a sink S such that $S \in [L, R]$.*

Proof. By Definition (2.12) we need to show that if $[L, R]$ is a trapping domain for ψ then there exists some $S \in [L, R]$ such that

$$\forall x \in (S - \epsilon, S + \epsilon), \lim_{n\to\infty} \psi^n(x) = S$$

Note that by Lemma (4.7), as long as x isn't a fixed point, $\psi^n(x)$ is monotone. Consider $\psi^n(x_0)$, the trajectory of ψ which starts at $x_0 \in [L, R]$. If x_0 is not a fixed point, then it is either monotone increasing or monotone decreasing.

Case 1: $\psi^n(x_0)$ is monotone increasing. Note that since $x_0 \leq R$, by Lemma (4.1), R forms an upper bound for $\psi^n(x_0)$. Thus since $\psi^n(x_0)$ is monotone increasing and R is an upper bound, the sequence defined by $\psi^n(x_0)$ converges to some point S between x_0 and R. Then S is a sink and we are done.

Case 2: $\psi^n(x_0)$ is monotone decreasing. Note that since $x_0 \geq L$, by Lemma (4.1), L forms a lower bound for $\psi^n(x_0)$. Thus since $\psi^n(x_0)$ is monotone decreasing and L is a lower bound, the sequence defined by $\psi^n(x_0)$ converges to some point S between x_0 and R. Then S is a sink and we are done.

If x_0 is a fixed point, then it is either a sink, or a source. If it is a sink, then we are done. But note that if x_0 is a source, it is still a right hand trapping point, since it is still a fixed point and thus $\psi(x_0) = x_0$. Additionally, if x_0 is a source, then there is an initial condition x_1 in the neighborhood of x_0 whose kick-flow trajectory $\psi^n(x_1)$ moves away from x_0 as n increases. Since that trajectory must be monotone, but is still bounded by L on the left and R on the right, it must converge so some point S. S is therefore a sink and we are done. □

We have shown that if a kick-flow system ψ has a trapping domain, then there is at least one sink on that domain. But when can we know that we have a trapping domain? By Definition (4.5), we need an interval (L, R) such that $\psi(L) \geq L$ and $\psi(R) \leq R$. The following the lemmas provide conditions which guarantee trapping domains.

Lemma 4.9. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$C \geq -f(x) \text{ for all } x \in [L, L+K]$$

where C is the disturbance rate, then L is a left hand trapping point.

Proof. By definitions (3.3) and (4.3) we want to show that if

$$\frac{K}{T} \geq -f(x) \text{ for all } x \in [L, L+K]$$

then

$$\psi(L) \geq L$$

Employing some algebra and using definitions (2.1) and (2.3), for all $x \in [L, L+K]$, we have

$$\begin{aligned}
\tfrac{K}{T} &\geq -f(x) \\
\int_0^T \tfrac{K}{T}\, dt &\geq \int_0^T -f(x)\, dt \\
\tfrac{K}{T}(T) &\geq -\int_0^T f(x)\, dt \\
K &\geq -\int_0^T f(x)\, dt \\
L + K + \int_0^T f(x)\, dt &\geq L \\
\phi(T, L+K) &\geq L \\
\psi(L) &\geq L
\end{aligned}$$

for all $x \in [L, L+K]$ as desired. □

Lemma (4.9) shows that if the disturbance rate is greater than the instantaneous rate of flow in the left direction between points $L + K$ and L, then L is a left hand trapping point. If we can find an upper bound for $-f(x)$, we can just compare that upper bound to the disturbance rate.

Corollary 4.10. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$C \geq \sup\{-f(x) : x \in [L, L+K]\}$$

where C is the disturbance rate, then L is a left hand trapping point.

We prove a similar lemma for right hand trapping points.

Lemma 4.11. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$C \leq -f(x) \text{ for all } x \in [R-K, R]$$

where C is the disturbance rate, then R is a right hand trapping point.

Proof. By definitions (4.4) and (3.3) we want to show that if

$$\frac{K}{T} \leq -f(x) \text{ for all } x \in [R-K, R]$$

then

$$\psi(R) \leq R$$

Employing some algebra and using definitions (2.1) and (2.3), for all $x \in [R-K, R]$, we have

$$\begin{aligned} \tfrac{K}{T} &\leq -f(x) \\ \int_0^T \tfrac{K}{T}\, dt &\leq \int_0^T -f(x)\, dt \\ \tfrac{K}{T}(T) &\leq -\int_0^T f(x)\, dt \\ K &\leq -\int_0^T f(x)\, dt \\ R + K + \int_0^T f(x)\, dt &\leq R \\ \phi(T, R+K) &\leq R \\ \psi(R) &\leq R \end{aligned}$$

for all $x \in [R-K, R]$ as desired. □

We can write a corollary for Lemma (4.11).

Corollary 4.12. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$C \leq \inf\{-f(x) : x \in [R-K, R]\}$$

where C is the disturbance rate, then R is a right hand trapping point.

We are now prepared to state and prove a theorem which guarantees the existence of a fixed point in a kick-flow system on a certain domain under certain conditions near the end points of that domain.

Theorem 4.13. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$C \geq -f(x) \forall x \in [L, L+K] \text{ and } C \leq -f(x) \forall x \in [R-K, R]$$

where C is the disturbance rate and $L + K \leq R - K$, then ψ has a fixed point S such that $S \in [L, R]$.

Proof. By lemmas (4.9) and (4.11), we know that L is a left hand trapping point and R is a right hand trapping point. Thus $[L, R]$ is a trapping domain, and theorem (4.6) tells us that ψ has a fixed point S such that $S \in [R, L]$, as desired. □

Theorem (4.13) shows that we need only understand the flow over small intervals at the end points of a domain of interest to determine whether that domain has any fixed points. We can write a corollary to Theorem (4.13) which is similar to corollaries (4.10) and (4.12).

Corollary 4.14. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$\sup\{-f(x) : x \in [L, L+K]\} \leq C \leq \inf\{-f(x) : x \in [R-K, R]\}$$

where C is the disturbance rate and $L \leq R$, then ψ has a fixed point S such that $S \in [L, R]$.

Now all we need to know about an interval is an upper bound for the flow within a specific neighborhood of the end points to show that the domain has at least one fixed point. We proceed to find requirements we can place on a differential equation $\dot{x} = f(x)$ which make finding the upper and lower bounds of f on intervals a simple task.

Lemma 4.15. *If the first derivative of a function $f(x)$ with respect to x is negative on the domain $[x_1, x_2]$, then*

$$\sup\{f(x) : x \in [x_1, x_2]\} = f(x_1)$$

and

$$\inf\{f(x) : x \in [x_1, x_2]\} = f(x_2)$$

Proof. Since

$$f'(x) < 0$$

f is monotone decreasing on $[x_1, x_2]$. Thus $f(x)$ attains a maximum at the smallest x, which in this case is the left end point $x = x_1$. By similar reasoning, $f(x)$ attains a minimum where x is largest, which occurs at $x = x_2$. So we are done. □

We can use Lemma (4.15) to show the existence of fixed points in kick-flow systems for which we know the sign of f in the associated differential equation.

Lemma 4.16. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$f'(x) < 0 \text{ for all } x \in ([L, L+K] \cup [R-K, R])$$

and

$$-f(L+K) \leq C \leq -f(R-K)$$

where C is the disturbance rate and $L < R$, then ψ has a fixed point S such that $S \in [L, R]$.

Proof. We want to show that if

$$f'(x) < 0 \text{ for all } x \in ([L, L+K] \cup [R-K, R])$$

and

$$-f(L+K) \leq C \leq -f(R-K)$$

then there is a fixed point between L and R. We know from Lemma (4.15) that

$$\sup\{-f(x) : x \in [L, L+K]\} = -f(L+K)$$

and

$$\inf\{-f(x) : x \in [R-K, R]\} = -f(R-K)$$

By assumption

$$-f(L+K) \leq C \leq -f(R-K)$$

and thus

$$\sup\{-f(x) : x \in [L, L+K]\} \leq C \leq \inf\{-f(x) : x \in [R-K, R]\}$$

and so by Corollary (4.14), ψ has a fixed point S such that $S \in [L, R]$, as desired. □

Lemma (4.16) tells us that if the derivative of the differential equation $\dot{x} = f(x)$ associated with a kick-flow system is negative within certain intervals, and certain conditions on the value of f are satisfied, then there is a fixed point between L and R. Of course, if $f'(x)$ is negative over our whole domain of interest, then it is certainly negative at the end intervals. This is is corollary (4.17).

Corollary 4.17. *Given a kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if*

$$f'(x) < 0 \text{ for all } x \in [L, R]$$

and

$$-f(L+K) \leq C \leq -f(R-K)$$

where C is the disturbance rate, then ψ has a fixed point S such that $S \in [L, R]$.

Consider the kick-flow system ψ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T) where $f'(x) < 0$ for all x. From Corollary (4.17), we can see that by showing certain requirements on the differential equation we can guarantee the existence of a fixed point. Another way to guarantee the existence of a fixed point of a kick-flow system is to show that $f(x)$ has a range of $(-\infty, \infty)$. That provides a fixed point, because fixed points occur when the kick balances the flow, so if the flow reaches every possible value, then it must balance the kick somewhere. Unfortunately, knowing that $-f'(x) < 0$ for all x does not necessarily imply that $f(x)$ reaches every real number because $f(x)$ could be monotone decreasing but converge to some value.

This concludes our discussion of fixed point existence. There are many ways to find out whether a kick-flow system has a fixed point on a given domain, but we move on now to a discussion of uniqueness. First we prove a lemma.

Theorem 4.18. *Given a kick-flow system ψ associated with a differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if $[L, R]$ is a trapping domain and $f'(x) \neq 0$ for all $x \in [L, R]$, then ψ has exactly one fixed point.*

Proof. We want to show that if $[L, R]$ is a trapping domain and

$$f'(x) < 0 \text{ for all } x \in [L, R]$$

then

$$\psi(S) = S$$

for some $S \in [L, R]$ and

$$x \neq S \implies \psi(x) \neq x$$

The first of these is proved by Theorem (4.6). The second we prove by contradiction. Assume that there exists an $x = x_0 \in [L, R]$ such that $x_0 \neq S$ and $\psi(x_0) = x_0$. Since S and x_0 are fixed points, we know from Lemma (2.6) that

$$\phi(T, S+K) = S \text{ and } \phi(T, x_0+K) = x_0$$

Thus by Definition (2.1)

$$S + K + \int_0^T f(x(t))dt = S$$

where $x(0) = S + K$. Subtracting S we obtain

$$K + \int_0^T f(x(t))dt = 0 \tag{22}$$

Also by Definition (2.1)

$$x_0 + K + \int_0^T f(x(t))dt = x_0$$

where $x(0) = x_0 + K$. Subtracting x_0 we obtain

$$K + \int_0^T f(x(t))dt = 0 \tag{23}$$

But since $f'(x) \neq 0$, we know that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Thus since $S \neq x_0$

$$\int_0^T f(x(t))dt \text{ with } x(0) = S + K$$

is not equal to

$$\int_0^T f(x(t))dt \text{ with } x(0) = x_0 + K$$

This gives rise to a contradiction since both equation (22) and (23) cannot be simultaneously satisfied. Therefore, $\psi(x)$ must not be equal to x if $x \neq S$, as desired. □

Theorem (4.18) shows that if the first derivative of the differential equation associated with a kick-flow system does not change sign, then trapping domains contain exactly one fixed point. Since we know that trapping domains must have at least one sink, we know that if the first derivative of the differential equation associated with a kick-flow system does not change sign, then the unique fixed point must be a sink.

We have established a few requirements on nonlinear differential equations that guarantee stable fixed points in the associated kick-flow system. Using the theorems presented in this paper, we can look at a kick-flow system and figure out whether a given domain contains stable fixed points or not. In this way we can analyze the resilience of a given system to disturbances. In the next section, we outline a few possible directions that this paper could go in the future.

5. Future work

The motivation for this project was to better understand the resilience of a dynamical system to disturbance. In Section (2), we describe the concept of and establish the notation for kick-flow systems. We apply those ideas in Section (3) to kick-flow systems associated with linear differential equation, proving the existence and uniqueness of stable fixed points given a few simple stipulations. In Section (4) we do the same for kick-flow systems associated with nonlinear differential equations. There are a few directions we could take to further understand the behavior of kick-flow systems.

The first topic of potential future work has to do with the disturbance rate. In Theorem (3.9) at the end of Section (3), we show that for a kick-flow system ψ associated with a linear differential equation $\dot{x} = -mx$, we can construct another differential equation whose dynamics are the same as the dynamics of ψ, where the kick size and time interval both approach zero while maintaining a constant disturbance rate C. That is to say that if the disturbance rate is held constant, then

$$\lim_{K,T\to 0} \psi^n(x_0) = \eta_{x_0}(R(n))$$

where ψ is the kick-flow system associated with $\dot{x} = -mx$ subjected to disturbance (K, T), $R(n)$ is the run time, and η is the dynamical system associated with $\dot{x} = -mx + C$. It would be nice to find the dynamical system which serves the same purpose as η above, but applied to a nonlinear system. For this, we have a conjecture.

Conjecture: Given a kick-flow system ψ and a flow-kick system σ associated with differential equation $\dot{x} = f(x)$ subjected to disturbance (K, T), if the cumulative disturbance C is held constant, then

$$\lim_{K,T\to 0} \psi^n(x_0) = \eta_{x_0}(R(n)) = \lim_{K,T\to 0} \sigma^n(x_0 + K)$$

where η is the dynamical system associated with $\dot{x} = f(x) + C$, $R(n)$ is the run time, and n is an integer.

Proving this conjecture would be an interesting result because it would give us a way to develop a lower bound for the resilience of a dynamical system to disturbance.

Another direction that the project could take is to generalize to systems of n differential equations. In that case, the kick-size would need to be a vector, instead of just a scaler and an understanding of resilience would need to incorporate both the magnitude and direction of the kick vector.

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References

[1] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*, volume 60. Academic press, 2004.

[2] Hans Kaper and Hans Engler. *Mathematics and climate*. Siam, 2013.

[3] Marten Scheffer. *Critical transitions in nature and society*. Princeton University Press, 2009.

[4] Brian Walker and David Salt. *Resilience thinking: sustaining ecosystems and people in a changing world*. Island Press, 2012.